Reduced Order Robust Output Regulation for Parabolic Systems

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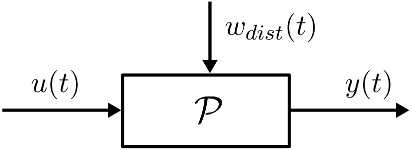
Main Objectives

Problem

*Study the **robust output regulation problem** for parabolic distributed parameter systems.*



Main Objectives

Problem

Study the ***robust output regulation problem*** *for parabolic distributed parameter systems.*

Objective: Design a controller such that the output $y(t)$ of the system converges to a reference signal

$$\|y(t) - y_{ref}(t)\| \to 0, \qquad \text{as} \quad t \to \infty$$

despite disturbance signals $w_{dist}(t)$.

The controller is required to be **robust** in the sense that it tolerates small perturbations in the parameters of the systems.

- Tolerance to **uncertainty** in models.
- Allows **approximate** controller parameters.

Main Objectives

Problem

*Study the **robust output regulation problem** for parabolic distributed parameter systems.*

Main results:

- Finite-dimensional low-order controller design method
- Application to tracking control in 2D room temperature models.

Main contributions:

- Earlier controllers of *unstable* systems **always ∞-dimensional**
- Novelty: Utilise Galerkin approximation theory and model reduction in robust output regulation

The Parabolic Linear System

$$\dot{x}(t) = Ax(t) + Bu(t) + B_d w_{\mathit{dist}}(t), \qquad x(0) = x_0 \in X$$
$$y(t) = Cx(t) + Du(t)$$

- A generates an **analytic** semigroup on the Hilbert space X
  - Convection-diffusion-reaction PDEs
  - Beams and plates with Kelvin-Voigt damping
- $u(t) \in U$ input, $y(t) \in Y$ output, $w_{\mathit{dist}}(t) \in U_d$ disturbance
- $B \in \mathcal{L}(U, X)$, $C \in \mathcal{L}(X, Y)$, $B_d \in \mathcal{L}(U_d, X)$ with U, U_d, Y finite-dimensional.

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Problem (The Robust Output Regulation Problem)

Choose a control law such that

- $\|y(t) - y_{ref}(t)\| \to 0$ *as* $t \to \infty$ *for all* $x_0 \in X$
- *Convergence holds even if* (A, B, B_d, C, D) *are perturbed.*

Earlier Work

Fact: Problem requires a dynamic feedback controller.

System stable (or stabilizable by output fb), **fin.-dim.** controller:

- Pohjolainen 1982, Hämäläinen–Pohjolainen 1996, 2000, 2011, Logemann–Townley 1997, Rebarber–Weiss 2003, Boulite–Idrissi–Ould Maaloum 2009, Humaloja–Kurula–Paunonen 2018, 2019, Deutscher–Gabriel 2018, . . .

System unstable, **∞-dimensional** controller:

- Byrnes et. al. 2000, Immonen 2006, 2007 Hämäläinen–Pohjolainen 2010, Paunonen 2013, 2016, 2017, Xu–Dubljevic 2016, 2017 (no robustness), B.Z. Guo–Meng 2020, 2021 (PDE models), . . .

Earlier Work

System unstable, **finite-dimensional** controller:

- Schumacher 1983, Curtain 1983 (quite strict conditions)
- Deutscher 2011, 2013 (no robustness, "spillover" possible)
- Lhachemi–Prieur 2020 arXiv (uses eigenmodes)

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Main Result (Paunonen-Phan IEEE TAC 2020)

Our controller design:

- *Internal model based robust design*
- *For general parabolic systems*
- *Direct: Uses Galerkin approximation and LQR/LQG methods*
- *Model reduction step ensures low order*

Robust Controller Design

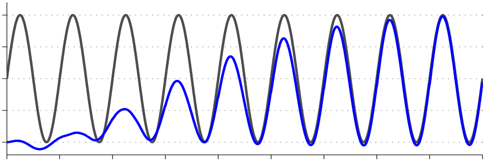
The Reference Signals

The reference signals we consider are of the form

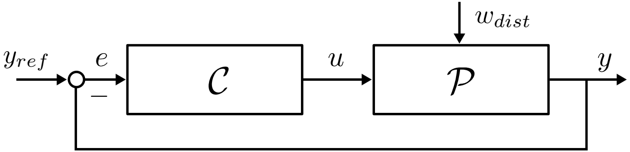
$$y_{ref}(t) = a_0 + \sum_{k=1}^{q} \left[a_k^1 \cos(\omega_k t) + a_k^2 \sin(\omega_k t) \right],$$

$$w_{dist}(t) = b_0 + \sum_{k=1}^{q} \left[b_k^1 \cos(\omega_k t) + b_k^2 \sin(\omega_k t) \right],$$

with **frequencies** $\{\omega_k\}_{k=0}^{q} \subset \mathbb{R}_+$, unknown amplitudes and phases.



The Dynamic Error Feedback Controller

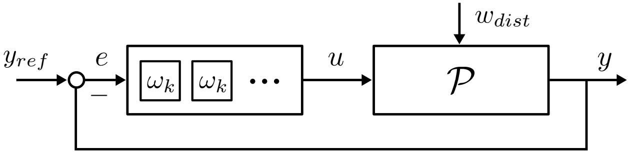


We consider an error feedback controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ of the form

$$\begin{aligned} \dot{z}(t) &= \mathcal{G}_1 z(t) + \mathcal{G}_2(y(t) - y_{\mathit{ref}}(t)) \qquad z(0) = z_0 \in Z, \\ u(t) &= K z(t), \end{aligned}$$

which is (ideally finite-dimensional) linear system on Z.

The Internal Model Principle



Theorem (Francis–Wonham, Davison 1970's, LP '10,'14)

The following are equivalent:

- *The controller solves the robust output regulation problem.*
- *Closed-loop system is stable and the controller has **an internal model** of the frequencies $\{\omega_k\}_k$ of $w_{\text{dist}}(t)$ and $y_{\text{ref}}(t)$.*

*"**Internal Model**": For every k, $\pm i\omega_k$ must be eigenvalues of $\mathcal{G}_1$ having $p = \dim Y$ linearly independent eigenvectors.*

Internal Model Based Controller Design

The robust output regulation problem can be solved in two parts:

Step 1° Include a suitable internal model into the controller

Step 2° Use the rest of the controller's parameters to stabilize the closed-loop system.

Internal model has fixed structure (easy), the closed-loop stability can be achieved in several ways (often the main challenge).

In this work: We utilise the well-developed theory of

"Galerkin approximations of parabolic systems"

[Banks–Kunish 1984, Ito 1990, Morris 1994, Banks–Ito 1997, Ito–Morris 1998, . . .]

Galerkin Approximations – Assumptions

$$\dot{x}(t) = Ax(t) + Bu(t) + B_d w_{dist}(t), \qquad x(0) = x_0 \in X$$
$$y(t) = Cx(t) + Du(t)$$

Assumption

There exists a sesquilinear form $a(\cdot,\cdot) : V \times V \to \mathbb{C}$ *such that*

$$\langle -A\phi, \psi \rangle = a(\phi, \psi), \qquad \forall \phi \in D(A), \psi \in V$$

and $a(\cdot,\cdot)$ *bounded and coercive, i.e.,* $\exists c_1, c_2, \lambda_0 > 0$ *s.t.*

$$|a(\phi,\psi)| \le c_1 \|\phi\|_V \|\psi\|_V \qquad \text{(boundedness)}$$
$$\operatorname{Re} a(\phi,\phi) \ge c_2 \|\phi\|_V^2 - \lambda_0 \|\phi\|_X^2 \qquad \text{(coercivity)}$$

- $\Rightarrow A - \lambda_0 I$ generates an analytic semigroup on X
- Typical for nD convection-diffusion-reaction equations

Galerkin Approximations

Assumption

$$\langle -A\phi, \psi\rangle = a(\phi, \psi), \qquad \forall \phi \in D(A), \psi \in V$$

Assumption (Approximating Subspaces V_N)

There are subspaces $(V_N)_N \subset V$, $\dim V_N < \infty$, *such that any* $\phi \in V$ *can be approximated by* $\phi_N \in V_N$ *in the sense*

$$\|\phi - \phi_N\|_V \to 0, \qquad \text{as} \quad N \to \infty.$$

V_N define **Galerkin approximations** (A^N, B^N, C^N) of (A, B, C),

$$\begin{aligned} \langle -A^N\phi, \psi\rangle &= a(\phi, \psi), && \forall \phi, \psi \in V_N, \\ \langle B^N u, \phi\rangle &= \langle u, B^*\phi\rangle, && \forall u \in U, \phi \in V_N, \\ C^N &= C|_{V_N} \end{aligned}$$

The Low Order Robust Controller

Aim: Use the approximations (A^N, B^N, C^N) in controller design.

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The end result: A finite-dimensional robust controller

$$\begin{bmatrix}\dot z_1(t)\\ \dot z_2(t)\end{bmatrix} = \begin{bmatrix}G_1 & 0\\ B_L^r K_1^N & A_L^r + B_L^r K_2^r\end{bmatrix}\begin{bmatrix}z_1(t)\\ z_2(t)\end{bmatrix} + \begin{bmatrix}G_2\\ -L^r\end{bmatrix}(y(t) - y_{\mathit{ref}}(t))$$

$$u(t) = \begin{bmatrix}K_1^N & K_2^r\end{bmatrix}\begin{bmatrix}z_1(t)\\ z_2(t)\end{bmatrix}$$

and a design algorithm for $(G_1, G_2, A_L^r, B_L^r, K_1^N, K_2^r, L^r)$ based on:

- The frequencies $\{\omega_k\}_k$ of $w_{\mathit{dist}}(t)$ and $y_{\mathit{ref}}(t)$.
- The Galerkin approximation (A^N, B^N, C^N, D) (**no** B_d!).
- A dimension parameter $r \in \mathbb{N}$ for the model reduction step.

Design Algorithm — Outline

Input parameters: $\{\omega_k\}_k$, (A^N, B^N, C^N, D), and $r \leq N$.

Step 1. Construct the "**internal model**" (G_1, G_2) based on $\{\omega_k\}_k$.

Step 2. Solve LQ Filter and Control Riccati equations for the pairs

$$(C^N, A^N) \qquad \text{and} \qquad \left(\begin{bmatrix} G_1 & G_2 C^N \\ 0 & A^N \end{bmatrix}, \begin{bmatrix} G_2 D \\ B^N \end{bmatrix} \right),$$

respectively, to obtain stabilizing output injection L^N and state feedback $K^N =: [K_1^N, K_2^N]$.

Step 3. Apply Balanced Truncation to the N-dimensional system

$$(A^N + L^N C^N, [B^N + L^N D, L^N], K_2^N)$$

to obtain a stable r-dimensional system $(A_L^r, [B_L^r, L^r], K_2^r)$.

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The Main Result

$$\begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} G_1 & 0 \\ B_L^r K_1^N & A_L^r + B_L^r K_2^r \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} G_2 \\ -L^r \end{bmatrix} (y(t) - y_{ref}(t))$$

$$u(t) = \begin{bmatrix} K_1^N & K_2^r \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$$

Theorem

Assume (A,B,C,D) is exponentially stabilizable and detectable and its transmission zeros do not coincide with $\{\pm i\omega_k\}_k \subset i\mathbb{R}$.

If $N \in \mathbb{N}$ and $r \leq N$ are sufficiently large, the controller solves the robust output regulation problem. There exist $M, \alpha > 0$ such that

$$\|y(t) - y_{ref}(t)\| \leq M e^{-\alpha t} \left(\|x(0)\| + \|z(0)\| + \|y_{ref}\|_\infty + \|w_{dist}\|_\infty \right)$$

for all initial states $x(0) \in X$ and $z(0) \in Z$ and $y_{ref}(t)$, $w_{dist}(t)$.

Discussion

Theorem

If $N \in \mathbb{N}$ and $r \leq N$ are sufficiently large, the controller solves the robust output regulation problem. There exist $M, \alpha > 0$ such that

$$\|y(t) - y_{ref}(t)\| \leq M e^{-\alpha t} \left(\|x(0)\| + \|z(0)\| + \|y_{ref}\|_\infty + \|w_{dist}\|_\infty\right)$$

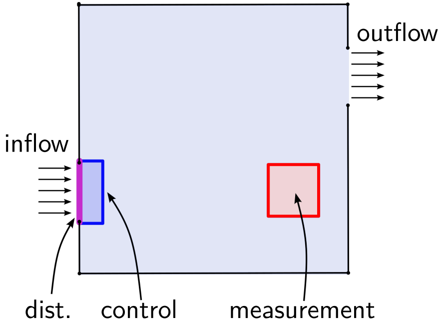
for all initial states $x(0) \in X$ and $z(0) \in Z$ and $y_{ref}(t)$, $w_{dist}(t)$.

- Uniform exponential convergence of $\|y(t) - y_{ref}(t)\| \to 0$, can achieve rate $\alpha > 0$ whenever $(A + \alpha I, B, C)$ is stab/detect.
- "N and r sufficiently large" inconvenient $\rightsquigarrow$ further research
- Required size of r depends (roughly) on decay of the Hankel singular values of (A^N, B^N, C^N) $\rightsquigarrow$ A fair amount of model reduction is typically possible for PDEs.

Room Temperature Control

Goal: Robust temperature tracking in a 2D room with steady state velocity field.

Thesis work by Konsta Huhtala (with Weiwei Hu).



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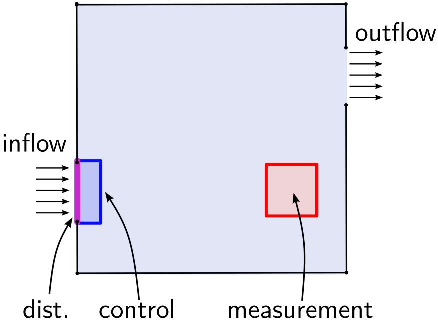


$$\dot{\theta}(\xi,t) = \frac{1}{RePr}\Delta\theta(\xi,t) - v(\xi)\cdot\nabla\theta(\xi,t) + b(\xi)u(t),$$

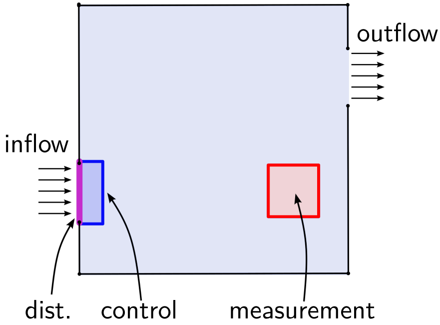
$$0 = \frac{1}{Re}\Delta v(\xi) - v(\xi)\cdot\nabla v(\xi) - \nabla p(\xi), \qquad \nabla\cdot v(\xi) = 0 \qquad \text{(NS)}$$

$$y(t) = \int c(\xi)\theta(\xi,t)d\xi$$

Room Temperature Control

Temperature tracking:

$$y_{ref}(t) = \sin(t) + 2\cos(2t),$$
$$w_{dist}(t) = 1.5\cos(3t).$$

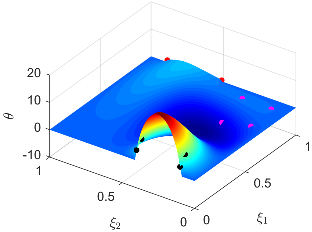


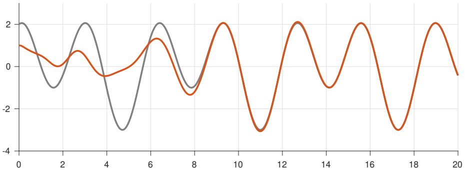
Reduced order controller design:

- Frequencies $\{\omega_k\}_k = \{1, 2, 3\}$, $\dim Y = 1$
- FEM approximation with 2nd order basis $\rightsquigarrow N = 1549$
- Controller design using Matlab (`are`, `balred`)
- Controller dimension $6 + 10 = 16$ (IM dim $+\ r$)

Simulation

- Higher order approximation ($N_{high} = 6297$) for the plant
- Exponential convergence of the output with $\alpha \approx 0.5$





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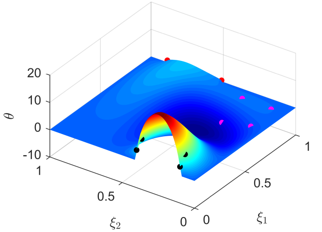

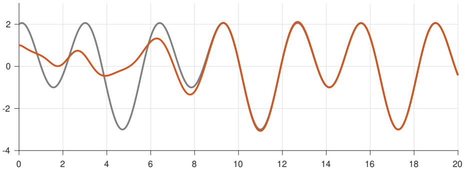

- The original PDE model is **stable**: Controller achieves radically faster convergence than "minimal order" controllers:

Low-Gain controller: $\dim = 6, \quad \max\ \alpha \leq 0.1,$

Our controller: $\dim = 16, \quad \alpha \approx 0.5$ (by design)

Simulation

Summary

- A low-order robust controller for parabolic systems based on Galerkin approximations
- A straightforward and easily implementable design algorithm
- Application: Robust temperature tracking in a 2D room model

LP and D. Phan, “Reduced order controller design for robust output regulation”, IEEE TAC 2020.

D. Phan and LP, “ Finite-dimensional controllers for robust regulation of boundary control systems”, MCRF, 2021.

K. Huhtala, LP, and Weiwei Hu, “Robust output tracking for a room temperature model with distributed control and observation” MTNS21

Implemented in: `https://github.com/rorpack-matlab`