On Robust Output Regulation for Continuous-Time Periodic Systems

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Main Objectives

Problem

Formulate and solve the robust output regulation problem for a stable continuous time periodic linear system.

Main tools:

- The *lifting technique* for representing the periodic system as an autonomous linear system
- Controller design with an infinite-dimensional internal model

Main Result

An autonomous infinite-dimensional robust controller for the periodic system.

Consider a stable periodic plant

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \qquad x(0) = x_0 \in X$$
$$y(t) = C(t)x(t) + D(t)u(t)$$

on $X = \mathbb{C}^n$, and $\sigma(\Phi_A(\tau, 0)) \subset \mathbb{D}$ (exponential stability).

Here

- $u(t) \in \mathbb{C}^m$ is the control input
- $y(t) \in \mathbb{C}^p$ is the measured output.
- $A(\cdot), B(\cdot), C(\cdot), D(\cdot)$ are continuous and τ-periodic.
- (The results extend to systems on Banach X.)

The Control Problem

Problem (Robust Output Regulation)

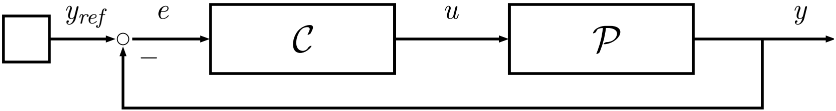
Choose a control law in such a way that

- *The closed-loop system is stable.*
- *The output* $y(t)$ *tracks a given reference signal* $y_{\mathit{ref}}(t)$ *asymptotically, i.e.*

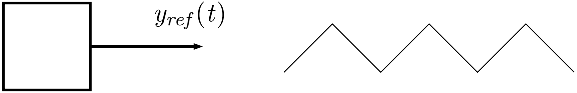
$$\lim_{t\to\infty} \|y(t) - y_{\mathit{ref}}(t)\| = 0$$

- *The above property is robust with respect to small perturbations in the parameters* $A(\cdot), B(\cdot), C(\cdot), D(\cdot)$.

The Error Feedback Control Scheme



The reference signal $y_{ref}(t)$ is τ-periodic and continuous.



General Approach

We aim to

- Use the *lifting technique* (Meyer & Burrus, 1975) to rewrite the plant and the exosystem as autonomous discrete-time systems.
- Design an internal model based controller to solve the discrete time control problem.
- Use the resulting control law to control the original system.

Literature Review

Our general approach has been successfully used for robust output regulation of periodic discrete-time systems:

- O. M. Grasselli and S. Longhi, 1991.
- O. M. Grasselli, S. Longhi, A. Tornambé, and P. Valigi, 1996.
- A. Langari, PhD Thesis, 1997.
- L. B. Jemaa and E. J. Davison, 2003.
- M. Nagahara and Y. Yamamoto, 2009.

Preliminary Comments

The lifting technique is well-known for both discrete and continuous-time systems. In both situations, the periodic plant can be rewritten as a system

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k, \qquad \mathbf{x}_0 = x_0 \\ \mathbf{y}_k &= \mathbf{C}\mathbf{x}_k + \mathbf{D}\mathbf{u}_k, \end{aligned}$$

on $X = \mathbb{C}^n$.

The challenge: For continuous time systems the output space, $Y = L^2(0, \tau; \mathbb{C}^p)$, is infinite-dimensional, and the classical internal model principle of Francis and Wonham, and Davison does not apply.

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The solution: The recent extensions of the internal model principle for infinite-dimensional systems (LP & Pohjolainen, 2010, 2013, 2014) also apply when $\dim Y = \infty$.

We use these results in designing the controller for the lifted system, for which the output tracking of y_{ref} corresponds to tracking of a *constant signal* $\mathbf{y}_k^{ref} \equiv y_{ref}(\cdot)$.

The controller contains an "infinite number of copies" of the frequency $\lambda = 1$.

Parameters of the Lifted System

For the lifted system we have $\mathbf{x}_k = x(k\tau)$, $\mathbf{y}_k(\cdot) = y(k\tau + \cdot)$ and

$$\begin{aligned}
\mathbf{A}\mathbf{x} &= \Phi_A(\tau, 0)\mathbf{x} \\
\mathbf{B}\mathbf{u} &= \int_0^\tau \Phi_A(\tau, s)B(s)\mathbf{u}(s)ds \\
(\mathbf{C}\mathbf{x})(\cdot) &= C(\cdot)\Phi_A(\cdot, 0)\mathbf{x} \\
(\mathbf{D}\mathbf{u})(\cdot) &= D(\cdot)\mathbf{u}(\cdot) + C(\cdot)\int_0^\cdot \Phi_A(\cdot, s)B(s)\mathbf{u}(s)ds.
\end{aligned}$$

The transfer function of the lifted system is

$$\mathbf{P}(\mu) = \mathbf{C}(\mu I - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \in \mathcal{L}(U, Y), \qquad \mu \in \rho(\mathbf{A}).$$

Main Result

Theorem

Assume $\mathbf{P}(1)$ *is boundedly invertible.*

Then there exists $\varepsilon^* > 0$ *such that for every* $0 < \varepsilon \le \varepsilon^*$ *the controller*

$$\mathbf{z}_{k+1} = \mathbf{z}_k + \varepsilon \mathbf{e}_k \qquad \mathbf{z}_0 = z^0 \in Z$$
$$\mathbf{u}_k = \mathbf{P}(1)^{-1} \mathbf{z}_k.$$

on $Z = L^2(0,\tau;\mathbb{C}^p)$ *solves the robust output regulation problem.*

Here at each step the regulation error $\mathbf{e}_k$ is defined as

$$\mathbf{e}_k = \mathbf{y}_k - \mathbf{y}_{\mathit{ref},k} = y(k\tau + \cdot) - y_{\mathit{ref}}(k\tau + \cdot)$$

Comments on the Controller

Theorem

Assume $\mathbf{P}(1)$ is invertible. For every $0 < \varepsilon \leq \varepsilon^*$

$$\mathbf{z}_{k+1} = \mathbf{z}_k + \varepsilon \mathbf{e}_k \qquad \mathbf{z}_0 = z^0 \in Z$$
$$\mathbf{u}_k = \mathbf{P}(1)^{-1}\mathbf{z}_k.$$

on $Z = L^2(0,\tau;\mathbb{C}^p)$ solves the robust output regulation problem.

- The invertibility of $\mathbf{P}(1) = \mathbf{C}(I - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$ is restrictive, and future research should be aimed at relaxing this condition.
- In addition, as the controller is ∞-dimensional, effective approximation schemes are needed for practical applications!

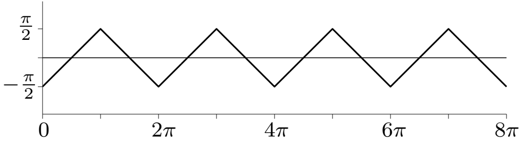
Example

Consider a stable periodic plant

$$\dot{x}(t) = a(t)x(t) + u(t), \qquad x(0) = x_0 \in \mathbb{C}$$
$$y(t) = x(t) + u(t),$$

with $\tau = 2\pi$, and $a(t) = \begin{cases} -1 & 0 \leq t < \pi \\ -2 & \pi \leq t < 2\pi. \end{cases}$

We aim to track the reference signal



The Inverse of $\mathbf{P}(1)$

Finding the inverse $\mathbf{u} = \mathbf{P}(1)^{-1}\mathbf{y}$ is equivalent to solving the Volterra–Fredholm equation

$$\mathbf{y}(t) = \mathbf{u}(t) + \frac{1}{1 - e^{-3\pi}} \int_0^{2\pi} K_F(t,s)\mathbf{u}(s)\,ds + \int_0^t K_V(t,s)\mathbf{u}(s)\,ds$$

with kernels

$$K_F(t,s) = e^{\int_0^t a(r)\,dr} e^{\int_s^{2\pi} a(r)\,dr}$$
$$K_V(t,s) = e^{\int_s^t a(r)\,dr}.$$

Can be solved numerically (here with Adomian decomposition). If the approximation is sufficiently accurate, robustness of the controller guarantees regulation.

Numerical Simulation

For numerical simulation we approximate the elements of $U = Y = L^2(0, 2\pi)$ with truncated Fourier series expansions

$$y(\cdot) = \sum_{k=-N}^{N} \langle y(\cdot), \phi_k \rangle_{L^2} \phi_k(\cdot),$$

$$u(\cdot) = \sum_{k=-N}^{N} \langle u(\cdot), \phi_k \rangle_{L^2} \phi_k(\cdot).$$

Simulation Results

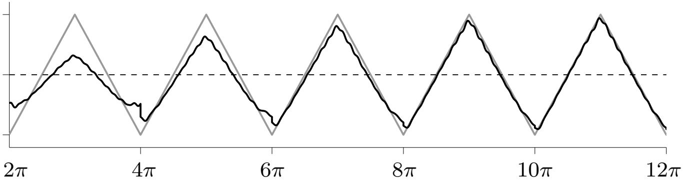


Figure: The output $y(t)$.

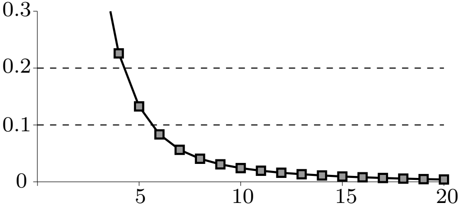


Figure: The regulation error $\mathbf{e}_k$.

Conclusions

In this presentation:

- Robust output regulation for continuous-time periodic systems.

Further research topics:

- Relaxing the assumption on invertibility of $\mathbf{P}(1)$
- Efficient methods for approximating the infinite-dimensional controller