Non-Uniform Stability of Damped Contraction Semigroups

For a skew-adjoint A and a possibly unbounded B, consider

$$\begin{cases} \dot{x}(t) = (A - BB^*)x(t), \\ x(0) = x_0 \end{cases}$$

or the special case

$$\begin{cases} \ddot{w}(t) + Lw(t) + DD^*\dot{w}(t) = 0 \\ w(0) = w_0, \quad \dot{w}(0) = w_1 \end{cases}$$

Goal (Resolvent Estimates for Non-Uniform Stability)

Introduce conditions on (A, B) *and* (L, D) *s.t.* $i\mathbb{R} \subset \rho(A - BB^*)$,

$$\|(is - A + BB^*)^{-1}\| \le N(|s|), \qquad s \in \mathbb{R}$$

for some known $N : \mathbb{R}_+ \to [0, \infty)$.

Motivation: Resolvent estimate implies "**Non-uniform stability**".

Polynomial and Non-Uniform Stability

Theorem (BT'10, RSS'19)

Assume $A - BB^*$ *generates a contraction s.g.,* $i\mathbb{R} \subset \rho(A - BB^*)$, *and*

$$\|(is - A + BB^*)^{-1}\| \le N(|s|), \qquad N \text{ non-decreasing.}$$

Then there exist $C, t_0 > 0$ *such that for* $t \ge t_0$

$$\|e^{(A-BB^*)t}x_0\| \le \frac{C}{M_T(t)}\|(A - BB^*)x_0\|, \qquad x_0 \in D(A - BB^*),$$

where "$M_T(t) = N^{-1}(t)$*" (e.g.,* $M_T(t) = t^{1/\alpha}$ *if* $N(s) = 1 + s^\alpha$*).*

[. . . , Liu–Rao '05, Batty–Duyckaerts '08, Borichev–Tomilov '10, Rozendaal–Seifert–Stahn '19]

Main Types of "Non-Uniform Observability" Conditions

1. Hautus-type condition with increasing functions M, m:

$$\|x\|^2 \le M(|s|)\|(is - A)x\|^2 + m(|s|)\|B^*x\|^2, \quad x \in D(A), s \in \mathbb{R}$$

2. Observability of the "Schrödinger group", increasing M_S, m_S:

$$\|w\|^2 \le M_S(s)\|(s^2 - L)w\|^2 + m_S(s)\|D^*w\|^2, \quad w \in D(L), s \ge 0$$

3. "Wavepacket condition", with spectral projection $P_{(a,b)}$ of A (for the interval $i(a,b) \subset i\mathbb{R}$), and decreasing γ, δ:

$$\|B^*x\| \ge \gamma(|s|)\|x\|, \qquad x \in \mathrm{Ran}(P_{(s-\delta(|s|), s+\delta(|s|))}), \; s \in \mathbb{R}.$$

Summary of the Main Results

Each condition implies $i\mathbb{R} \subset \rho(A - BB^*)$ and a resolvent bound determined by the parameters (M, m), (M_S, m_S), or (γ, δ).

Our Results in Context of Earlier Research

(B^*, A) **exactly** observable $\quad \Leftrightarrow \quad$ $A - BB^*$ exponentially stable

$\boxed{(B^*, A) \textbf{ non-uniformly obs.} \quad \Leftrightarrow \quad A - BB^* \textbf{ non-uniformly stable}}$

(B^*, A) **approx.** observable $\quad \Leftrightarrow^* \quad$ $A - BB^*$ strongly/weakly stable

Exact and approximate observability: [Slemrod, Levan, Russell, Benchimol, Guo–Luo, Lasiecka–Triggiani, Curtain–Weiss . . .]

Non-uniform observability: [Ammari–Tucsnak '01, Ammari et.al.]

R. Chill, LP, D. Seifert, R. Stahn, Y. Tomilov, "Non-Uniform Stability of Damped Contraction Semiroups," *in review* (https://arxiv.org/abs/1911.04804)

Applications and examples

Can be especially applied in

- 2D Wave equations with viscous damping
  - Reproduces known model-specific results (literature extensive)
- 1D Wave and beam equations with pointwise or distributed damping
  - Analysis simple since A has uniform spectral gap
- Su–Tucsnak–Weiss "Stabilizability properties of a linearized water waves system," *Systems Control Lett.*, 2020.
  - Interesting case: eigenvalues of A behave like $\lambda_k \approx i\sqrt{k}$

A "Non-uniform Hautus test"

Consider the Hautus-type condition [Miller 2012]

$$\|x\|^2 \leq M(|s|)\|(is - A)x\|^2 + m(|s|)\|B^*x\|^2, \quad x \in D(A), s \in \mathbb{R},$$

for some non-decreasing $M, m\colon [0,\infty) \to [r_0, \infty)$.

Theorem

If the above condition holds, then $i\mathbb{R} \subset \rho(A - BB^*)$. If $N(s) := M(s) + m(s)$ has positive increase, then

$$\|e^{(A-BB^*)t}x_0\| \leq \frac{C}{N^{-1}(t)}\|(A - BB^*)x_0\|, \quad x_0 \in D(A - BB^*)$$

Observability of the Schrödinger Group

For

$$\ddot{w}(t) + Lw(t) + DD^*\dot{w}(t) = 0,$$

and $M_S, m_S\colon [0,\infty) \to [r_0,\infty)$ consider ($s \geq 0$)

$$\|w\|^2 \leq M_S(s)\|(s^2 - L)w\|^2 + m_S(s)\|D^*w\|^2, \quad w \in D(L)$$

This is **observability of the "Schrödinger group"** (D^*, iL) (generalises Anantharaman–Leataud 2014, Joly–Laurent 2019)

Theorem

A similar result, decay rate determined by $N^{-1}(t)$*, where*

$$N(s) := M_S(s)m_S(s)(1 + s^2).$$

A "Wavepacket Condition"

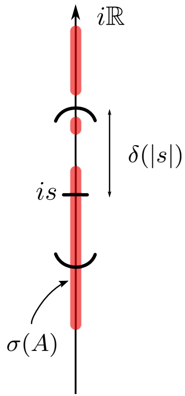
For A skew-adjoint with spectral projection $P_{(a,b)}$ (for $i(a,b) \subset i\mathbb{R}$)

$$\|B^*x\| \geq \gamma(|s|)\|x\|, \qquad x \in \operatorname{Ran}(P_{(s-\delta(|s|),s+\delta(|s|))}), \; s \in \mathbb{R}$$

for some non-increasing $\delta, \gamma \colon [0,\infty) \to (0, r_0]$.

Such x are often called **"wavepackets"** of A.

(Used for exact observability, e.g., in Ramdani et. al. 2005, Miller 2012, Tucsnak–Weiss 2009.)



Theorem

If $A^* = -A$ and if $N(s) := \delta(s)^{-2}\gamma(s)^{-2}$ has positive increase, then

$$\|e^{(A-BB^*)t}x_0\| \leq \frac{C}{N^{-1}(t)}\|(A-BB^*)x_0\|.$$

We consider $A : D(A) \subset X \to X$ generating e^{At} on Hilbert X.

Assumption

For $D(A) \subset V \hookrightarrow X$, let $V \hookrightarrow X \hookrightarrow V^*$ (Gelfand triple)

(1) e^{At} *is contractive and*

$$\mathrm{Re}\langle A_{-1}x, x\rangle_{V^*,V} \leq 0 \quad \textit{for } x \in V \textit{ satisfying } A_{-1}x \in V^*.$$

(2) $B \in \mathcal{L}(U, X_{-1})$ ***and*** $B \in \mathcal{L}(U, V^*)$, $\mathrm{Ran}((1 - A_{-1})^{-1}B) \subset V$.

(3) $B^* \in \mathcal{L}(V, U)$ *is defined as the adjoint of* $B \in \mathcal{L}(U, V^*)$, *i.e.*

$$\langle Bu, x\rangle_{V^*,V} = \langle u, B^*x\rangle_U, \qquad u \in U,\ x \in V.$$

(4) $\sup_{s\in\mathbb{R}} \|B^*(1 + is - A_{-1})^{-1}B\| < \infty.$

- (4) can be removed, but resolvent growth changes (up to s^4)
- Satisfied for second order systems, $D \in \mathcal{L}(U, \mathrm{Dom}(L^{1/2})^*)$.