Non-uniform Stability of Coupled Systems and PDEs

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Main Objectives

Problem

*Consider the stability of different types of **coupled systems and PDEs**.*

The focus is on couplings leading to **non-uniform stability**.

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The focus is on couplings leading to **non-uniform stability**.

Motivation:

- Coupling of stable and unstable PDEs and ODEs often leads to rational decay of energy, i.e., polynomial stability.
- Situation also appears in control applications.

Main results:

- New stability results for coupled PDEs.
- Disclaimer: Will not solve all your problems!

Outline

(1) Discussion: Passive systems and feedback in coupled PDEs

(2) Introduction to polynomial and non-uniform stability

(3) Main stability results.

- General conditions for polynomial and nonuniform stability of coupled PDEs and systems.

Coupled PDE-PDE and PDE-ODE systems appear in models of

- Fluid-structure interactions
- Thermo-elasticity
- Mechanical systems, e.g., beams with tip masses
- Magnetohydrodynamics
- Acoustics

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Couplings may either be

- Through the **boundary** (Fluid-structure, acoustics), or
- **inside a shared domain** (Thermo-elasticity, MHD)

Motivation 1: Coupled Wave–Heat Systems

Models for fluid–structure and heat–structure interactions:

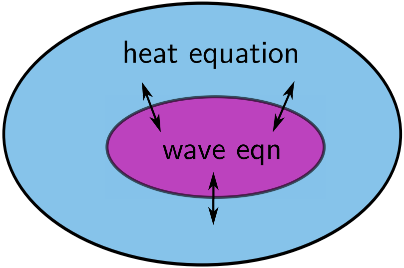
$$\frac{\partial^2 u}{\partial t^2}(x,t) = \Delta u(x,t)$$

$$\updownarrow$$

coupling BCs

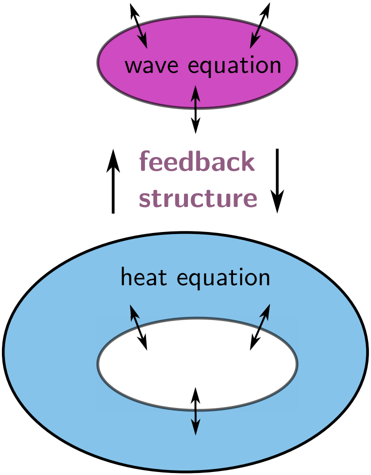
$$\updownarrow$$

$$\frac{\partial w}{\partial t}(x,t) = \Delta w(x,t)$$



References: Avalos & Triggiani, Duyckaerts, Zhang & Zuazua, Mercier, Nicaise, Ammari, Guo, and many others.

Coupled Wave–Heat Systems



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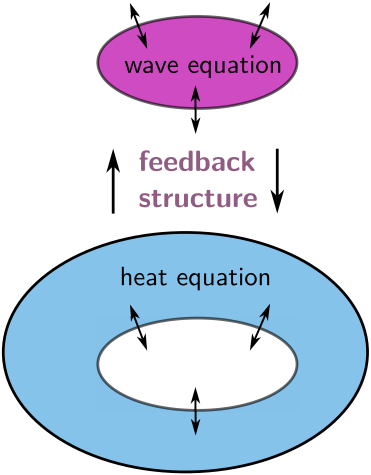

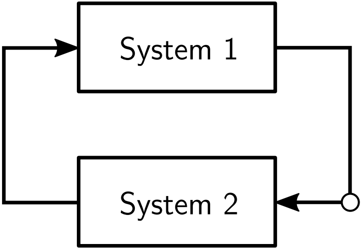

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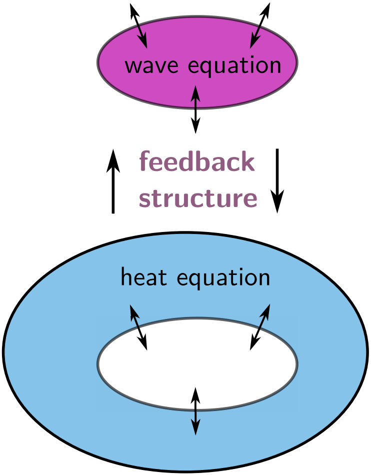
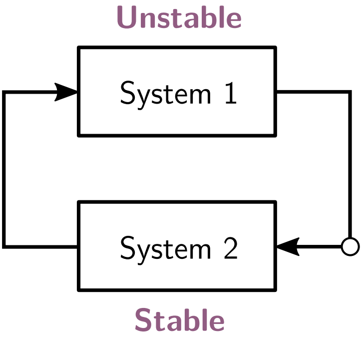
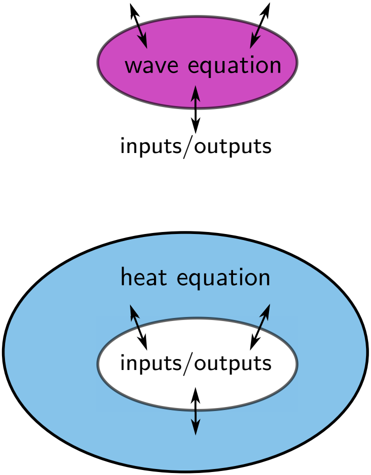
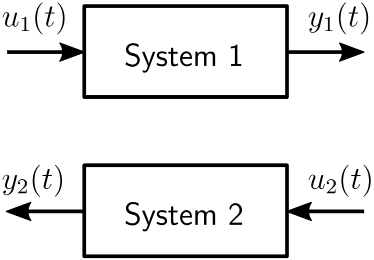




Inputs and Outputs

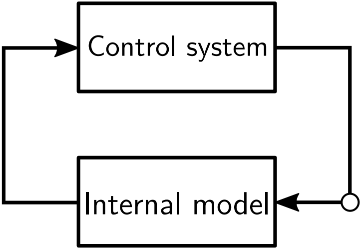




Motivation 2: Internal Model Based Control

Problem

Closed-loop stabilization in ***Robust Output Tracking and Disturbance Rejection*** *for stable systems.*



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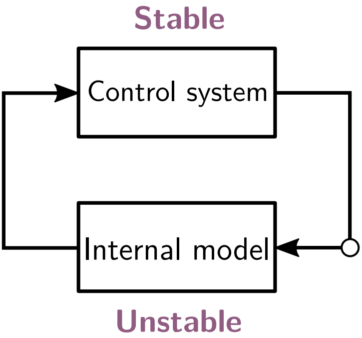
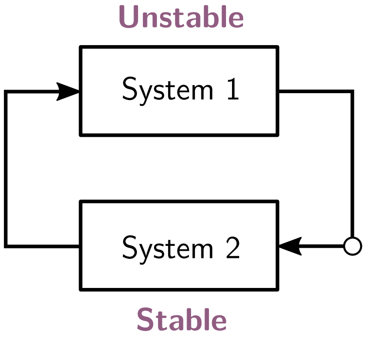


Problem

Use the properties of the two systems to deduce stability of the coupled system.



Polynomial and Non-Uniform Stability

Theorem (Borichev & Tomilov '10)

Let $T(t)$ be a uniformly bounded C_0-semigroup on a Hilbert space X. Let A be the generator of $T(t)$ and $\sigma(A) \cap i\mathbb{R} = \emptyset$.

For any constant $\alpha > 0$, the following are equivalent:

$$\|T(t)x_0\| \leq \frac{M}{t^{1/\alpha}} \|Ax_0\| \qquad \text{for some } M > 0$$

$$\|(is - A)^{-1}\| \leq M_R(1 + |s|^\alpha), \qquad \text{for some } M_R > 0$$

Application: $E(t) \sim \|T(t)x_0\|^2$ for many PDE systems.

Polynomial and Non-Uniform Stability

Theorem (Rozendaal, Seifert & Stahn 2017, on Hilbert X)

Assume $T(t)$ bounded, $i\mathbb{R} \subset \rho(A)$. Define an increasing $M(\cdot)$ by

$$M(s) = \sup_{|r| \leq s} \|(ir - A)^{-1}\|, \quad s > 0.$$

If $M(\cdot)$ "positive increase", then for some $c, C > 0$

$$\frac{c}{M^{-1}(t)} \|Ax_0\| \leq \|T(t)x_0\| \leq \frac{C}{M^{-1}(t)} \|Ax_0\|, \qquad x_0 \in \mathcal{D}(A)$$

Polynomial and Non-Uniform Stability

Theorem (Batty & Duyackerts 2008, on Banach X)

Assume $T(t)$ bounded, $i\mathbb{R} \subset \rho(A)$. Define an increasing $M(\cdot)$ by

$$M(s) = \sup_{|r| \le s} \|(ir - A)^{-1}\|, \quad s > 0.$$

Then for some $c, C > 0$

$$\|T(t)x_0\| \le \frac{C}{M_{log}^{-1}(ct)} \|Ax_0\|, \qquad x_0 \in \mathcal{D}(A)$$

where $M_{log}(s) = M(s)(\log(1 + M(s)) + \log(1 + s))$.

This is **optimal** for general Banach X (Borichev & Tomilov '10).

Take-Home Message

If your system is contractive or bounded, then

*"Non-uniform stability **only** requires a resolvent estimate on $i\mathbb{R}$"*

Polynomial and Non-Uniform Stability Appear in . . .

- Multidimensional damped wave equations (non-GCC)
- Wave equations on *exterior domains*
- Platoon-type systems
- Here: Coupled PDE and PDE-ODE systems

Problem

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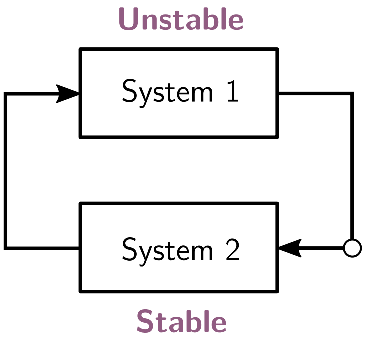

Impedance Passive Systems

Consider (regular) linear systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0 \in X$$
$$y(t) = B^* x(t)$$

where X is Hilbert, A generates a **contraction semigroup**, and $B \in \mathcal{L}(U, V^*)$ for some suitable spaces U and $V^* \supseteq X$.

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Such systems are "**impedance passive**", which in particular means they have "**no internal sources of energy**",

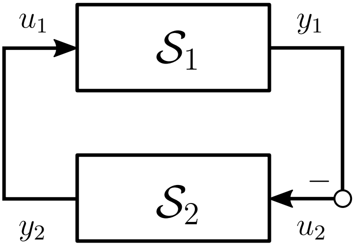
$$\frac{d}{dt}\|x(t)\|^2 \leq 2\,\mathrm{Re}\langle u(t), y(t)\rangle_Y$$

Examples:

- Many mechanical systems, RLC circuits, ...

Feedback Theory of Passive Systems

Property: "Power-preserving interconnection" preserves passivity.



$\Rightarrow$ Closed-loop semigroup contractive on Hilbert $X_1 \times X_2$.

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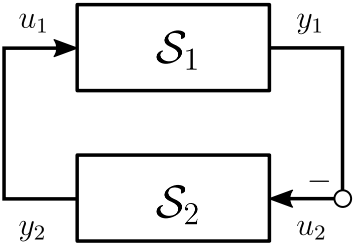

$\Rightarrow$ Closed-loop semigroup contractive on Hilbert $X_1 \times X_2$.

Exponential and strong stability results:

- Rebarber-Weiss '03, Ramirez-Le Gorrec-Macchelli-Zwart '14, Guiver-Logemann-Opmeer '17, Zhao-Weiss '17, ...

Coupled Passive Systems

If for $k = 1, 2$ we let

$$\dot{x}_k(t) = A_k x_k(t) + B_k u_k(t), \qquad x_k(0) \in X_k$$
$$y_k(t) = B_k^* x_k(t),$$

then the "**power-preserving interconnection**" leads to

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A_1 & B_1 B_2^* \\ -B_2 B_1^* & A_2 \end{bmatrix}}_{=: A} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Example: 1D Wave–Heat Model

$$\begin{cases} v_{tt}(\xi,t) = v_{\xi\xi}(\xi,t), & \xi \in (-1,0),\ t>0, \\ w_t(\xi,t) = w_{\xi\xi}(\xi,t), & \xi \in (0,1),\ t>0, \\ v_\xi(0,t) = w_\xi(0,t), \quad v_t(0,t) = w(0,t), & t>0, \end{cases}$$

- [Xu Zhang & Zuazua, Batty, Paunonen & Seifert, (2D version: Avalos, Triggiani & Lasiecka)]
- Known: Closed-loop polynomially stable, $\|R(is,A)\| = O(\sqrt{|s|})$.

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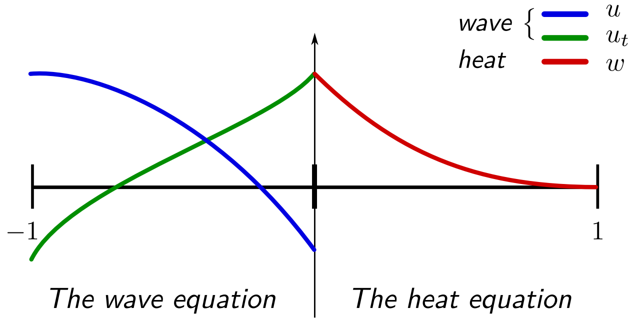


Example: 1D Wave-Heat — Open-Loop Splitting

Wave system on $(-1,0)$:

$$\begin{aligned} v_{tt}(\xi,t) &= v_{\xi\xi}(\xi,t) \\ y_1(t) &= v_\xi(0,t) \\ u_1(t) &= v_t(0,t) \end{aligned}$$

Unstable

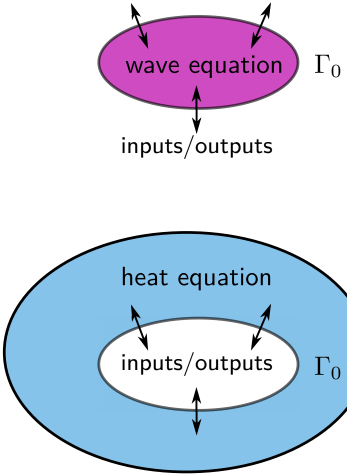
Heat system on $(0,1)$:

$$\begin{aligned} w_t(\xi,t) &= w_{\xi\xi}(\xi,t) \\ y_2(t) &= w(0,t) \\ u_2(t) &= -w_\xi(0,t) \end{aligned}$$

Stable

The systems **are** impedance passive. We have $U=\mathbb{C}$ and B_1 and B_2 are unbounded.

Inputs and Outputs



2D systems are more complicated to set up.

For boundary couplings U is a function space on Γ_0.

In in-domain couplings, space on Ω or $\Omega_0 \subset \Omega$.

Problem

Derive a resolvent estimate for

$$A := \begin{bmatrix} A_1 & B_1 B_2^* \\ -B_2 B_1^* & A_2 \end{bmatrix}$$

in terms of the properties of

- (A_1, B_1, B_1^*) [**Unstable**]
- (A_2, B_2, B_2^*) [**Stable**]

Assumption

- A_1 *is diagonalizable and skew-adjoint,* $A_1 = \sum_{k\in\mathbb{Z}} i\omega_k \langle \cdot, \phi_k \rangle \phi_k$
- *Uniform gap:* $\inf_{k\neq l} |\omega_k - \omega_l| > 0$ *(for simplicity).*
- $T_2(t)$ *gen. by* A_2 *is exponentially stable.*

Conditions for Non-Uniform Stability

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- $T_2(t)$ *gen. by* A_2 *is exponentially stable.*
- $\|B_1^* \phi_k\| \neq 0$ *for all* k *(i.e.,* (A_1, B_1) *is "approx. controllable").*
- *Denoting* $P_2(\lambda) = B_2^*(\lambda - A_2)^{-1} B_2$ *(transfer function),*

$$P_2(i\omega_k) + P_2(i\omega_k)^* > 0 \qquad \forall k \in \mathbb{Z}.$$

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Proposition

The closed-loop is strongly stable and $\sigma(A) \subset \mathbb{C}_-$.

- Denote $P_2(\lambda) = B_2^*(\lambda - A_2)^{-1}B_2$ (transfer function)

Theorem

Let $\gamma, \eta : \mathbb{R}_+ \to (0,1)$ *be decreasing (and "nice") so that*

$$\|B_1^*\phi_k\| \geq c_1\gamma(|\omega_k|) \qquad \forall k$$

$$\mathrm{Re}\langle P_2(is)u, u\rangle \geq c_2\eta(|s|)\|u\|^2 \qquad s \approx \omega_k$$

for some constants $c_1, c_2, s_0 > 0$.
Then the closed-loop system is non-uniformly stable so that

$$\|(is - A)^{-1}\| \lesssim \frac{M_R}{\gamma(|s|)^2\eta(|s|)}, \qquad |s| \text{ large.}$$

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Thus

$$\|T(t)x\| \leq \frac{M_T}{M^{-1}(t)}\|Ax\|, \qquad x \in \mathcal{D}(A)$$

where $M(s) \sim \gamma(s)^{-2}\eta(s)^{-1}$.

Theorem

Let $\beta, \gamma \geq 0$ *such that*

$$\|B_1^* \phi_k\| \geq c_1 |\omega_k|^{-\beta} \qquad \forall k$$

$$\mathrm{Re}\langle P_2(is)u, u\rangle \geq c_2 |s|^{-\gamma} \|u\|^2 \qquad s \approx \omega_k$$

for some constants $c_1, c_2, s_0 > 0$.
Then the closed-loop system is non-uniformly stable so that

$$\|(is - A)^{-1}\| \leq M_R(1 + |s|^{2\beta+\gamma}), \qquad |s| \text{ large.}$$

Thus

$$\|T(t)x\| \leq \frac{M_T}{t^{1/\alpha}} \|Ax\|, \qquad x \in \mathcal{D}(A)$$

for $\alpha = 2\beta + \gamma \geq 0$.

Comments:

- Theorem requires some admissiblity and well-posedness assumptions (swept under the carpet here). Limits 2D-nD BC.
- Lack of spectral gap and repeated eigenvalues are allowed in a more general version (affects the rate).

Optimality

- Obtained rate is not always optimal, especially if
  - A_1 has no spectral gap (2D, nD waves) or
  - eigenvalues diverge as $|\omega_k| \to \infty$ (beams and plates).
- A nice way of getting (possibly) suboptimal rates easily.

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References:

- Paunonen Arxiv June '17
- Partly based on joint work with Chill, Stahn & Tomilov

Example: 1D Wave-Heat

Wave system on $(-1,0)$:

$$v_{tt}(\xi,t) = v_{\xi\xi}(\xi,t)$$
$$y_1(t) = v_\xi(0,t)$$
$$u_1(t) = v_t(0,t)$$

Heat system on $(0,1)$:

$$w_t(\xi,t) = w_{\xi\xi}(\xi,t)$$
$$y_2(t) = w(0,t)$$
$$u_2(t) = -w_\xi(0,t)$$

- A_1 diagonalizable, $\omega_k \sim k\pi$, ϕ_k trigonometric
- $B_1^*\phi_k \neq 0$, and $|B_1^*\phi_k| \gtrsim 1 = |\omega_k|^0$
- $P_2(is) = B_2^*(is - A_2)^{-1}B_2$ satisfies $|P_2(is)| \sim |s|^{-1/2}$.

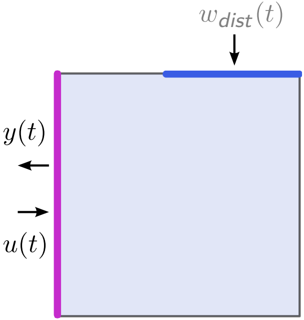
Thus the closed-loop system is polynomially stable,

$$\|(is-A)^{-1}\| = O(|s|^{1/2}) \qquad \text{and} \qquad \|T(t)x\| \le \frac{M}{t^2}\|Ax\|.$$

Reproduces results of [Zhang-Zuazua, Batty-Paunonen-Seifert].

Application: Robust Periodic Tracking

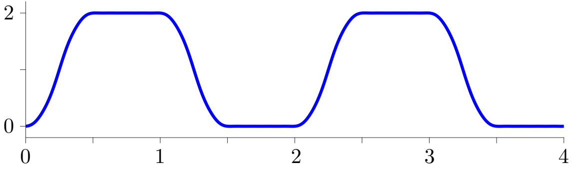
$$x_t(\xi,t) = \Delta x(\xi,t),$$
$$\frac{\partial x}{\partial n}(\xi,t)|_{\Gamma_1} = u(t), \quad \frac{\partial x}{\partial n}(\xi,t)|_{\Gamma_0} = 0$$
$$y(t) = \int_{\Gamma_1} x(\xi,t)d\xi,$$



Defines a regular linear system,

$$|P(is)| = O(\frac{1}{\sqrt{|s|}}) \qquad \text{for large } |s|.$$

Objective: Track a Reference Signal $y_{\textsf{ref}}(t)$



Consider tracking of a nonsmooth 2-periodic reference signal

$$y_{\textsf{ref}}(t) = \sum_{k\in\mathbb{Z}} \hat{y}_{ref}(k) e^{i\pi k t}$$

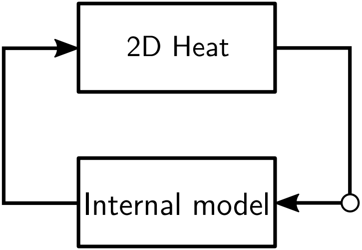
where $|\hat{y}_{ref}(k)| = O(|k|^{-3})$.

Internal Model Based Control

Theorem (Internal Model Principle, LP '10,'14)

Robust tracking is achieved if

- *Controller has "**an internal model**" of frequencies* $\{ik\pi\}_{k\in\mathbb{Z}}$
- ***The closed-loop system is stable**.*



$\sim$ periodic transport/wave eqn, or a delay line

Robust Controller Construction

Construct an **internal model based controller** (A_c, B_c, B_c^*)

- $A_c = \mathrm{diag}(ik\pi)_{k\in\mathbb{Z}}$ on $\ell^2(\mathbb{C})$, cf. periodic transport eqn.
- $B_c = (b_k)_k \in \mathcal{L}(\mathbb{C}, \ell^2(\mathbb{C}))$, choose $b_k = (1+|k|)^{-(1/2+\varepsilon)}$.

The controller is passive, (A_c, B_c) approximately controllable.

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Proposition

The closed-loop is polynomially stable so that

$$\|T_{cl}(t)x_{cl}(0)\| \le \frac{M}{t^{1/\alpha}}\|A_{cl}x_{cl}(0)\|, \qquad \forall x_{cl}(0) \in \mathcal{D}(A_{cl}),$$

where $\alpha = 3/2 + 2\varepsilon$.

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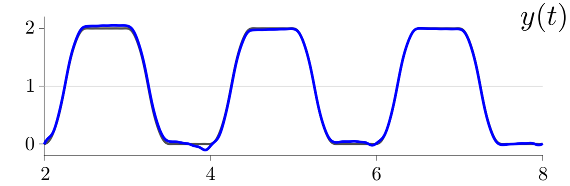
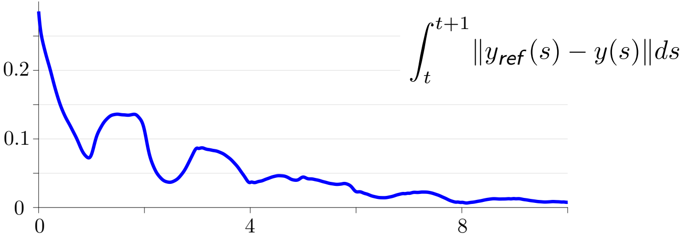
where $\alpha = 3/2 + 2\varepsilon$. *If* $0 < \varepsilon < 1/2$, *then*

$$\int_t^{t+1} \|y(s) - y_{ref}(s)\| ds = O\left(\frac{1}{t^{1/\alpha}}\right)$$

for "suitable" intial states $x_{cl}(0) \in X \times Z$ *(∼ classical solutions).*





Approximations:

- Finite Differences 20×20 grid,
- A_c truncated to a 31×31-matrix.

Conclusions

In this presentation:

- Discussion of coupled systems and PDEs
- General conditions for non-uniform and polynomial stability of coupled systems.

LP, "Stability and Robust Regulation of Passive Linear Systems," http://arxiv.org/abs/1706.03224