Damping and Negative Feedback in Infinite-Dimensional Systems

Lassi Paunonen

Tampere University, Finland

July 2025

Funded by the Research Council of Finland grant 349002

Damping in Abstract Infinite-Dimensional Systems

Consider an abstract Cauchy problem

$$\dot{x}(t) = (A - BB^*)x(t), \qquad x(0) = x_0$$

on a Hilbert space X.

- A generates a contraction semigroup $T(t)$ (often $A^* = -A$)
- $-BB^*$ represents damping

Problem

Find conditions on A and B such that $\|x(t)\| \to 0$ as $t \to \infty$ for all x_0, and investigate the rates of convergence.

Plan of the talk

- Introduction and examples
- Main results on sufficient conditions for stability
- Nonlinear dampings

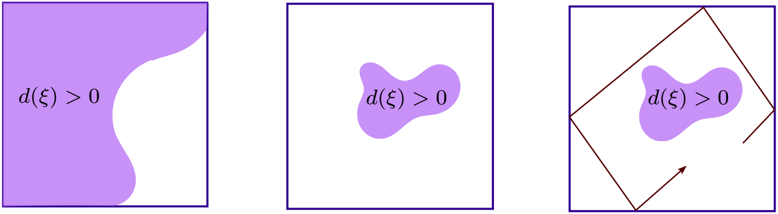
Disclaimer: The presentation contains over-simplifications and some technical assumptions are hidden. Details can be found in the articles.

Primary Motivation: Damped Wave Equations

Consider the wave equation on a "nice" domain $\Omega \subset \mathbb{R}^2$,

$$\begin{aligned} \ddot{w}(\xi,t) - \Delta w(\xi,t) + d(\xi)\dot{w}(\xi,t) &= 0, & \xi \in \Omega, \quad t > 0 \\ w(\xi,t) &= 0 & \xi \in \partial\Omega \end{aligned}$$

Then the stability of the wave equation depends on geometry of Ω and $\omega := \{ \xi \in \Omega \mid d(\xi) > 0 \}$:



Exponential stability | Non-uniform stability | Geometric Control Condition

Primary Motivation: Damped Wave Equations

The wave equation on $\Omega \subset \mathbb{R}^2$,

$$\begin{aligned} \ddot{w}(\xi,t) - \Delta w(\xi,t) + d(\xi)\dot{w}(\xi,t) &= 0, \qquad && \xi \in \Omega, \quad t>0 \\ w(\xi,t) &= 0 && \xi \in \partial\Omega \end{aligned}$$

has the form $\dot{x}(t) = (A - BB^*)x(t)$ on $X = H_0^1(\Omega) \times L^2(\Omega)$ with $x(t) = (w(\cdot,t), \dot{w}(\cdot,t))^\top$,

$$A = \begin{bmatrix} 0 & I \\ -\Delta & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ \sqrt{d(\cdot)} \end{bmatrix},$$

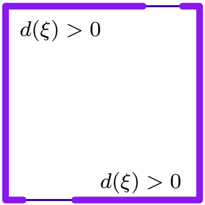
and $D(A) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$. Here $A^* = -A$ and $B \in \mathcal{L}(L^2(\Omega), X)$.

Wave Equations with Boundary Damping

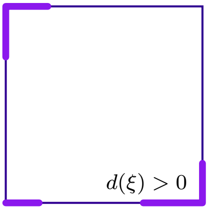
Consider the wave equation on a domain $\Omega \subset \mathbb{R}^2$, with boundary Γ,

$$\begin{aligned} \ddot{w}(\xi,t) - \Delta w(\xi,t) &= 0, & \xi \in \Omega, \quad t > 0 \\ \nu \cdot \nabla w(\xi,t) + d(\xi)\dot{w}(\xi,t) &= 0 & \xi \in \Gamma \end{aligned}$$

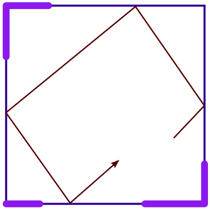
Then the stability of the wave equation again depends on Ω and now on $\omega := \{ \xi \in \Gamma \mid d(\xi) > 0 \}$ [Bardos, Lebeau, Rauch '92]:



Exponential stability



Non-uniform stability



GCC

A (Very) Practical Example

A (Very) Practical Example

A (Very) Practical Example

Well-posedness

Does the equation

$$\dot{x}(t) = (A - BB^*)x(t), \qquad x(0) = x_0$$

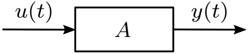
have solutions?

- When $B \in \mathcal{L}(U, X)$ (i.e., B is "bounded"), then $A - BB^*$ generates a contraction semigroup.
- Boundary dampings lead to situations where $\mathrm{Ran}(B) \not\subset X$, and the semigroup property of $A - BB^*$ is more delicate.

Connection to Systems with Feedback

Consider the **linear system** (A, B, B^*)

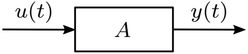
$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = B^*x(t)$$



Connection to Systems with Feedback

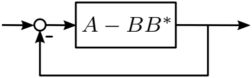
Consider the **linear system** (A, B, B^*)

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = B^*x(t)$$



Defining $u(t) = -y(t) + v(t)$, we (formally) obtain

$$\dot{x}(t) = (A - BB^*)x(t) + Bv(t),$$
$$y(t) = B^*x(t)$$



This is known as negative feedback.

$\rightsquigarrow$ This explains how **feedback theory** can be used to investigate the well-posedness of the damped equation!

Result

Under very mild assumptions $(A - BB^*, B, B^*)$ is "well-posed" and $A - BB^*$ generates a semigroup $T_B(t)$.

Stability Analysis

When $A - BB^*$ generates a contraction semigroup, the asymptotic behaviour of solutions of

$$\dot{x}(t) = (A - BB^*)x(t), \qquad x(0) = x_0$$

can be studied in terms of the resolvent $(\lambda - A + BB^*)^{-1}$ on $i\mathbb{R}$:

Result (Exponential and polynomial stability)

If $\|(is - A + BB^*)^{-1}\| \leq M_R$, *then* $\|x(t)\| \leq Me^{-\alpha t}\|x_0\|$ *for all* $x_0 \in X$ *with* $\alpha > 0$.

Stability Analysis

When $A - BB^*$ generates a contraction semigroup, the asymptotic behaviour of solutions of

$$\dot{x}(t) = (A - BB^*)x(t), \qquad x(0) = x_0$$

can be studied in terms of the resolvent $(\lambda - A + BB^*)^{-1}$ on $i\mathbb{R}$:

Result (Exponential and polynomial stability)

If $\|(is - A + BB^*)^{-1}\| \leq M_R$, *then* $\|x(t)\| \leq Me^{-\alpha t}\|x_0\|$ *for all* $x_0 \in X$ *with* $\alpha > 0$.

If $\|(is - A + BB^*)^{-1}\| \leq M(1 + |s|^\alpha)$, *then* $\|x(t)\| = o(t^{-1/\alpha})$ *for all* $x_0 \in D(A - BB^*)$.

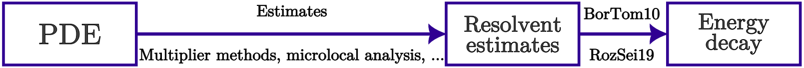
Stability Analysis

- Our feedback structure is **also useful in stability analysis**
- Roughly: "Equation $\dot{x}(t) = (A - BB^*)x(t)$ is stable if the system (A, B, B^*) is **observable**"
- **Observability** measures how easy it is to reconstruct the state $x(t)$ of the system

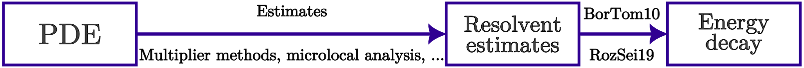
$$\dot{x}(t) = Ax(t)$$
$$y(t) = B^*x(t)$$

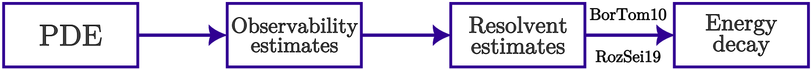
  based only on the "observation" $y(t)$ over some time interval.
- This way observability is related to how well B^* "can see" (especially the unstable) modes of A.
  $\rightsquigarrow$ can measure how effectively $-BB^*$ can damp these modes!

Deriving Decay Rates



Deriving Decay Rates





- "Observability estimates" aim to reduce the derivation of resolvent estimates to a simpler problem.
- Instead of the damped problem, these involve the undamped equation with an output $y(t) = B^*x(t)$.
- For polynomial stability: Ammari–Tucsnak 2001, Ammari et. al., Anantharaman–Léautaud 2014, Joly–Laurent 2019, ...

Main Results

Observability-Type Conditions for Stability

A Non-Uniform Hautus Test

Consider the Hautus-type condition [Miller 2012]

$$\|x\|^2 \le M(|s|)\|(is - A)x\|^2 + m(|s|)\|B^*x\|^2, \quad x \in D(A), s \in \mathbb{R},$$

for some non-decreasing $M, m\colon [0,\infty) \to [r_0,\infty)$.

A Non-Uniform Hautus Test

Consider the Hautus-type condition [Miller 2012]

$$\|x\|^2 \leq M(|s|)\|(is - A)x\|^2 + m(|s|)\|B^*x\|^2, \quad x \in D(A), s \in \mathbb{R},$$

for some non-decreasing $M, m \colon [0, \infty) \to [r_0, \infty)$.

Theorem (Chill–P–Seifert–Stahn–Tomilov '23)

If the above condition holds, then $i\mathbb{R} \subset \rho(A - BB^*)$ and

$$\|(is - A + BB^*)^{-1}\| \lesssim (m(|s|) + M(|s|))(1 + |s|^\eta)$$

where $\eta \geq 0$ is such that $\|B^*(1 + is - A)^{-1}B\|^2 \lesssim 1 + |s|^\eta$.

- η measures the "level of unboundedness" of B, and $\eta \leq 4$.
- If $m(s) + M(s) \lesssim 1 + s^\beta$, then $\|x(t)\| = o(t^{-1/(\beta+\eta)})$ for all $x_0 \in D(A - BB^*)$.

Other Observability Conditions

Other results in [Chill–P–Seifert–Stahn–Tomilov '23]:

- Resolvent estimate based on a "wavepacket condition" with variable parameters
- Hautus test for the "Schrödinger group" for second order systems
- A time-domain weighted observability condition

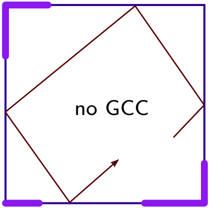
Result (Chill–P–Seifert–Stahn–Tomilov '23)

The above conditions imply resolvent bounds for $A - BB^*$.

Wave Equations with Boundary Damping

Wave equation with boundary damping:

$$\ddot{w}(\xi,t) - \Delta w(\xi,t) = 0, \qquad \xi \in \Omega$$
$$\nu \cdot \nabla w(\xi,t) + d(\xi)\dot{w}(\xi,t) = 0, \quad \xi \in \Gamma$$



- Lack of exponential stability understood in the non-GCC case
- Concrete decay rates in the non-GCC have not been investigated much (unlike for in-domain dampings!)

The Wave Equation with Boundary Damping

The wave equation on $\Omega \subset \mathbb{R}^2$ with boundary Γ, $d \in L^\infty$, $d \geq 0$

$$\ddot{w}(\xi,t) - \Delta w(\xi,t) = 0, \qquad \xi \in \Omega, \quad t > 0$$

$$\nu \cdot \nabla w(\xi,t) + d(\xi)\dot{w}(\xi,t) = 0 \qquad \xi \in \Gamma.$$

Plan:

1. Verify an observability condition
2. Find $\eta \geq 0$ such that $\|B^*(1+is-A)^{-1}B\|^2 \lesssim 1+|s|^\eta$.

The Wave Equation with Boundary Damping

The wave equation on $\Omega \subset \mathbb{R}^2$ with boundary Γ, $d \in L^\infty$, $d \geq 0$

$$\ddot{w}(\xi,t) - \Delta w(\xi,t) = 0, \qquad \xi \in \Omega, \quad t > 0$$

$$\nu \cdot \nabla w(\xi,t) + d(\xi)\dot{w}(\xi,t) = 0 \qquad \xi \in \Gamma.$$

Plan:

1. Verify an observability condition
2. Find $\eta \geq 0$ such that $\|B^*(1+is-A)^{-1}B\|^2 \lesssim 1 + |s|^\eta$.

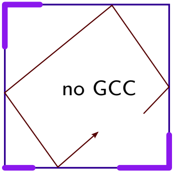
Proposition (LP, D. Seifert, N. Vanspranghe, '24)

We have $\|B^*(1+is-A)^{-1}B\|^2 \lesssim 1 + |s|^\eta$ *in the following cases:*

$\eta = 1 + \varepsilon$	*for* Ω *rectangle*
$\eta = 1$	*when* Γ *is smooth and flat*
$\eta = 2/3$	*when* Γ *is smooth and concave*
$\eta = 4/3$	*when* Γ *is smooth*

Wave Equation on a Rectangle

$$\ddot{w}(\xi,t) - \Delta w(\xi,t) = 0, \qquad \xi \in \Omega$$
$$\nu \cdot \nabla w(\xi,t) + d(\xi)\dot{w}(\xi,t) = 0, \qquad \xi \in \Gamma$$



Proposition

Assume that Ω *is a rectangle and there exists* $\omega \subset \Gamma$ *such that* $\operatorname{ess\,sup}_{\xi\in\omega} d(\xi) > 0$. *Then for any* $\varepsilon > 0$ *we have*

$$\|\nabla w(\cdot,t)\|_{L^2} + \|\dot{w}(\cdot,t)\|_{L^2} = o\left(\frac{1}{t^{1/\alpha}}\right)$$

with $\alpha = 3 + \varepsilon$ *for all classical solutions.*

Previous [Abbas–Nicaise '15]: $\alpha = 2$ if damping on single full edge

Wave Equation on a Rectangle

$$\ddot{w}(\xi,t) - \Delta w(\xi,t) = 0, \qquad \xi \in \Omega$$
$$\nu \cdot \nabla w(\xi,t) + d(\xi)\dot{w}(\xi,t) = 0, \quad \xi \in \Gamma$$

Proposition

Assume that Ω is a rectangle and $\exists \omega \subset \Gamma$ s.t. $\operatorname{ess\,sup}_{\xi\in\omega} d(\xi) > 0$. Then for any $\varepsilon > 0$ we have $\|\nabla w(\cdot,t)\|_{L^2} + \|\dot{w}(\cdot,t)\|_{L^2} = o(t^{-1/\alpha})$ with $\alpha = 3+\varepsilon$ for all classical solutions.

Proof.

An observability condition contributes to a term $1+s^2$ in the resolvent growth rate, and $\eta = 1+\varepsilon$ for the rectangle. In total, $\|(is - A + BB^*)^{-1}\| \lesssim 1 + |s|^{2+1+\varepsilon}$. □

Comments on (Sub-)Optimality

- The optimal rate in the rectangle case is most likely $\alpha = 2$.
- In our results, the actual observability estimate and the measure η of unboundedness are **decoupled**.
- This **explains** suboptimality in several cases:
  - The observability conditions **need to prepare for the worst**.
  - In reality, the "observability" and "unboundedness" aspects interact, and they may compensate for each other beneficially.

Despite these comments, the take-home message could be:

> *Observability estimates combined with accurate analysis of η can lead to reasonably sharp energy decay rates in the 2D boundary damped wave equations.*

Nonlinear Dampings

Nonlinear Dampings

Consider a model with nonlinear damping,

$$\dot{x}(t) = Ax(t) + B\phi(-B^*x(t)), \qquad x(0) = x_0$$

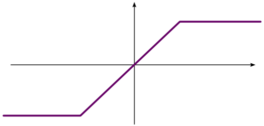
where A generates a contraction semigroup and $\phi : U \to U$ is a continuous and monotone function.

- ϕ is **monotone** if

$$\mathrm{Re}\langle \phi(u_2) - \phi(u_1), u_2 - u_1 \rangle \geq 0, \qquad u_1, u_2 \in U.$$

- Ensures that the damping "acts in the right direction"
- Typical example is the **saturation function**

$$\phi(u) = \begin{cases} u & \text{if } \|u\| \leq 1 \\ \frac{1}{\|u\|}u & \text{if } \|u\| > 1 \end{cases}$$



Nonlinear Dampings

Consider a model with nonlinear damping,

$$\dot{x}(t) = Ax(t) + B\phi(-B^*x(t)), \qquad x(0) = x_0$$

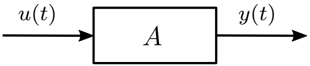
where $\phi : U \to U$ is a continuous and monotone function.

- When $B \in \mathcal{L}(U, X)$ and ϕ is locally Lipschitz, the equation is "semilinear" and existence of solutions follows easily.
- Solutions determined by a **nonlinear** semigroup of contractions.
- Again: boundary dampings lead to $\mathrm{Ran}(B) \not\subset X$, and the well-posedness is much more challenging!
- Results on well-posedness: Seidman–Li '01, Berrahmoune '12, Tucsnak–Weiss '14, Augner '19, Guiver et. al. '19, Hastir et. al. '19, Marx–Weiss '25 (in addition, several PDE results).

Feedback Approach To Well-Posedness

Consider

$$\dot{x}(t) = Ax(t) + B\phi(u(t)),$$
$$y(t) = B^* x(t)$$



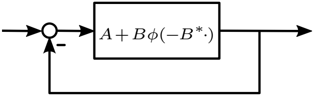
Feedback Approach To Well-Posedness

Consider

$$\dot{x}(t) = Ax(t) + B\phi(u(t)),$$
$$y(t) = B^*x(t)$$

Defining $u(t) = -y(t) + v(t)$, we (formally) obtain

$$\dot{x}(t) = Ax(t) + B\phi(-B^*x(t) + v(t)),$$
$$y(t) = B^*x(t)$$



$\rightsquigarrow$ Again **feedback theory** can be used to investigate the well-posedness (and stability) of the damped equation!

Well-Posedness Result

Theorem (Hastir–P '25)

Let $\phi : U \to U$ be a continuous monotone function. Under mild additional assumptions

$$\dot{x}(t) = Ax(t) + B\phi(-B^*x(t)), \qquad x(0) = x_0$$

has a well-defined generalised solution for every $x_0 \in X$. If x_1, x_2 are two solutions, then

$$\|x_2(t) - x_1(t)\| \le \|x_2(0) - x_1(0)\|, \qquad t \ge 0.$$

Well-Posedness Result

Theorem (Hastir–P '25)

Let $\phi : U \to U$ be a continuous monotone function. Under mild additional assumptions

$$\dot{x}(t) = Ax(t) + B\phi(-B^*x(t)), \qquad x(0) = x_0$$

has a well-defined generalised solution for every $x_0 \in X$. If x_1, x_2 are two solutions, then

$$\|x_2(t) - x_1(t)\| \le \|x_2(0) - x_1(0)\|, \qquad t \ge 0.$$

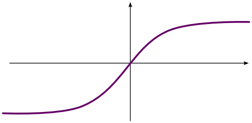
- Also existence of classical solutions.
- The result is also applicable when (A, B, B^*) is replaced with a **impedance passive well-posed system** or a **system node**.
- Also for $y(t) = -B^*x(t) + v(t)$ with external input $v(t)$.

To investigate the stability of

$$\dot{x}(t) = Ax(t) + B\phi(-B^*x(t)), \qquad x(0) = x_0, \qquad (*)$$

we assume that there exist $\alpha, \beta, \delta > 0$ such that

$$\mathrm{Re}\langle \phi(u), u\rangle \geq \begin{cases} \alpha\|u\|^2 & \text{if } \|u\| \leq \delta \\ \beta & \text{if } \|u\| > \delta \end{cases}$$



Theorem (Hastir–P '25)

Assume that $\phi : U \to U$ is continuous and monotone and that $\alpha, \beta, \delta > 0$ exist (+ mild additional assumptions). Assume further that the semigroup generated by $A - BB^*$ is strongly stable.

Then 0 is a globally asymptotically stable equilibrium point of $(*)$, i.e., $\|x(t)\| \to 0$ as $t \to \infty$ for all $x_0 \in X$.

Timoshenko Beam with Nonlinear Boundary Damping

Consider a Timoshenko beam model on $[0,1]$,

$$\rho \ddot{w} = \left(K\left(w' - \theta\right)\right)' \qquad I_\rho \ddot{\theta} = \left(EI\theta'\right)' + K\left(w' - \theta\right)$$

$$w(0,t) = 0, \qquad \theta(0,t) = 0$$

$$EI(1)\theta'(1,t) = \phi_{\mathrm{sat}}\left(-\dot{\theta}(1,t)\right)$$

$$K(1)\left(w'(1,t) - \theta(1,t)\right) = \phi_{\mathrm{sat}}\left(-\dot{w}(1,t)\right)$$

with displacement profile $w(\cdot,t)$ and rotation angle $\theta(\cdot,t)$. Here ϕ_{sat} is the scalar saturation function such that $\phi_{\mathrm{sat}}(u) = u$ for $|u| \leq 1$ and $\phi_{\mathrm{sat}}(u) = u/|u|$ for $|u| > 1$.

Timoshenko Beam with Nonlinear Boundary Damping

Consider a Timoshenko beam model on $[0,1]$,

$$\rho\ddot{w} = \left(K\left(w' - \theta\right)\right)' \qquad I_\rho\ddot{\theta} = \left(EI\theta'\right)' + K\left(w' - \theta\right)$$

$$EI(1)\theta'(1,t) = \phi_{\text{sat}}\left(-\dot{\theta}(1,t)\right)$$

$$K(1)\left(w'(1,t) - \theta(1,t)\right) = \phi_{\text{sat}}\left(-\dot{w}(1,t)\right)$$

Proposition

The beam model has well-defined generalised solutions for all initial conditions $w(\cdot,0), \theta(\cdot,0) \in H^1(0,1)$ *and* $\dot{w}(\cdot,0), \dot{\theta}(\cdot,0) \in L^2(0,1)$.

Moreover, 0 *is a globally asymptotically stable equilibrium point of the system, i.e., all generalised solutions satisfy*

$$\|w\|_{H^1(0,1)} + \|\theta\|_{H^1(0,1)} + \|\dot{w}\|_{L^2(0,1)} + \|\dot{\theta}\|_{L^2(0,1)} \to 0$$

as $t \to \infty$.

Conclusion

In this presentation:

- The feedback-theoretic viewpoint to well-posedness and stability of damped equations
- New results on well-posedness and stability for nonlinear damped models

R. Chill, LP, D. Seifert, R. Stahn, and Y. Tomilov, "Non-Uniform Stability of Damped Contraction Semigroups," *Analysis & PDE*, 2023, https://arxiv.org/abs/1911.04804

LP, D. Seifert, and N. Vanspranghe, "Admissibility theory in abstract Sobolev scales and transfer function growth at high frequencies," arXiv, https://arxiv.org/abs/2412.14786

A. Hastir and LP, "Well-posedness and stability of infinite-dimensional systems under monotone feedback," arXiv, https://arxiv.org/abs/2503.16092

Contact: lassi.paunonen@tuni.fi, paunonenmath.com