Non-Uniform Stability of Damped Contraction Semigroups

Lassi Paunonen

Tampere University, Finland

joint work with R. Chill, D. Seifert, R. Stahn and Y. Tomilov.

Workshop on Analysis, Control, and Operator Theory

University of Bordeaux

July 8th, 2021

Goal of the Talk

*Introduce general conditions for non-uniform stability of **damped** hyperbolic Cauchy problems (and PDEs).*

Goal of the Talk

Introduce general conditions for non-uniform stability of ***damped*** *hyperbolic Cauchy problems (and PDEs).*

$$\begin{cases} \dot{x}(t) = (A - BB^*)x(t) \\ x(0) = x_0 \end{cases}$$

and

$$\begin{cases} \ddot{w}(t) + A_0 w(t) + B_0 B_0^* \dot{w}(t) = 0 \\ w(0) = w_0, \quad \dot{w}(0) = w_1 \end{cases}$$

Problem

Formulate conditions on (A, B) *and* (A_0, B_0) *such that*

$$\|x(t)\| \to 0, \qquad \text{or} \qquad \|w(t)\| \to 0 \qquad \text{as} \quad t \to \infty$$

and especially study the ***rate*** *of the convergence.*

Goal of the Talk

*Introduce general conditions for non-uniform stability of **damped** hyperbolic Cauchy problems and PDEs.*

Damped systems of the form

$$\dot{x}(t) = (A - BB^*)x(t) \quad \text{and} \quad \ddot{w}(t) + A_0 w(t) + B_0 B_0^* \dot{w}(t) = 0$$

Motivation:

- So-called "polynomial" and "non-uniform" stability often arise in wave/beam/plate equations with weak or partial dampings
- Most of the current literature based on case-by-case analysis

Main results:

- General **observability-type** sufficient conditions for stability

(B^*, A) **exactly** observable $\quad\Leftrightarrow\quad$ $A - BB^*$ exponentially stable

$$\|x(t)\| \le Me^{-\omega t}\|x_0\| \;\forall x_0$$

(B^*, A) **approx.** observable $\quad\Leftrightarrow^*\quad$ $A - BB^*$ strongly/weakly stable

$$x(t) \to 0 \;\forall x_0$$

[Slemrod, Levan, Russell, Benchimol, Guo–Luo, Lasiecka–Triggiani, Curtain–Weiss . . .]

(B^*, A) **exactly** observable $\Leftrightarrow$ $A - BB^*$ exponentially stable

$$\|x(t)\| \le Me^{-\omega t}\|x_0\| \ \forall x_0$$

(B^*, A) **non-uniformly** obs. $\Leftrightarrow$ $A - BB^*$ non-uniformly stable

(B^*, A) **approx.** observable $\Leftrightarrow^*$ $A - BB^*$ strongly/weakly stable

$$x(t) \to 0 \ \forall x_0$$

[Slemrod, Levan, Russell, Benchimol, Guo–Luo, Lasiecka–Triggiani, Curtain–Weiss . . .]

Earlier work: Ammari–Tucsnak 2001, Ammari et. al.

Main Assumptions (roughly, to keep things simple)

- A generates a contraction semigroup e^{At} on X Hilbert, i.e., $t \mapsto e^{At}$ is strongly continuous and $\|e^{At}\| \le 1$.
- Either $B \in \mathcal{L}(U,X)$, or (A,B,B^*) is a "well-posed system".
- $\Rightarrow A - BB^*$ generates a contraction semigroup $e^{(A-BB^*)t}$

Main case:

$$\ddot{w}(t) + A_0 w(t) + B_0 B_0^* \dot{w}(t) = 0, \qquad \text{on} \quad X_0$$

where $A_0 > 0$, $B_0 \in \mathcal{L}(U, D(A_0^{1/2})^*)$ leads to

$$A = \begin{bmatrix} 0 & I \\ A_0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ B_0 \end{bmatrix}, \qquad \text{on} \quad X = D(A_0^{1/2}) \times X_0.$$

"Well-posedness" $\Leftrightarrow \lambda \mapsto \lambda B_0^*(\lambda^2 + A_0)^{-1} B_0$ bounded for $\lambda = 1 + is$

Polynomial and Non-Uniform Stability

Definition

$e^{(A-BB^*)t}$ generated by $A - BB^*$ is **non-uniformly stable** if there exist an increasing $M_T \colon [t_0,\infty) \to \mathbb{R}_+$ and $C > 0$ such that

$$\|e^{(A-BB^*)t}x_0\| \leq \frac{C}{M_T(t)}\|(A-BB^*)x_0\| \qquad x_0 \in D(A-BB^*)$$

[. . . , Liu–Rao '05, Batty–Duyckaerts '08, Borichev–Tomilov '10, Rozendaal–Seifert–Stahn '19]

Application: $E(t) \sim \|e^{(A-BB^*)t}x_0\|^2$ for many PDE systems.

Polynomial and Non-Uniform Stability

Definition

$e^{(A-BB^*)t}$ generated by $A-BB^*$ is **non-uniformly stable** if there exist an increasing $M_T\colon [t_0,\infty)\to\mathbb{R}_+$ and $C>0$ such that

$$\|e^{(A-BB^*)t}x_0\| \le \frac{C}{M_T(t)}\|(A-BB^*)x_0\| \qquad x_0\in D(A-BB^*)$$

[. . . , Liu–Rao '05, Batty–Duyckaerts '08, Borichev–Tomilov '10, Rozendaal–Seifert–Stahn '19]

Theorem (BT'10, RSS'19)

Assume $e^{(A-BB^*)t}$ *is contractive,* $i\mathbb{R}\subset\rho(A-BB^*)$*, and*

$$\|(is-A+BB^*)^{-1}\| \le M(|s|), \qquad M \text{ non-decreasing.}$$

- *If* $M(s)\lesssim 1+s^\alpha$*, then* $M_T(t)=t^{1/\alpha}$
- *If* M *has "positive increase", then* $M_T(t)=M^{-1}(t)$.

Main Problem

Damped systems of the form

$$\dot{x}(t) = (A - BB^*)x(t) \quad \text{and} \quad \ddot{w}(t) + A_0 w(t) + B_0 B_0^* \dot{w}(t) = 0$$

Problem

How do (A, B) or (A_0, B_0) determine the stability of the system?

Main results:

*Conditions based on **observability-type properties** of (B^*, A) and (B_0^*, iA_0).*

A “Non-uniform Hautus test”

Consider the Hautus-type condition [Miller 2012]

$$\|x\|^2 \le M_o(|s|)\|(is-A)x\|^2 + m_o(|s|)\|B^*x\|^2, \quad x \in D(A), s \in \mathbb{R},$$

for some non-decreasing $M_o, m_o \colon [0,\infty) \to [r_0,\infty)$.

A "Non-uniform Hautus test"

Consider the Hautus-type condition [Miller 2012]

$$\|x\|^2 \le M_o(|s|)\|(is - A)x\|^2 + m_o(|s|)\|B^*x\|^2, \quad x \in D(A), s \in \mathbb{R},$$

for some non-decreasing $M_o, m_o \colon [0,\infty) \to [r_0,\infty)$.

Theorem

If the above condition holds, then $i\mathbb{R} \subset \rho(A - BB^*)$. If $M(s) := M_o(s) + m_o(s)$ has positive increase, then

$$\|e^{(A-BB^*)t}x_0\| \le \frac{C}{M^{-1}(t)}\|(A - BB^*)x_0\|, \quad x_0 \in D(A - BB^*)$$

Observability of the Schrödinger Group

For

$$\ddot{w}(t) + A_0 w(t) + B_0 B_0^* \dot{w}(t) = 0, \qquad \text{on} \quad X_0$$

and $M_S, m_S \colon [0,\infty) \to [r_0,\infty)$ consider ($s \geq 0$)

$$\|w\|^2 \leq M_S(s) \|(s^2 - A_0)w\|^2 + m_S(s)\|B_0^* w\|^2, \quad w \in D(A_0)$$

This is **observability of the "Schrödinger group"** (B_0^*, iA_0) (generalises Anantharaman–Leataud 2014, Joly–Laurent 2019)

Theorem

A similar result, decay rate determined by $M^{-1}(t)$, where

$$M(s) := M_S(s) m_S(s)(1+s^2).$$

A "Wavepacket Condition"

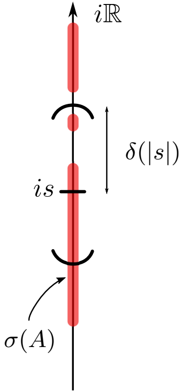
For A skew-adjoint with spectral projection $P_{(a,b)}$ (for $i(a,b) \subset i\mathbb{R}$)

$$\|B^*x\| \geq \gamma(|s|)\|x\|, \qquad x \in \mathrm{Ran}(P_{(s-\delta(|s|),s+\delta(|s|))}), \ s \in \mathbb{R}$$

for some non-increasing $\delta, \gamma \colon [0,\infty) \to (0, r_0]$.

Such x are often called **"wavepackets"** of A.

(Used for exact observability, e.g., in Ramdani et. al. 2005, Miller 2012, Tucsnak–Weiss 2009.)



A "Wavepacket Condition"

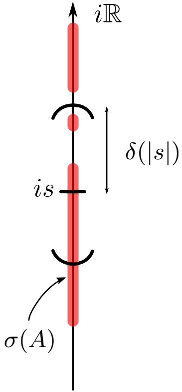
For A skew-adjoint with spectral projection $P_{(a,b)}$ (for $i(a,b) \subset i\mathbb{R}$)

$$\|B^*x\| \geq \gamma(|s|)\|x\|, \qquad x \in \operatorname{Ran}(P_{(s-\delta(|s|),s+\delta(|s|))}), \ s \in \mathbb{R}$$

for some non-increasing $\delta, \gamma \colon [0,\infty) \to (0, r_0]$.

Such x are often called **"wavepackets"** of A.

(Used for exact observability, e.g., in Ramdani et. al. 2005, Miller 2012, Tucsnak–Weiss 2009.)



Theorem

If $A^* = -A$ and if $M(s) := \delta(s)^{-2}\gamma(s)^{-2}$ has positive increase, then

$$\|e^{(A-BB^*)t}x_0\| \leq \frac{C}{M^{-1}(t)}\|(A-BB^*)x_0\|.$$

Time-Domain Non-Uniform Observability

Time-domain observability conditions:

If $0 \in \rho(A)$, $\tau, c_\tau, \beta > 0$:

$$c_\tau \|(-A)^{-\beta} x_0\|^2 \leq \int_0^\tau \|B^* e^{At} x_0\|^2 dt, \qquad x_0 \in D(A).$$

(cf. generalised observability conditions by Ammari–Tuscnak 2001, Ammari–Bchatnia–El Mufti 2017)

Theorem

Assume $A^* = -A$, $0 \in \rho(A)$, $B \in \mathcal{L}(U,X)$ and $0 < \beta \leq 1$. If the above condition holds, then $i\mathbb{R} \subset \rho(A - BB^*)$, and

$$\|e^{(A-BB^*)t} x_0\| \leq \frac{C}{t^{1/(2\beta)}} \|Ax_0\|, \quad x_0 \in D(A)$$

Examples: 2D Wave Equations

A wave equation with viscous damping on a convex $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary, $b \in L^\infty(\Omega)$

$$
\begin{aligned}
&w_{tt}(\xi,t) - \Delta w(\xi,t) + b(\xi)^2 w_t(\xi,t) = 0, && \xi \in \Omega,\ t > 0,\\
&w(\xi,t) = 0, && \xi \in \partial\Omega,\ t > 0,\\
&w(\cdot,0) = w_0(\cdot) \in H^2(\Omega) \cap H_0^1(\Omega), && w_t(\cdot,0) = w_1(\cdot) \in H_0^1(\Omega).
\end{aligned}
$$

- Several results exist for the exact observability of the Schrödinger group $(b, i\Delta)$ (Jaffard '90, Burq–Zworski '19) for rectangles/tori. Leads to polynomial decay $1/\sqrt{t}$.
- Precise lower bounds on b lead to generalised observability of the Schrödinger group via Burq–Zuily 2016.
- In general our results are sub-optimal, since conditions do not take into account the **smoothness** of b! (Burq–Hitrik '07)

1D Wave Equations

Consider a wave equation with weak damping (and Dirichlet BC)

$$w_{tt}(\xi,t) - w_{\xi\xi}(\xi,t) + b(\xi)\int_0^1 b(r)w_t(r,t)dr = 0, \quad \xi \in (0,1), \; t > 0,$$

- The wavepacket condition characterises (optimal) stability via lower bounds of the sine Fourier coefficients, e.g., $(c, \alpha > 0)$

$$\left|\int_0^1 b(\xi)\sin(n\pi\xi)d\xi\right| \geq \frac{c}{n^\alpha}$$

- Pointwise damping possible (formally $b(\xi) = \delta(\xi - \xi_0)$).
- Analogous results for Euler–Bernoulli / Timoshenko beams

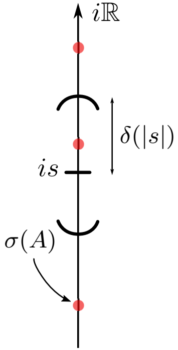
Application: Water Waves System

In the reference

Su–Tucsnak–Weiss "Stabilizability properties of a linearized water waves system," *Systems & Control Letters*, 2020.

the results were applied to prove polynomial stabilizability of a "water waves system" in a 2D rectangular domain.

- Models small amplitude gravity water waves
- A has discrete spectrum $\subset i\mathbb{R}$, but no uniform gap, $\lambda_k \approx i\sqrt{k}$
- Decay rate derived using the "Wavepacket condition"
- $\delta(s) \to 0$ so that $(s-\delta(|s|), s+\delta(|s|)) \cap \sigma(A)$ are singleton sets for all $s \in \mathbb{R}$.
- Optimality possible (..?)



Conclusions

In this presentation:

- General sufficient conditions for non-uniform stability of the semigroup generated by $A - BB^*$.
- Discussion of PDE examples and optimality of the results

R. Chill, LP, D. Seifert, R. Stahn, Y. Tomilov, "Non-Uniform Stability of Damped Contraction Semiroups," *in review* (https://arxiv.org/abs/1911.04804)

Contact: lassi.paunonen@tuni.fi