# Non-Uniform Stability of Damped Contraction Semigroups

Lassi Paunonen

Tampere University, Finland

joint work with R. Chill, D. Seifert, R. Stahn and Y. Tomilov.

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# Goal of the Talk

Consider the asymptotic behaviour of solutions of the "abstract (damped) wave equation"

$$\begin{cases} \ddot{w}(t) + Lw(t) + DD^*\dot{w}(t) = 0 \\ w(0) = w_0, \quad \dot{w}(0) = w_1 \end{cases}$$

on a Hilbert space $H$.

## Problem

*Formulate conditions on $(L, D)$ such that for all initial conditions*

$$\|w(t)\| \to 0 \qquad \textit{as} \quad t \to \infty$$

*and especially study the* ***rate*** *of the convergence.*

# Assumptions

$$\ddot{w}(t) + Lw(t) + DD^*\dot{w}(t) = 0, \qquad w(0) = w_0, \quad \dot{w}(0) = w_1$$

Throughout the presentation:

- $L : \mathrm{Dom}(L) \subset H \to H$ is self-adjoint, positive, and boundedly invertible. Operator $D$ is bounded, $D \in \mathcal{L}(U, H)$.

## Example

In the case of the $n$D wave equation with viscous damping $d \geq 0$

$$\begin{aligned} \ddot{w}(\xi,t) - \Delta w(\xi,t) + d(\xi)\dot{w}(\xi,t) &= 0, && \xi \in \Omega, \quad t > 0 \\ w(\xi,t) &= 0 && \xi \in \partial\Omega \end{aligned}$$

- $H = L^2(\Omega)$, $w(t) := w(\cdot,t)$, and $L = -\Delta$ (Dirichlet BC's)
- $U = H$ and $(Du)(\xi) = \sqrt{d(\xi)}u(\xi)$.

- $w_0 \in \mathrm{Dom}(L^{1/2})$ and $w_1 \in H$ $\quad$ ($\rightsquigarrow$ mild/weak solutions)

# Semigroup Formulation

The equation

$$\ddot{w}(t) + Lw(t) + DD^*\dot{w}(t) = 0, \qquad w(0) = w_0, \quad \dot{w}(0) = w_1$$

can rewritten as a first order abstract Cauchy problem

$$\dot{x}(t) = (A - BB^*)x(t), \qquad x(0) = (w_0, w_1)^T$$

with $x(t) = (w(t), \dot{w}(t))^T \in X := \mathrm{Dom}(L^{1/2}) \times H$,

$$A = \begin{bmatrix} 0 & I \\ -L & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ D \end{bmatrix}$$

with $\mathrm{Dom}(A) = \mathrm{Dom}(L) \times \mathrm{Dom}(L^{1/2})$. Here $A - BB^*$ generates a contraction semigroup on $X$.

# Stability

Consider

$$\ddot{w}(t) + Lw(t) + DD^*\dot{w}(t) = 0, \qquad w(0) = w_0, \quad \dot{w}(0) = w_1.$$

## Definition (Stability)

The abstract wave equation is **stable** if

$$\|L^{1/2}w(t)\| + \|\dot{w}(t)\| \to 0, \qquad \text{as} \quad t \to \infty \qquad (*)$$

for all initial conditions $w_0 \in \mathrm{Dom}(L^{1/2})$ and $w_1 \in H$.

For PDEs, the quantity in $(*)$ is typically proportional to the square root of the **energy** of the solution $w(t)$.

# Non-Uniform Stability

$$\ddot{w}(t) + Lw(t) + DD^*\dot{w}(t) = 0, \qquad w(0) = w_0, \quad \dot{w}(0) = w_1$$

**Definition (Non-Uniform Stability)**

There exists an increasing unbounded $M(\cdot) : [t_0, \infty) \to (0, \infty)$ s.t.

$$\|L^{1/2}w(t)\| + \|\dot{w}(t)\| \leq \frac{1}{M(t)} \left( \|Lw_0\| + \|L^{1/2}w_1\| \right), \quad t \geq t_0$$

for all initial conditions $w_0 \in \mathrm{Dom}(L)$, $w_1 \in \mathrm{Dom}(L^{1/2})$.

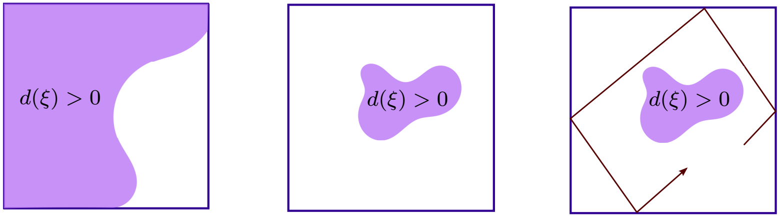
- $w_0, w_1$ correspond to **classical solutions** of the PDE.
- In **Strong (Asymptotic) Stability** all mild solutions decay (without guaranteed rate).
- In **Uniform Exponential Stability** all (mild) solutions decay at an exponential rate for all $w_0 \in \mathrm{Dom}(L^{1/2})$ and $w_1 \in H$.

# Damped Wave Equations

**Non-uniform stability** is encoutered in wave/beam/plate equations with **partial** or **weak** dampings. In the $2$D wave equation

$$\begin{aligned} \ddot{w}(\xi,t) - \Delta w(\xi,t) + d(\xi)\dot{w}(\xi,t) &= 0, && \xi \in \Omega, \quad t > 0 \\ w(\xi,t) &= 0 && \xi \in \partial\Omega \end{aligned}$$

stability depends on geometry of $\Omega$ and $\omega := \{\, \xi \in \Omega \mid d(\xi) > 0 \,\}$:




Exponential stability — Non-uniform stability — Geometric Control Condition

# Our Results

*Introduce general conditions for non-uniform stability of abstract damped wave equations.*

## Motivation:

- "Non-uniform" stability is encountered in wave/beam/plate equations with weak or partial dampings
- Most of the current literature based on case-by-case analysis

## Main results:

- General **observability-type** sufficient conditions for $(L, D)$ to guarantee non-uniform stability and to identify the decay rate.
- Proofs based on resolvent estimates on $i\mathbb{R}$ combined with general results of non-uniform stability of semigroups [Liu–Rao, Batty–Duyackerts, Borichev–Tomilov, Rozendaal–Seifert–Stahn].

# Observability-Type Conditions vs. Stability

$$\ddot{w}(t) + Lw(t) + DD^*\dot{w}(t) = 0, \qquad w(0) = w_0, \quad \dot{w}(0) = w_1$$

**Exact** observability $\Leftrightarrow$ Exponential stability

**Approximate** observability $\Leftrightarrow$ Weak/Strong stability

[Slemrod '72, Levan, Russell, Benchimol, Guo–Luo, Lasiecka–Triggiani, Curtain–Weiss . . . ]

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"**Non-uniform observability**" $\Leftrightarrow$ Non-Uniform stability

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[Slemrod '72, Levan, Russell, Benchimol, Guo–Luo, Lasiecka–Triggiani, Curtain–Weiss ...]

Earlier work: Ammari–Tucsnak 2001, Ammari et. al., Anantharaman–Leataud 2014, Joly–Laurent 2019

# Main Results

R. Chill, LP, D. Seifert, R. Stahn, Y. Tomilov, "Non-Uniform Stability of Damped Contraction Semiroups," *Analysis & PDE*, accepted (https://arxiv.org/abs/1911.04804)

# A Non-Uniform Hautus Test

Consider the Hautus-type condition [Miller 2012]

$$\|w\|^2 \le M_0(s)\|(s^2 - L)w\|^2 + m_0(s)\|D^* w\|^2, \quad w \in \mathrm{Dom}(L), s \ge 0$$

for some non-decreasing $M_0, m_0 \colon [0,\infty) \to [r_0,\infty)$.

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for some non-decreasing $M_0, m_0 \colon [0, \infty) \to [r_0, \infty)$.

## Theorem

*If the above condition holds and* $N(s) := M_0(s) m_0(s)(1 + s^2)$, *then*

$$\|L^{1/2} w(t)\| + \|\dot{w}(t)\| \leq \frac{C}{N^{-1}(t)} \left( \|L w_0\| + \|L^{1/2} w_1\| \right), \quad t \geq t_0$$

*for some* $C, t_0 > 0$ *and for all* $w_0 \in \mathrm{Dom}(L)$, $w_1 \in \mathrm{Dom}(L^{1/2})$.

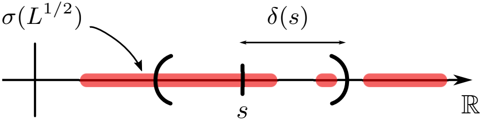
- Generalises Anantharaman–Leautaud 2014, Joly–Laurent 2019

# A "Wavepacket Condition"

Operator $L^{1/2} > 0$ has spectral projections $P_{(a,b)}$ (for $(a,b) \subset \mathbb{R}_+$). Assume

$$\|D^* w\| \geq \gamma(s)\|w\|, \qquad w \in \mathrm{Ran}(P_{(s-\delta(s),s+\delta(s))}), \ s > 0$$

for some non-increasing $\delta, \gamma \colon [0,\infty) \to (0, r_0]$.




Such $w$ are **"wavepackets"** of $L^{1/2}$, previously used for exact observability.

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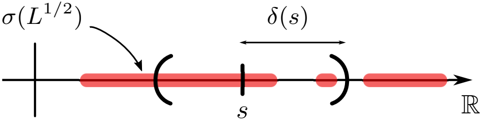



Such $w$ are **"wavepackets"** of $L^{1/2}$, previously used for exact observability.

## Theorem

*If $N(s) := \gamma(s)^{-2}\delta(s)^{-2}$ has "positive increase", then* $\exists C, t_0 > 0$,

$$\|L^{1/2} w(t)\| + \|\dot{w}(t)\| \leq \frac{C}{N^{-1}(t)} \left( \|L w_0\| + \|L^{1/2} w_1\| \right), \quad t \geq t_0$$

The results are presented in:

R. Chill, LP, D. Seifert, R. Stahn, Y. Tomilov, "Non-Uniform Stability of Damped Contraction Semiroups," *Analysis & PDE*, accepted (https://arxiv.org/abs/1911.04804)

Additional results:

- Additional and alternative observability-type conditions
- Analogous theory for first-order systems

$$\dot{x}(t) = (A - BB^*)x(t), \qquad x(0) = x_0$$

- Unbounded $D \in \mathcal{L}(U, \mathrm{Dom}(L^{1/2})^*)$ ($\rightsquigarrow$ boundary damping)

# Examples: 2D Wave Equations

A wave equation with viscous damping on a convex $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary, $d \in L^\infty(\Omega)$, $d \geq 0$

$$
\begin{aligned}
&w_{tt}(\xi,t) - \Delta w(\xi,t) + d(\xi) w_t(\xi,t) = 0, && \xi \in \Omega,\ t > 0,\\
&w(\xi,t) = 0, && \xi \in \partial\Omega,\ t > 0,\\
&w(\cdot,0) = w_0(\cdot) \in H^2(\Omega) \cap H_0^1(\Omega), && w_t(\cdot,0) = w_1(\cdot) \in H_0^1(\Omega).
\end{aligned}
$$

- Several results exist for our Hautus-type condition with **constant** $M_0(s)$ and $m_0(s)$ (Jaffard '90, Burq–Zworski '19) for rectangles/tori. Leads to **rational decay** $1/\sqrt{t}$.
- Precise lower bounds on $d$ lead to non-uniform stability using the Hautus-type condition with [Burq–Zuily 2016].
- In general our results are sub-optimal, since conditions do not take into account the **smoothness** of $d$! (Burq–Hitrik '07)

# 1D Wave Equations

Consider a wave equation with weak damping (and Dirichlet BC)

$$w_{tt}(\xi,t) - w_{\xi\xi}(\xi,t) + d(\xi)\int_0^1 d(r)w_t(r,t)dr = 0, \quad \xi \in (0,1), \; t>0,$$

- The wavepacket condition characterises (optimal) stability via lower bounds of the sine Fourier coefficients, e.g., $(c,\alpha>0)$

$$\left|\int_0^1 d(\xi)\sin(n\pi\xi)d\xi\right| \geq \frac{c}{n^\alpha}$$

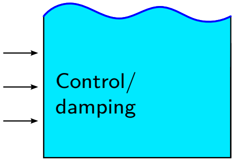
- Pointwise damping possible (formally $d(\xi)=\delta(\xi-\xi_0)$).
- Analogous results for Euler–Bernoulli / Timoshenko beams

# Application: Water Waves System

In the reference

Su–Tucsnak–Weiss "Stabilizability properties of a linearized water waves system," *Systems & Control Letters*, 2020.

the results were applied to prove non-uniform stabilizability of a "**water waves system**" in a 2D domain.




- The PDE system models small amplitude water waves
- Stability and convergence rate proved using the "Wavepacket condition"
- $\delta(s) \to 0$ so that $(s - \delta(s), s + \delta(s))$ reduce to 1D spectral subspaces.
- The stability result is likely to be optimal.

# Conclusions

In this presentation:

- General sufficient conditions for non-uniform stability of abstract damped wave equations
- Discussion of PDE examples and optimality of the results

R. Chill, LP, D. Seifert, R. Stahn, Y. Tomilov, "Non-Uniform Stability of Damped Contraction Semiroups," *Analysis & PDE*, accepted (https://arxiv.org/abs/1911.04804)

Contact: lassi.paunonen@tuni.fi