Robustness of Strong and Polynomial Stability of Semigroups

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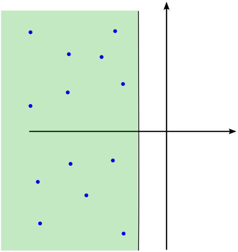
Stability of Semigroups

- X is Hilbert, $A : \mathcal{D}(A) \subset X \to X$
- A generates a uniformly bounded semigroup $T(t)$

Exponential:



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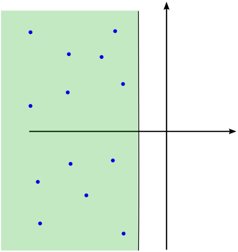

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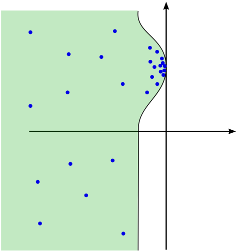

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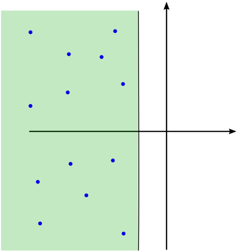

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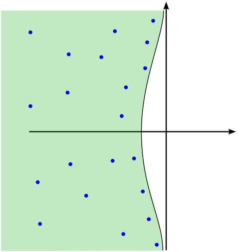

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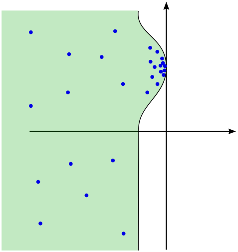

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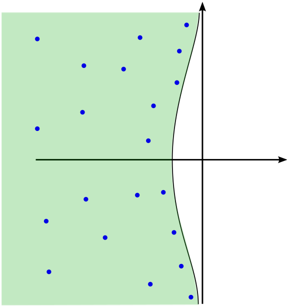
Polynomial Stability



Definition

$T(t)$ is called *polynomially stable* if

- $T(t)$ is uniformly bounded,
- $i\mathbb{R} \subset \rho(A)$,
- There exist $\alpha > 0$ and $M > 0$ s.t.

$$\|T(t)A^{-1}\| \leq \frac{M_A}{t^{1/\alpha}} \qquad \forall t > 0.$$

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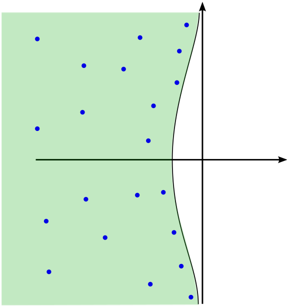

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Since uniform boundedness is required, a polynomially stable semigroup is also strongly stable.

Characterization on a Hilbert Space

Theorem

If $T(t)$ is a uniformly bounded semigroup and $i\mathbb{R} \subset \rho(A)$. For a fixed $\alpha > 0$ the following are equivalent.

(a) $$\|T_A(t)A^{-1}\| \le \frac{M_A}{t^{1/\alpha}}, \qquad \forall t > 0$$

(b) $$\|R(i\omega, A)\| \le M(1 + |\omega|^\alpha)$$

Borichev & Tomilov (2010), and Batty & Duyckaerts (2008).

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(c) $$\sup_{\operatorname{Re}\lambda \geq 0} \|R(\lambda, A)(-A)^{-\alpha}\| < \infty.$$

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Robustness of Polynomial Stability

Assume $T(t)$ and $\alpha > 0$ are such that

$$\|T(t)x\| \leq \frac{M_A}{t^{1/\alpha}} \|Ax\| \qquad \forall x \in \mathcal{D}(A).$$

Problem

Consider stability of the semigroup generated by

$$A + BC,$$

where $B \in \mathcal{L}(\mathbb{C}^p, X)$ *and* $C \in \mathcal{L}(X, \mathbb{C}^p)$.

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Challenge: There exist B, C with arbitrarily small $\|B\|, \|C\|$ s.t. $A + BC$ is unstable.

Main Results on Polynomial Stability

Assume perturbation $A + BC$ satisfies

$$\mathcal{R}(B) \subset \mathcal{D}((-A)^{\beta}) \quad \text{and} \quad \mathcal{R}(C^*) \subset \mathcal{D}((-A^*)^{\gamma}) \tag{1}$$

for some $\beta, \gamma \geq 0$ such that $\beta + \gamma \geq \alpha$.

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The operators B and C are "more than bounded".

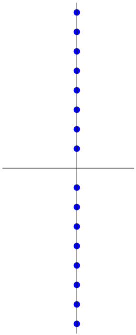
Theorem (LP 2012, 2013)

Assume $\beta + \gamma \geq \alpha$. There exists $\delta > 0$ such that if B and C satisfy (1) *and*

$$\|(-A)^\beta B\| < \delta, \qquad \textit{and} \qquad \|(-A^*)^\gamma C^*\| < \delta,$$

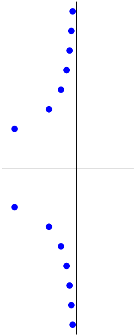
then the semigroup generated by $A + BC$ is strongly and polynomially stable (with the same exponent $\alpha > 0$).

Example: 1D Wave Equation



Pole placement

$$\xrightarrow{u = \mathcal{K}z}$$



Undamped wave equation

$$\dot{x} = \mathcal{A}x + \mathcal{D}u$$

where $\mathcal{A}^* = -\mathcal{A}$.

Polynomially stabilized

$$\dot{x} = (\mathcal{A} + \mathcal{D}\mathcal{K})x$$

with $\alpha = 2$ (dep'n on $\mathcal{D}$).

For the Wave Equation

In the original wave equation on $[0,1]$

$$\frac{\partial^2 w}{\partial t^2}(z,t) = \frac{\partial^2 w}{\partial z^2}(z,t) + d_0(z)u(t) + b_0\left(\langle w, c_1 \rangle_{L^2} + \langle \frac{\partial w}{\partial t}, c_2 \rangle_{L^2} \right)$$

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the polynomial stability is preserved if

$$b_0, c_2 \in \mathcal{D}\left(\frac{\partial^2}{\partial z^2}\right) \qquad \text{and} \qquad c_1 \in L^2(0,1),$$

and if the L^2-norms

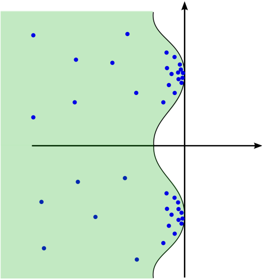
$$\|b_0\|_{L^2}, \quad \|b_0'\|_{L^2}, \quad \|c_1\|_{L^2}, \quad \|c_2\|_{L^2}, \quad \|c_2'\|_{L^2}$$

are sufficiently small.

Part II:

Preservation of Strong Stability

Finite Spectral Points on $i\mathbb{R}$



Restriction:

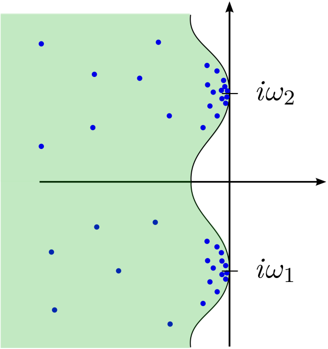
Polynomial stability implies $\sigma(A) \cap i\mathbb{R} = \varnothing$

Problem

How to handle spectrum on $i\mathbb{R}$?

Solution

Study the situation where $T(t)$ is strongly stable, $\sigma(A) \cap i\mathbb{R}$ is finite, and the resolvent growth is suitably bounded on $i\mathbb{R}$.



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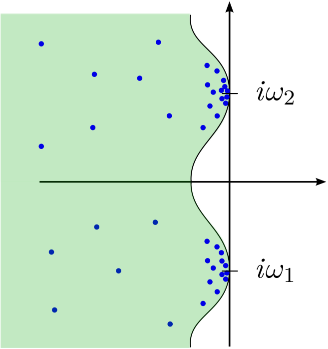


For a fixed $\alpha \geq 1$:

$$\|R(i\omega, A)\| \leq \frac{M}{|\omega - \omega_2|^{\alpha}}$$

$$\|R(i\omega, A)\| \leq \frac{M}{|\omega - \omega_1|^{\alpha}}$$

Main Problem

Problem

For a fixed $\alpha \geq 1$, consider stability of the semigroup generated by

$$A + BC,$$

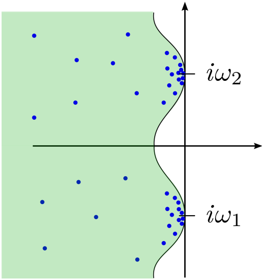
where $B \in \mathcal{L}(\mathbb{C}^p, X)$ and $C \in \mathcal{L}(X, \mathbb{C}^p)$.

General aim: Define suitable graph norms to measure the sizes of B and C.

Properties

The operators $i\omega_1 - A$ and $i\omega_2 - A$ have unbounded sectorial inverses

$$(i\omega_1 - A)^{-1} \qquad \text{and} \qquad (i\omega_2 - A)^{-1}$$



Use graph norms of the **inverses** in studying robustness of stability!

Robustness of Stability

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$$\mathcal{R}(B) \subset \mathcal{D}((i\omega_1 - A)^{-\beta}) \quad \text{and} \quad \mathcal{R}(C^*) \subset \mathcal{D}((-i\omega_1 - A^*)^{-\gamma})$$

$$\mathcal{R}(B) \subset \mathcal{D}((i\omega_2 - A)^{-\beta}) \quad \text{and} \quad \mathcal{R}(C^*) \subset \mathcal{D}((-i\omega_2 - A^*)^{-\gamma})$$

for some $\beta, \gamma \geq 0$ such that $\beta + \gamma \geq \alpha$.

Robustness of Stability

Assume

$$\mathcal{R}(B) \subset \mathcal{D}((i\omega_k - A)^{-\beta}), \quad \mathcal{R}(C^*) \subset \mathcal{D}((-i\omega_k - A^*)^{-\gamma}) \quad (2)$$

for some $\beta, \gamma \geq 0$ and $k = 1, 2$ such that $\beta + \gamma \geq \alpha$.

Theorem

Assume $\beta + \gamma \geq \alpha$. There exists $\delta > 0$ such that if B and C satisfy (2) *and*

$$\|B\| + \|(i\omega_k - A)^{-\beta} B\| < \delta, \qquad \textit{and}$$

$$\|C\| + \|(-i\omega_k - A^*)^{-\gamma} C^*\| < \delta$$

for $k = 1, 2$, then the semigroup generated by $A + BC$ is strongly stable.

Example

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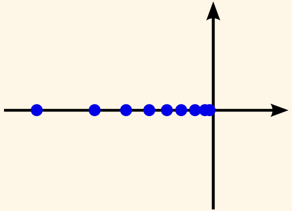
Consider $X = \ell^2(\mathbb{C})$ and $A \in \mathcal{L}(X)$ by

$$A = \sum_{k=1}^{\infty} -\frac{1}{k} \langle \cdot, e_k \rangle e_k \in \mathcal{L}(X)$$

and $A + \langle \cdot, c \rangle b$ with $b, c \in X$.



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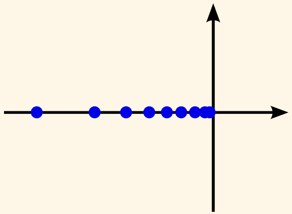

Now $\sigma(A) \cap i\mathbb{R} = \{0\}$ and $\alpha = 1$.

Inverse $(-A)^{-1}$ unbounded, self-adjoint, positive. For $\beta \geq 0$

$$(-A)^{-\beta} x = \sum_{k=1}^{\infty} k^{\beta} \langle x, e_k \rangle e_k,$$

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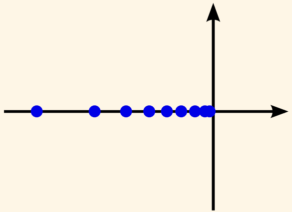

and $A + \langle \cdot, c \rangle b$ with $b, c \in X$.

Conclusion: $A + \langle \cdot, c \rangle b$ is stable for $\beta + \gamma = 1$, and for small norms

$$\|(-A)^{-\beta} b\|^2$$

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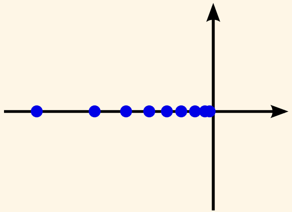

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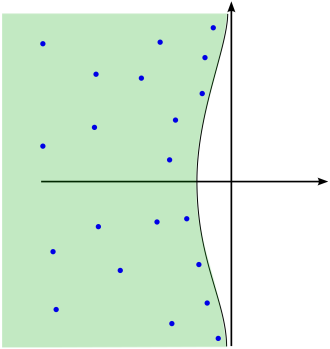
$$\|(-A)^{-\beta} b\|^2 = \sum_{k=1}^{\infty} k^{2\beta} |\langle b, e_k\rangle|^2$$

$$\|(-A^*)^{-\gamma} c\|^2 = \sum_{k=1}^{\infty} k^{2\gamma} |\langle c, e_k\rangle|^2$$

Compare

Polynomial stability:

$A : \mathcal{D}(A) \subset X \to X$



Strong stability:

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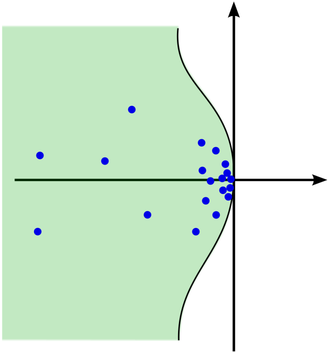

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$$\|R(i\omega, A)\| \leq M|\omega|^\alpha$$

for $|\omega|$ large.

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for $\lambda \in \mathbb{C}^+$.

(Borichev & Tomilov '10)

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for all $t > 0$.

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Conditions for $A + BC$:

Graph norms with $\beta + \gamma = \alpha$

$$\|(-A)^{\beta} B\|, \ \|(-A^*)^{\gamma} C^*\|$$

small $\Rightarrow$ Robustness.

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References

L. Paunonen, Robustness of strong and polynomial stability of semigroups, *Journal of Functional Analysis*, 2012.

L. Paunonen, Robustness of strong stability of semigroups, *ArXiv/Submitted*, 2013.

Conclusions

- Conditions for the preservation of strong and polynomial stabilities of a semigroup
- Comparison of results.

Thank You!