Reduced Order Internal Models in Robust Output Regulation

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Main Objectives

Problem

Study the robust output regulation problem in the case where robustness is not required with respect to all perturbations.

Main results:

- A test to determine robustness with respect to a given set of perturbations.
- Refine the Internal Model Principle: A "full" internal model is not always necessary if the class of perturbations is restricted.

Consider a plant

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0$$
$$y(t) = Cx(t) + Du(t)$$

where

- $u(t) \in \mathbb{C}^m$ is the control input
- $y(t) \in \mathbb{C}^p$ is the measured output.

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The transfer function is denoted by

$$P(\lambda) = CR(\lambda, A)B + D.$$

The Control Problem

Problem (Robust Output Regulation)

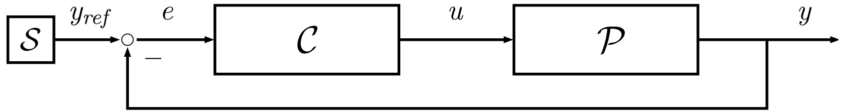
Choose a control law in such a way that

- *The output $y(t)$ tracks a given reference signal $y_{\mathit{ref}}(t)$ asymptotically, i.e.*

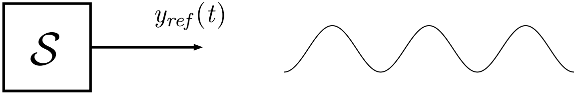
$$\lim_{t\to\infty} \|y(t) - y_{\mathit{ref}}(t)\| = 0$$

- *The above property is robust with respect to small perturbations in the operators (A, B, C, D) of the plant.*

The Exosystem and the Control Scheme



The Exosystem



$$\dot{v}(t) = Sv(t), \qquad v_0 \in \mathbb{C}^q$$
$$y_{ref}(t) = Fv(t)$$

S is a diagonal matrix

$$S = \begin{pmatrix} i\omega_1 & & & \\ & i\omega_2 & & \\ & & \ddots & \\ & & & i\omega_q \end{pmatrix}$$

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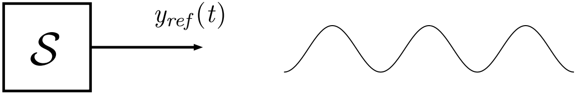

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The eigenvalues $i\omega_k \in i\mathbb{R}$ of S determine the frequencies ω_k in the reference signals $y_{ref}(t)$.

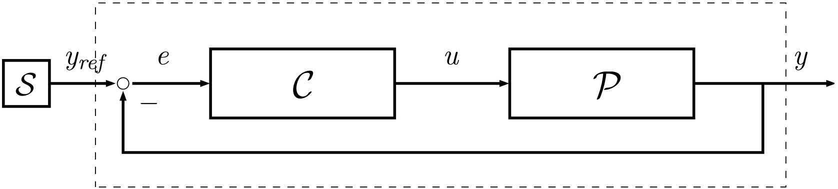
The Dynamic Error Feedback Controller

We consider an error feedback controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ of the form

$$\begin{aligned} \dot{z}(t) &= \mathcal{G}_1 z(t) + \mathcal{G}_2(y(t) - y_{ref}(t)), \qquad z(0) = z_0 \in Z \\ u(t) &= Kz(t) \end{aligned}$$

Feedback controllers are known to be essential in achieving robustness.

The Closed-Loop System



Closed-loop system

"Closed-loop stability" means that *without input, the states of the plant and the controller decay to zero asymptotically*.

The Internal Model Principle

Theorem (Francis & Wonham, 1970's, LP & SP 2010)

A stabilizing feedback controller solves the robust output regulation problem if and only if it contains p copies of the dynamics of the signal generator.

Here $p = \dim Y$, the number of outputs.

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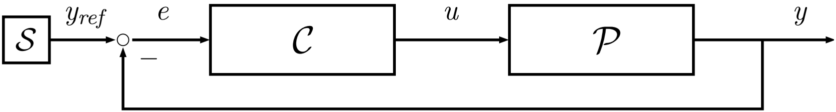
The p-copy for an exosystem with $S = \mathrm{diag}(i\omega_1, \ldots, i\omega_q)$:

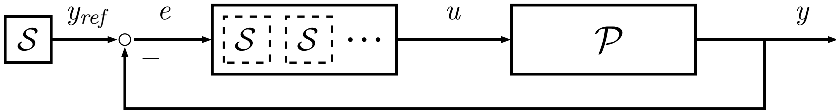
Any eigenvalue $i\omega_k$ of S must be an eigenvalue of $\mathcal{G}_1$ with p linearly independent eigenvectors associated to it, i.e.,

$$\dim \mathcal{N}(i\omega_k - \mathcal{G}_1) \geq p.$$

Feedback Controller



The p-Copy Internal Model Principle



Remarks on the Internal Model Principle

Remark

The proof of the Internal Model Principle is largely based on requiring robustness with respect to perturbations to the output operators of the exosystem.

Allowing such perturbations is often unnecessary (in particular, if reference signals are known accurately).

Motivates the study of robustness with respect to “smaller” classes of perturbations, and for individual perturbations.

Basic Assumptions on the Perturbations

Denote by $\mathcal{O}$ the class of all admissible perturbations of the plant:

$$(A, B, C, D) \quad \longrightarrow \quad (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) \in \mathcal{O}.$$

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$$(A, B, C, D) \quad \longrightarrow \quad (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) \in \mathcal{O}.$$

The perturbations in $\mathcal{O}$ are assumed be "small" so that

- The perturbed closed-loop system is exponentially stable
- The eigenvalues $\{i\omega_k\}$ of the exosystem satisfy $i\omega_k \in \rho(\tilde{A})$.

Denote

$$\tilde{P}(\lambda) = \tilde{C}R(\lambda, \tilde{A})\tilde{B} + \tilde{D}.$$

Aim

Problem (Robust Output Regulation)

The controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ *is such that*

- *The output* $y(t)$ *tracks the reference signal* $y_{ref}(t)$, *i.e.*

$$\lim_{t\to\infty} \|y(t) - y_{ref}(t)\| = 0 \qquad (1)$$

- *If the operators of the plant are changed s.t.*

$$(A, B, C, D) \quad \longrightarrow \quad (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) \in \mathcal{O},$$

the property (1) *is still true.*

If the second part is true for some $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$, we say that the controller is *robust* w.r.t. to these perturbations.

Testing Robustness for Perturbations in $\mathcal{O}$

Theorem

A stabilizing controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ *is robust with respect to given perturbations* $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) \in \mathcal{O}$ *if and only if the equations*

$$\tilde{P}(i\omega_k)Kz^k = -Fe_k$$

$$(i\omega_k - \mathcal{G}_1)z^k = 0$$

have a solution $z^k \in \mathcal{D}(\mathcal{G}_1)$ *for all* $k \in \{1, \ldots, q\}$.

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Here: e_k is an Euclidean basis vector, F is the output operator of the exosystem, $\mathcal{G}_1$ is the system operator and K the output operator of the controller.

Theorem

The controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ *is robust w.r.t* $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) \in \mathcal{O}$ *iff*

$$\tilde{P}(i\omega_k) K z^k = -F e_k \tag{2a}$$

$$(i\omega_k - \mathcal{G}_1) z^k = 0 \tag{2b}$$

have a solution $z^k \in \mathcal{D}(\mathcal{G}_1)$ *for all* $k \in \{1, \dots, q\}$.

The perturbations are only visible through the change of the transfer function at the frequencies $i\omega_k$

$$P(i\omega_k) \quad \longrightarrow \quad \tilde{P}(i\omega_k)$$

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We have robustness in particular if the perturbations do not change the value of $P(\lambda)$ at the frequencies $\lambda = i\omega_k$.

Robustness w.r.t. a Restricted Class of Perturbations

Problem

If we are only interested in robustness with respect to a specific class of perturbations, we can then ask

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If we are only interested in robustness with respect to a specific class of perturbations, we can then ask

how big an internal model do we need?

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number of copies of $i\omega_k$ in the controller
$\longleftrightarrow$ # of lin. indep't eigenvectors of $\mathcal{G}_1$ corresponding to $i\omega_k$
$\longleftrightarrow$ $\dim \mathcal{N}(i\omega_k - \mathcal{G}_1)$

Theorem

The controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ *is robust w.r.t* $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) \in \mathcal{O}$ *iff*

$$\tilde{P}(i\omega_k)Kz^k = -Fe_k$$

$$(i\omega_k - \mathcal{G}_1)z^k = 0$$

have a solution $z^k \in \mathcal{D}(\mathcal{G}_1)$ *for all* $k \in \{1, \ldots, q\}$.

For a fixed k the theorem implies that

$$\dim \mathcal{N}(i\omega_k - \mathcal{G}_1)$$

must be at least the number of linearly independent solutions z^k corresponding to different perturbations.

Lower Bound for the Size of the Internal Model

Especially easy, if $\dim U = \dim Y = p$ and $\tilde{P}(i\omega_k)$ are invertible:

Theorem

Define

$$\tilde{p}_k = \dim \operatorname{span}\{\, \tilde{P}(i\omega_k)^{-1} F e_k \mid (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) \in \mathcal{O}_0 \,\},$$

where $\mathcal{O}_0 \subset \mathcal{O}$. Then robustness w.r.t. perturbations in $\mathcal{O}_0$ implies

$$\dim \mathcal{N}(i\omega_k - \mathcal{G}_1) \geq \tilde{p}_k.$$

Classes Requiring a Full Internal Model

A *full* internal model is necessary for robustness with respect to all small perturbations in any one of the operators.

Theorem

If the control law is robust with respect to all small rank one perturbations in any one of the operators A, B, C, or D of the plant, then the controller necessarily incorporates a p-copy internal model of the exosystem.

A Quick Recap

So far in theory. . .

- A method for testing robustness with respect to given perturbations
- A lower bound for the size of the internal model
- Some “small” classes of perturbations that require a full internal model

Example: A MIMO Wave Equation

Set-point regulation ($y_{ref}(t) \equiv y_r \in \mathbb{C}^p$ constant, $p > 1$) for

$$\frac{\partial^2 w}{\partial t^2}(z,t) - \alpha \frac{\partial w}{\partial t}(z,t) = \frac{\partial^2 w}{\partial z^2}(z,t) + Bu(t)$$

$$y(t) = Cw(\cdot,t).$$

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We can build a 1-dimensional controller that is robust with respect to all sufficiently small perturbations in α.

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Example

We can build a 1-dimensional controller that is robust with respect to all sufficiently small perturbations in α.

Key: Exosystem has $i\omega_0 = 0$, and for perturbations in α we have $\tilde{P}(0) = P(0)$. Thus one copy of the exosystem is sufficient for robustness.

References

LP & S. Pohjolainen - *Reduced order internal models in robust output regulation*, Transactions on Automatic Control, to appear.

LP & S. Pohjolainen - *The internal model principle for systems with unbounded control and observation*, submitted.

Extensions

The results in this presentation are also valid for

- Non-diagonal exosystems (i.e., S has Jordan blocks)
- Infinite-dimensional exosystems (nonsmooth reference signals)
- Infinite-dimensional plants with unbounded control and observation.

Conclusions

In this presentation.

- Robust output regulation with restricted classes of perturbations.
- A method for testing robustness with respect to given perturbations.
- Some small classes of perturbations requiring a full internal model.