The Internal Model Principle for Regular Linear Systems

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Main Objectives

Problem

Introduce new results on robust output regulation for regular linear systems.

Main results:

- Characterization of controllers achieving output regulation via the *regulator equations*
- Characterization of robust controllers via the Internal Model Principle. In addition:
    - A test to determine robustness with respect to a given set of perturbations.
    - Theory of Reduced Order Internal Models
- Robust controllers for regular linear systems

The Infinite-Dimensional Plant

Consider a plant

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad x(0) = x_0 \in X \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

where

- $u(t) \in U = \mathbb{C}^m$ is the control input
- $y(t) \in Y = \mathbb{C}^p$ is the measured output.

For simplicity, we assume:

- No disturbance signals to the state of the plant, i.e., $E = 0$
- Diagonal signal generator, i.e., only bounded reference signals.

The Infinite-Dimensional Plant

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One option for more general assumptions on B and C:

- $B \in \mathcal{L}(\mathbb{C}^m, X_{-1})$ and $C \in \mathcal{L}(X_1, \mathbb{C}^p)$ are admissible
- Define $C_\Lambda x = \lim_{\lambda \to \infty} \lambda C R(\lambda, A) x$ in $\mathcal{D}(C_\Lambda) = \{\, x \mid \text{lim exists} \,\}$
- (A, B, C, D) is regular, i.e., for one/all $\lambda \in \rho(A)$

$$P(\lambda) = C_\Lambda R(\lambda, A_{-1}) B + D$$

is well-defined and $P(\cdot) \in H^\infty(\mathbb{C}_\beta^+)$ for some $\beta \in \mathbb{R}$.

The Infinite-Dimensional Plant

Consider a plant

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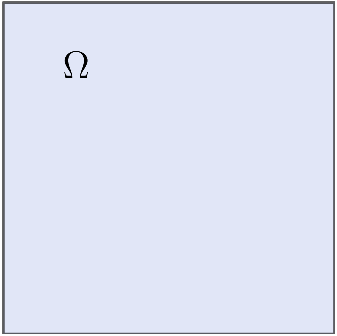
Properties: For any $\beta > \omega_0(T(t))$ we have

$$\begin{aligned} P(\cdot) &\in H^\infty(\mathbb{C}_\beta^+, \mathcal{L}(U,Y)), \\ C_\Lambda R(\cdot, A) &\in H^\infty(\mathbb{C}_\beta^+, \mathcal{L}(X,Y)), \\ R(\cdot, A_{-1})B &\in H^\infty(\mathbb{C}_\beta^+, \mathcal{L}(U,X)), \end{aligned}$$

Example: Controlled Heat Equation



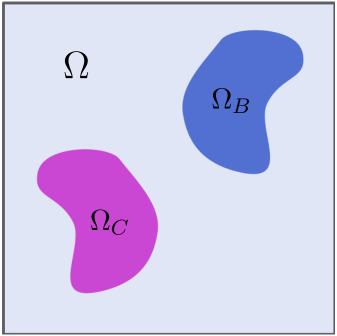
A 2D heat equation on Ω with Dirichlet BC's:

$$u_t = \Delta u, \quad u(\xi,0) = u_0(\xi)$$
$$u(t,\xi)|_{\partial\Omega} = 0.$$

The system on $X = L^2(\Omega)$ with

$$A = \Delta : \mathcal{D}(A) \subset X \to X$$
$$\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$$

Example: Controlled Heat Equation



The system on $X = L^2(\Omega)$ with

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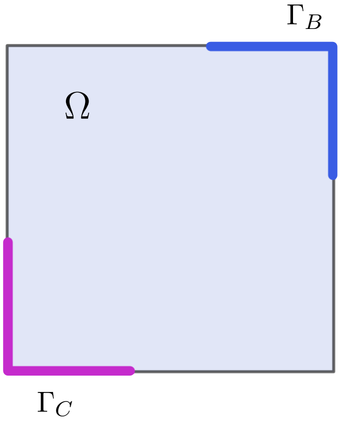
Distributed control and obs.:

$$Cx(t) = \int_{\Omega_C} x(\xi, t) d\xi \in \mathbb{C}$$
$$Bu(t) = \chi_{\Omega_B}(\cdot) u(t), \quad u(t) \in \mathbb{C}$$

Here B and C are bounded.

Example: Heat Equation with Boundary Control



The system on $X = L^2(\Omega)$ with

$$A = \Delta : \mathcal{D}(A) \subset X \to X$$
$$\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$$

Control and observation also possible through the boundary (Dirichlet/Neumann/Robin)

Here B and C are unbounded, and the plant is regular under suitable assumptions.

The Control Problem

Problem (Robust Output Regulation)

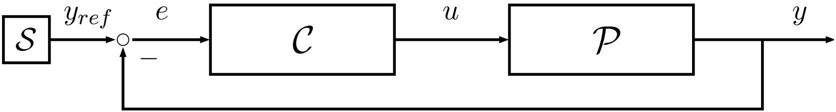
Choose a control law in such a way that

- *The output* $y(t)$ *tracks a given reference signal* $y_{\textit{ref}}(t)$ *asymptotically, i.e.*

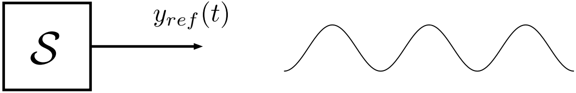
$$\lim_{t\to\infty} \|y(t) - y_{\textit{ref}}(t)\| = 0$$

- *The above property is robust with respect to small perturbations in the operators* (A, B, C, D) *of the plant.*

The Exosystem and the Control Scheme



The Exosystem

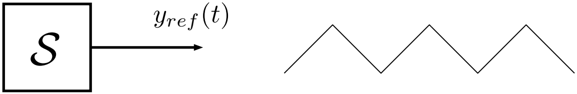


The reference signals can be trigonometric polynomials

$$y_{\mathsf{ref}}(t) = \sum_{k=1}^{q} c_k e^{i\omega_k t}$$

The signal $y_{ref}(t)$ contains the frequencies $(i\omega_k)_{k=1}^{q} \in i\mathbb{R}$.

The Exosystem

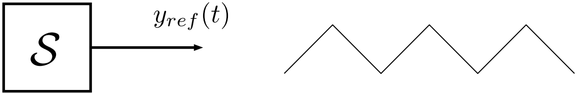


Alternatively, we can consider nonsmooth periodic or almost periodic functions

$$y_{ref}(t) = \sum_{k \in \mathbb{Z}} c_k e^{i\omega_k t}$$

The signal $y_{ref}(t)$ contains the frequencies $(i\omega_k)_{k\in\mathbb{Z}} \in i\mathbb{R}$.

The Exosystem



The references y_{ref} are obtained as outputs of the *exosystem*

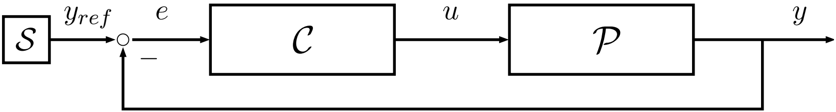
$$\begin{aligned} \dot v(t) &= Sv(t), \qquad v_0 \in W \\ y_{ref}(t) &= Fv(t) \end{aligned}$$

where S is an unbounded diagonal operator

$$S = \operatorname{diag}(i\omega_k)_{k\in\mathbb{Z}}, \qquad \mathcal{D}(S) = \left\{ (v_k) \in W \;\middle|\; (\omega_k v_k) \in \ell^2(\mathbb{C}) \right\}$$

on the Hilbert space $W = \ell^2(\mathbb{C})$ and $F \in \mathcal{L}(W, Y)$. The eigenvalues $i\omega_k \in i\mathbb{R}$ of S are the frequencies in $y_{ref}(t)$.

The Exosystem and the Control Scheme



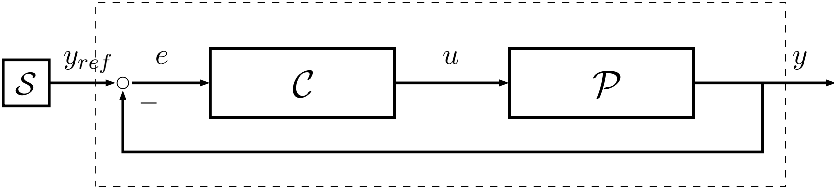
The Dynamic Error Feedback Controller

We consider an error feedback controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ of the form

$$\begin{aligned}\dot{z}(t) &= \mathcal{G}_1 z(t) + \mathcal{G}_2(y(t) - y_{ref}(t)), \qquad z(0) = z_0 \in Z\\ u(t) &= K z(t),\end{aligned}$$

where $\mathcal{G}_1$ generates a semigroup on a Banach space Z, $\mathcal{G}_2 \in \mathcal{L}(Y, Z)$ and $K \in \mathcal{L}(Z_1, U)$ is admissible.

The Closed-Loop System



Closed-loop system

Theorem

The closed-loop system (A_e, B_e, C_e, D_e) consisting of the plant and the controller is a regular linear system.

Proof.

Theory of feedback for regular linear systems, by Weiss (1994). □

The Control Problem

The Robust Output Regulation Problem

Choose a controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ in such a way that

- The closed-loop system is strongly/polynomially/exponentially stable.
- The output $y(t)$ tracks a given reference signal $y_{\textit{ref}}(t)$ asymptotically, i.e.

$$\int_t^{t+1} \|y(s) - y_{\textit{ref}}(s)\|_Y ds \to 0 \qquad \text{as} \quad t \to \infty$$

  for all initial states $x_{e0} = (x_0, z_0) \in X_e$ and $v_0 \in W$
- The above property is robust with respect to small perturbations in the operators (A, B, C, D) of the plant.

The Internal Model Principle

Theorem (Francis & Wonham, 1970's, LP & SP 2010)

A stabilizing feedback controller solves the robust output regulation problem if and only if it contains p copies of the dynamics of the signal generator.

Here $p = \dim Y$, the number of outputs.

The Internal Model Principle

Theorem (Francis & Wonham, 1970's, LP & SP 2010)

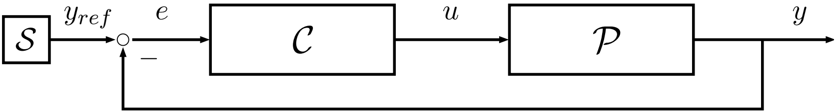
A stabilizing feedback controller solves the robust output regulation problem if and only if it contains p copies of the dynamics of the signal generator.

Here $p = \dim Y$, the number of outputs.

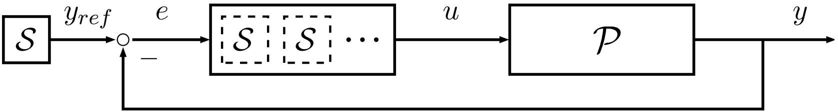
The p-copy for an exosystem with $S = \operatorname{diag}(i\omega_k)_{k\in\mathbb{Z}}$:

> *Any eigenvalue $i\omega_k$ of S must be an eigenvalue of $\mathcal{G}_1$ with p linearly independent eigenvectors associated to it, i.e.,*
>
> $$\dim \mathcal{N}(i\omega_k - \mathcal{G}_1) \geq \dim Y = p.$$

Feedback Controller



The p-Copy Internal Model Principle



Various Forms of the Internal Model

Definition

The controller has a *p-copy internal model* if for every $k \in \mathbb{Z}$

$$\dim \mathcal{N}(i\omega_k - \mathcal{G}_1) \geq \dim Y = p.$$

Various Forms of the Internal Model

Definition

The controller has a *p-copy internal model* if for every $k \in \mathbb{Z}$

$$\dim \mathcal{N}(i\omega_k - \mathcal{G}_1) \geq \dim Y = p.$$

Definition

The controller *satisfies the $\mathcal{G}$-conditions* if for every $k \in \mathbb{Z}$

$$\begin{aligned} \mathcal{R}(i\omega_k - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) &= \{0\} \\ \mathcal{N}(\mathcal{G}_2) &= \{0\}. \end{aligned}$$

Various Forms of the Internal Model

Definition

The controller *has the PK-property* if for every $k \in \mathbb{Z}$

$$(P(i\omega_k)K)|_{\mathcal{N}(i\omega_k - \mathcal{G}_1)} \in \mathcal{L}(\mathcal{N}(i\omega_k - \mathcal{G}_1), Y) \qquad \text{is surjective.}$$

Various Forms of the Internal Model

Definition

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Note that if $P(i\omega_k)$ is assumed to be surjective (as is usually done here) and $U = Y = \mathbb{C}^p$, then the PK-property is equivalent to

$$K|_{\mathcal{N}(i\omega_k - \mathcal{G}_1)} \in \mathcal{L}(\mathcal{N}(i\omega_k - \mathcal{G}_1), U) \qquad \text{is surjective.}$$

Also the PK-property clearly implies the p-copy internal model.

Alternate Form of the PK-Conditions

Denote by $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{F})$ and $(\tilde{A}_e, \tilde{B}_e, \tilde{C}_e, \tilde{D}_e)$ the perturbed plant and closed-loop system, and $\tilde{P}(\lambda) = \tilde{C}_\Lambda R(\lambda, \tilde{A})\tilde{B} + \tilde{D}$.

Theorem (Roughly)

The controller has the PK-property if and only if for all perturbations of the plant

$$\tilde{P}(i\omega_k)Kz^k = -\tilde{F}e_k$$

$$(i\omega_k - \mathcal{G}_1)z^k = 0$$

have a solution $z^k \in \mathcal{D}(\mathcal{G}_1)$ *for all* $k \in \mathbb{Z}$.

Proof of the Internal Model Principle for bounded B and C

Robust Regulation

$\Updownarrow$ $\quad \sigma(A) \cap \sigma(S) = \varnothing$

$\mathcal{G}$-Conditions

$\Updownarrow$ $\quad \sigma(A_e) \cap \sigma(S) = \varnothing$

PK-property

$\Updownarrow$ $\quad \sigma(A_e) \cap \sigma(S) = \varnothing$, $\dim Y < \infty$

p-Copy Internal Model

$\mathcal{G}$-Conditions vs. Robust Regulation

Robust Regulation

$\Uparrow$

no assumptions required

$\mathcal{G}$-Conditions

Extensions

The results in this presentation are also valid for

- Also for disturbance rejection
- Non-diagonal exosystems (i.e., S has Jordan blocks)
- Infinite-dimensional exosystems (nonsmooth reference signals)
- Infinite-dimensional plants with unbounded control and observation.