Non-uniform Stability of Coupled PDEs: A Systems Theory Approach

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October 1st, 2018

Funded by Academy of Finland grants
298182 (2016-2019) and 310489 (2017-2021)

Main Objectives

Problem

Consider different types of ***coupled PDE systems*** *from the point of view of* ***control theory****.*

Our focus is on systems exhibiting **non-uniform stability**.

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Motivation:

- Coupling of stable and unstable PDEs and ODEs often leads to rational decay of energy, i.e., polynomial stability.

Main results:

- Discussion and a (hopefully new) viewpoint.
- New stability results for coupled PDEs.
- Disclaimer: Will not solve all your problems!

Outline

(1) "Passive feedback structures" in coupled PDE systems

- Highlight parallels in coupled PDEs and linear systems

(2) Conversion from coupled PDEs to coupled systems

- Examples on "how to set it up".

(3) What do we get?

- General conditions for polynomial and nonuniform stability of coupled PDEs and systems.

Coupled PDE-PDE and PDE-ODE systems appear in models of

- Fluid-structure interactions
- Thermo-elasticity
- Mechanical systems, e.g., beams with tip masses
- Magnetohydrodynamics
- Acoustics

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Couplings may either be

- Through the **boundary** (Fluid-structure, acoustics), or
- **inside a shared domain** (Thermo-elasticity, MHD)

We will focus on couplings that are **passive** (details later).

Example: Coupled Wave–Heat Systems

Models for fluid–structure and heat–structure interactions:

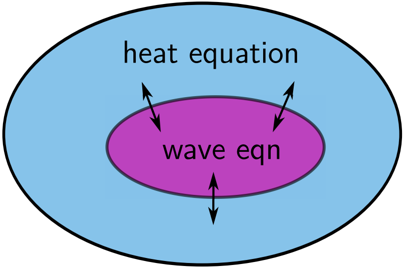
$$\frac{\partial^2 u}{\partial t^2}(x,t) = \Delta u(x,t)$$

$$\updownarrow$$

coupling BCs

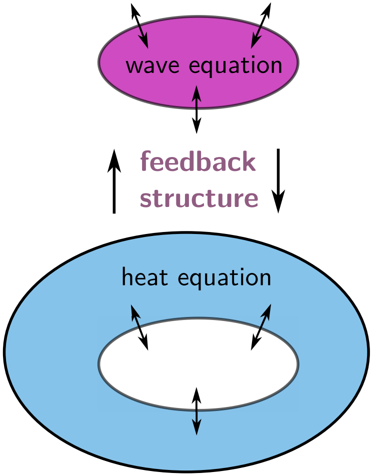
$$\updownarrow$$

$$\frac{\partial w}{\partial t}(x,t) = \Delta w(x,t)$$



References: Avalos & Triggiani, Duyckaerts, Mercier, Nicaise, Ammari, Zuazua, Guo, and many others.

Coupled Wave–Heat Systems



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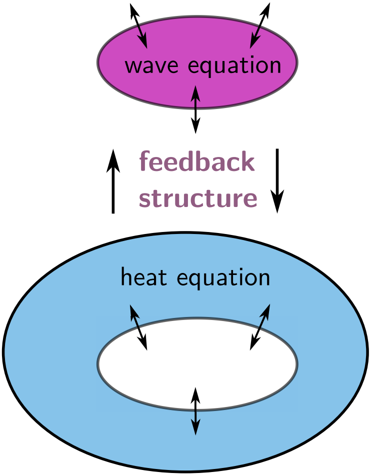

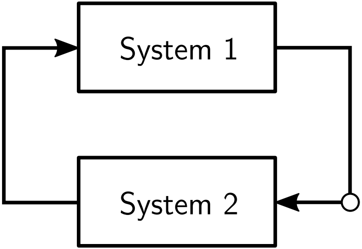

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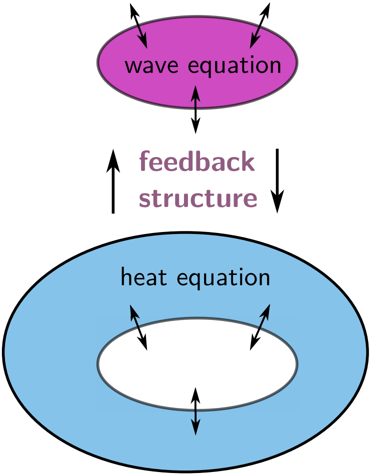
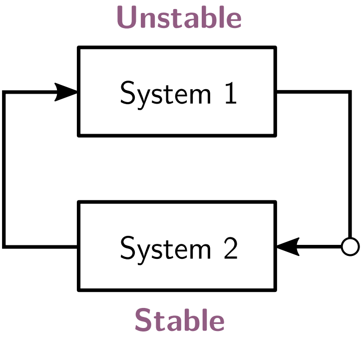
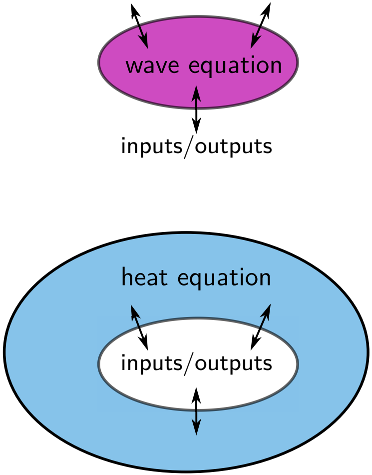
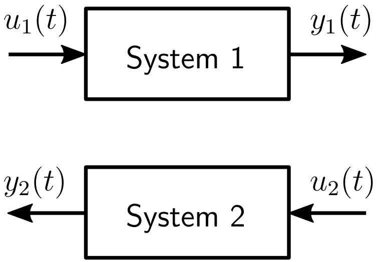




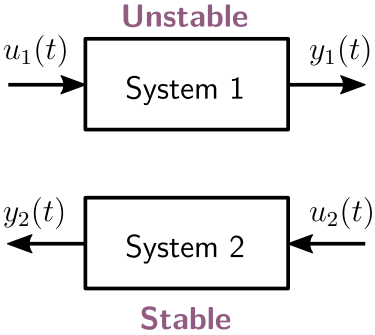
Inputs and Outputs





Problem

Use the properties of the two systems to deduce stability of the coupled PDE.



Linear Control Systems

Consider the **linear control system**:

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t), \qquad x(0) = x_0 \in X \\
y(t) &= Cx(t) + Du(t)
\end{aligned}
$$

where X is Hilbert, A generates a semigroup, and B and C are either bounded or unbounded.

"Passive" Systems

To keep things simple, we only focus on systems

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0 \in X$$
$$y(t) = B^*x(t)$$

where X is Hilbert, A generates a **contraction semigroup**, and $B \in \mathcal{L}(U, V^*)$ for some suitable spaces U and $V^* \supseteq X$.

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Such systems are "**impedance passive**", which in particular means they have "**no internal sources of energy**",

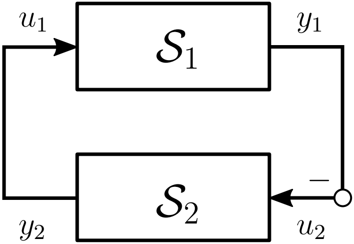
$$\frac{d}{dt}\|x(t)\|^2 \leq 2\,\mathrm{Re}\langle u(t), y(t)\rangle_Y$$

Examples:

- Many mechanical systems, RLC circuits, ...

Feedback Theory of Passive Systems

Property: "Power-preserving interconnection" preserves passivity!



$\Rightarrow$ Closed-loop semigroup contractive on Hilbert $X_1 \times X_2$.

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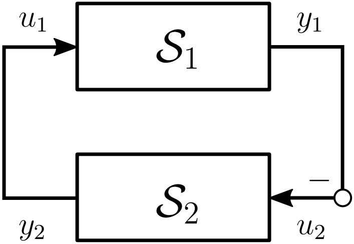

$\Rightarrow$ Closed-loop semigroup contractive on Hilbert $X_1 \times X_2$.

Some results exist on exponential stability, here we focus on **non-uniform stability** $\rightarrow$ decay rates for total energy.

Coupled Passive Systems

If for $k = 1, 2$ we let

$$\begin{aligned} \dot{x}_k(t) &= A_k x_k(t) + B_k u_k(t), \qquad x_k(0) \in X_k \\ y_k(t) &= B_k^* x_k(t), \end{aligned}$$

then the "**power-preserving interconnection**" leads to

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A_1 & B_1 B_2^* \\ -B_2 B_1^* & A_2 \end{bmatrix}}_{=:\, A} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Question

How to choose U, B_1, and B_2?

Example: 1D Wave–Heat Model

$$\begin{cases} v_{tt}(\xi,t) = v_{\xi\xi}(\xi,t), & \xi \in (-1,0),\ t>0, \\ w_t(\xi,t) = w_{\xi\xi}(\xi,t), & \xi \in (0,1),\ t>0, \\ v_\xi(0,t) = w_\xi(0,t), \quad v_t(0,t) = w(0,t), & t>0, \end{cases}$$

- [Xu Zhang & Zuazua, Batty, Paunonen & Seifert, (2D version: Avalos, Triggiani & Lasiecka)]
- Known: Non-uniform stability with $\alpha = 1/2$.

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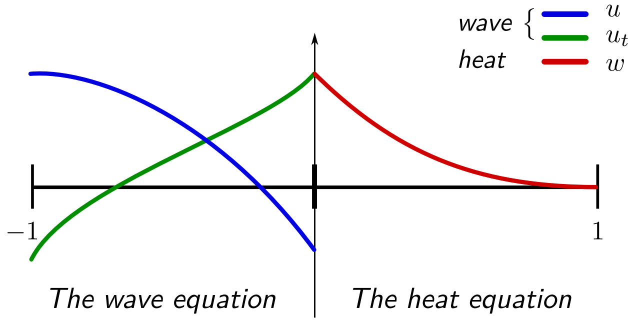


Example: 1D Wave-Heat — Open-Loop Splitting

Wave system on $(-1,0)$:

$$\begin{aligned} v_{tt}(\xi,t) &= v_{\xi\xi}(\xi,t) \\ y_1(t) &= v_\xi(0,t) \\ u_1(t) &= v_t(0,t) \end{aligned}$$

Unstable

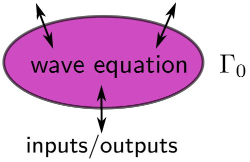
Heat system on $(0,1)$:

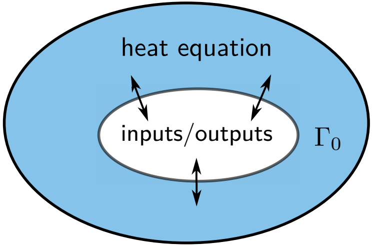
$$\begin{aligned} w_t(\xi,t) &= w_{\xi\xi}(\xi,t) \\ y_2(t) &= w(0,t) \\ u_2(t) &= -w_\xi(0,t) \end{aligned}$$

Stable

The systems **are** impedance passive. We have $U=\mathbb{C}$ and B_1 and B_2 are unbounded.

Inputs and Outputs





2D systems are a bit more complicated to set up.

For boundary couplings U is a function space on Γ_0.

In in-domain couplings, space on Ω or $\Omega_0 \subset \Omega$.

Polynomial and Non-Uniform Stability

Theorem (Borichev & Tomilov '10)

Let $T(t)$ be a uniformly bounded C_0-semigroup on a Hilbert space X. Let A be the generator of $T(t)$ and $\sigma(A) \cap i\mathbb{R} = \emptyset$.

For any constant $\alpha > 0$, the following are equivalent:

$$\|T(t)x_0\| \leq \frac{M}{t^{1/\alpha}}\|Ax_0\| \qquad \text{for some } M > 0$$

$$\|(i\omega - A)^{-1}\| \leq M_R(1 + |s|^\alpha), \qquad \text{for some } M_R > 0$$

General: Batty & Duyckaerts '08, Rozendaal, Seifert & Stahn '17.

Application: $E(t) \sim \|T(t)x_0\|^2$ for many PDE systems.

Polynomial and Non-Uniform Stability

Since our coupled systems are contractive by default,

"Non-uniform stability ***only*** *requires a resolvent estimate"*

Problem

Derive a resolvent estimate for

$$A := \begin{bmatrix} A_1 & B_1 B_2^* \\ -B_2 B_1^* & A_2 \end{bmatrix}$$

in terms of the properties of

- (A_1, B_1, B_1^*) [**Unstable**]
- (A_2, B_2, B_2^*) [**Stable**]

Assumption

- A_1 *is diagonalizable and skew-adjoint,* $A_1 = \sum_{k \in \mathbb{Z}} i\omega_k \langle \cdot, \phi_k \rangle \phi_k$
- *Uniform gap:* $\inf_{k \neq l} |\omega_k - \omega_l| > 0$ *(for simplicity).*
- $T_2(t)$ *gen. by* A_2 *is exponentially stable.*

Conditions for Non-Uniform Stability

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- *Uniform gap:* $\inf_{k \neq l} |\omega_k - \omega_l| > 0$.
- $T_2(t)$ *gen. by* A_2 *is exponentially stable.*
- $\|B_1^* \phi_k\| \neq 0$ *for all* k *(i.e.,* (A_1, B_1) *is "approx. controllable").*
- *Denoting* $P_2(\lambda) = B_2^*(\lambda - A_2)^{-1} B_2$ *("transfer function"),*

$$P_2(i\omega_k) + P_2(i\omega_k)^* > 0 \qquad \forall k \in \mathbb{Z}.$$

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- *A_1 is diagonalizable and skew-adjoint, $A_1 = \sum i\omega_k \langle \cdot, \phi_k \rangle \phi_k$*
- *Uniform gap: $\inf_{k \neq l} |\omega_k - \omega_l| > 0$.*
- *$T_2(t)$ gen. by A_2 is exponentially stable.*
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$$P_2(i\omega_k) + P_2(i\omega_k)^* > 0 \qquad \forall k \in \mathbb{Z}.$$

Proposition

The closed-loop is strongly stable and $\sigma(A) \subset \mathbb{C}_-$.

- Denote $P_2(\lambda) = B_2^*(\lambda - A_2)^{-1}B_2$ ("transfer function")

Theorem

Let $\gamma, \eta : \mathbb{R}_+ \to (0,1)$ be decreasing (and "nice") so that

$$\|B_1^*\phi_k\| \geq c_1\gamma(|\omega_k|) \qquad \forall k$$

$$\mathrm{Re}\langle P_2(is)u, u\rangle \geq c_2\eta(|s|)\|u\|^2 \qquad s \approx \omega_k$$

for some constants $c_1, c_2, s_0 > 0$.
Then the closed-loop system is non-uniformly stable so that

$$\|(is - A)^{-1}\| \lesssim \frac{M_R}{\gamma(|s|)^2\eta(|s|)}, \qquad |s| \text{ large.}$$

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Thus

$$\|T(t)x\| \leq \frac{M_T}{M^{-1}(t)}\|Ax\|, \qquad x \in \mathcal{D}(A)$$

where $M(s) \sim \gamma(s)^{-2}\eta(s)^{-1}$.

Theorem

Let $\beta, \gamma \geq 0$ *such that*

$$\|B_1^* \phi_k\| \geq c_1 |\omega_k|^{-\beta} \qquad \forall k$$

$$\mathrm{Re}\langle P_2(is)u, u\rangle \geq c_2 |s|^{-\gamma} \|u\|^2 \qquad s \approx \omega_k$$

for some constants $c_1, c_2, s_0 > 0$.
Then the closed-loop system is non-uniformly stable so that

$$\|(is - A)^{-1}\| \leq M_R(1 + |s|^{2\beta+\gamma}), \qquad |s| \text{ large.}$$

Thus

$$\|T(t)x\| \leq \frac{M_T}{t^{1/\alpha}} \|Ax\|, \qquad x \in \mathcal{D}(A)$$

for $\alpha = 2\beta + \gamma \geq 0$.

Comments:

- Theorem requires some admissiblity and well-posedness assumptions (swept under the carpet here). Limits 2D-nD BC.
- Lack of spectral gap and repeated eigenvalues are allowed in a more general version (affects the rate).

Optimality

- Obtained rate is not always optimal, especially if
  - A_1 has no spectral gap (2D, nD waves) or
  - eigenvalues diverge as $|\omega_k| \to \infty$ (beams and plates).
- A nice way of getting (possibly) suboptimal rates easily.

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References:

- Paunonen Arxiv June '17
- Partly based on joint work with Chill, Stahn & Tomilov

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- A_1 diagonalizable, $\omega_k \sim k\pi$, ϕ_k trigonometric
- $B_1^*\phi_k \neq 0$, and $|B_1^*\phi_k| \gtrsim 1 = |\omega_k|^0$
- $P_2(is) = B_2^*(is-A_2)^{-1}B_2$ satisfies $|P_2(is)| \sim |s|^{-1/2}$.

Thus the closed-loop system is polynomially stable,

$$\|(is-A)^{-1}\| = O(|s|^{1/2}) \qquad \text{and} \qquad \|T(t)x\| \leq \frac{M}{t^2}\|Ax\|.$$

Reproduces results of [Zhang-Zuazua, Batty-Paunonen-Seifert].

Conclusions

In this presentation:

- Discussion of coupled PDE and PDE-ODE systems from the viewpoint of systems theory
- General conditions for non-uniform and polynomial stability of coupled systems.

LP, "Stability and Robust Regulation of Passive Linear Systems," http://arxiv.org/abs/1706.03224