Robust Regulation for Port-Hamiltonian Systems

Jukka-Pekka Humaloja, Lassi Paunonen and Seppo Pohjolainen

Tampere University of Technology

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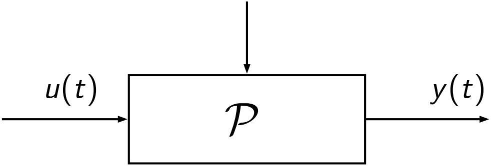
Outline

The Classes of Systems



Consider an infinite-dimensional linear system with *input* $u(t)$, *output* $y(t)$.

In particular, we may consider

- Systems with bounded input and output operators
- Regular linear systems
- Boundary control systems
- **Port–Hamiltonian systems**

Goals and Main Results

- Consider robust output tracking for port–Hamiltonian systems of first, second and even order
- Introduce a simple controller structure for impedance energy preserving port–Hamiltonian systems.
  - The systems in our class are unstable, and existing controllers would require observer design.

Port–Hamiltonian Systems in Brief

- The class of port-Hamiltonian systems covers Hamiltonian PDEs with 1 spatial dimension. Interacting with the environment via the boundaries of the spatial domain.
- Examples of port-Hamiltonian systems are, i.a., wave equation, Timoshenko beam (first order systems), Euler–Bernoulli beam and Schrödinger equation (second order systems).
- Enables a natural expressions for energy and change of energy of the system in terms of inputs and outputs.

An Example: A Schrödinger Equation

A Schrödinger equation on the spatial interval $\zeta \in [0,1]$:

$$\frac{\partial}{\partial t} w(\zeta,t) = i \frac{\partial^2}{\partial \zeta^2} w(\zeta,t), \quad t \geq 0$$

is a (second-order) port-Hamiltonian system

We can consider inputs and outputs (more precise conditions later)

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} x'(0,t) \\ x(1,t) \end{bmatrix},$$
$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} ix(0,t) \\ ix'(1,t) \end{bmatrix}.$$

Another Example: A 1D Wave Equation

A model for a vibrating string on the spatial interval $\zeta \in [0,1]$

$$\frac{\partial^2}{\partial t^2} p(\zeta,t) = \frac{\partial^2}{\partial \zeta^2} p(\zeta,t),$$

can be written as a first-order port-Hamiltonian system

$$\frac{\partial}{\partial t} \begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial \zeta} \begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t) \end{bmatrix},$$

where $x_1(\zeta,t) = \partial_t p(\zeta,t)$ and $x_2(\zeta,t) = \partial_\zeta p(\zeta,t)$.

We can consider inputs and outputs

$$u(t) = \begin{bmatrix} x_2(1,t) \\ x_1(0,t) \end{bmatrix}, \qquad y(t) = \begin{bmatrix} x_1(1,t) \\ -x_2(0,t) \end{bmatrix}.$$

The Control Problem

Problem (Robust Output Regulation)

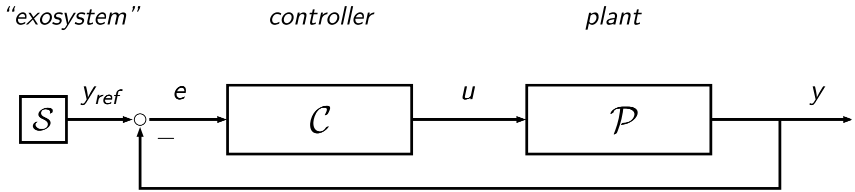
Choose a control law in such a way that

- *The output $y(t)$ tracks a given reference signal $y_{ref}(t)$ asymptotically, i.e.*

$$\lim_{t\to\infty} \|y(t) - y_{ref}(t)\| = 0$$

- *The above property is robust with respect to small perturbations in the parameters of the plant.*

The Control Scheme



The Internal Model Principle

Theorem (Francis & Wonham, 1970's, LP & SP 2010)

A stabilizing feedback controller solves the robust output regulation problem if and only if it contains p copies of the dynamics of the exosystem.

Here $p = \dim Y$, the number of outputs.

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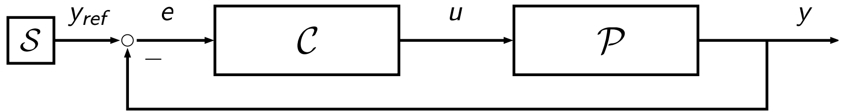
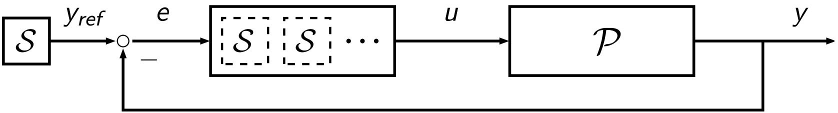
The p-copy for bounded $y_{ref}(t) = \sum_{k=1}^{N} a_k e^{i\omega_k t}$:

> *Any frequency $i\omega_k$ of the reference signal must be an eigenvalue of the controller with p linearly independent eigenvectors associated to it.*

Feedback Controller



The p-Copy Internal Model Principle



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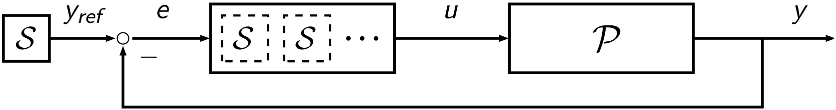

The internal model principle for classes of systems:

- Bounded input and output operators (2010)
- Regular linear system with distributed disturbances (2014)
- Regular linear systems with boundary disturbance (2016)
- Boundary control systems (2016, (2002))

Port-Hamiltonian Systems

A linear port-Hamiltonian system of order $N = 1$ or $N \in 2\mathbb{N}$ on the spatial interval $\zeta \in [a, b]$ is given by

$$\begin{aligned} \frac{\partial}{\partial t} x(\zeta, t) &= \mathcal{A} x(\zeta, t), \quad x(0) = x_0, \\ u(t) &= \mathcal{B} x(\cdot, t), \\ y(t) &= \mathcal{C} x(\cdot, t), \end{aligned}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A} x(\zeta, t) := \sum_{k=0}^{N} P_k \frac{\partial^k (\mathcal{H}(\zeta) x(t, \zeta))}{\partial \zeta^k},$$

where $P_k \in \mathbb{C}^{n \times n}$, $P_k^* = (-1)^{k+1} P_k$ for $k \geq 0$, and P_N invertible.

Port-Hamiltonian Systems

- The matrix function $\mathcal{H} : [a, b] \to \mathbb{C}^{n\times n}$ is measurable, and there exists $0 < m \leq M$ such that $m|\zeta|^2 \leq \zeta^* \mathcal{H}(\zeta)\zeta \leq M|\zeta|^2$ and $\mathcal{H}(\zeta) = \mathcal{H}(\zeta)^*$ for $\zeta \in \mathbb{C}^n$.
- We choose state space $X = L^2([a, b]; \mathbb{C}^n)$ with inner product

$$\langle f, g\rangle_X = \frac{1}{2}\int_a^b g(\zeta)^* \mathcal{H}(\zeta) f(\zeta) d\zeta.$$

- The domain of the operator $\mathcal{A}$ is given by

$$\mathcal{D}(\mathcal{A}) = \{x \in X \mid \mathcal{H}x \in H^N([a, b]; \mathbb{C}^n)\}$$

- The Hamiltonian of the port-Hamiltonian system is

$$E(t) = \langle x(t), x(t)\rangle_X = ||x(t)||_X^2.$$

Port-Hamiltonian Systems

- Let

$$\Phi : H^N([a,b];\mathbb{C}^n) \to \mathbb{C}^{2nN},$$
$$\Phi(x) := (x(b),\dots,x^{(N-1)}(b),x(a),\dots,x^{(N-1)}(a))^T.$$

- Define the boundary port variables f_∂, e_∂ by

$$\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} := \frac{1}{\sqrt{2}} \begin{bmatrix} Q & -Q \\ I & I \end{bmatrix} \Phi(\mathcal{H}x),$$

where Q is a block matrix given by

$$Q_{ij} := \begin{cases} (-1)^{j-1} P_{i+j-1}, & i+j \leq N+1 \\ 0, & \text{else} \end{cases}.$$

Port-Hamiltonian Systems

- Define inputs and outputs $\mathcal{B} : \mathcal{H}^{-1}(H^N([a,b],\mathbb{C}^n)) \to \mathbb{C}^n$ and $\mathcal{C} : \mathcal{H}^{-1}(H^N([a,b],\mathbb{C}^n)) \to \mathbb{C}^n$ by

$$u(t) = \mathcal{B}x(t) := W_B \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix},$$

$$y(t) = \mathcal{C}x(t) := W_C \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix},$$

  where $W_B, W_C \in \mathbb{C}^{nN\times 2nN}$.
- Main idea: $e_\partial(t)$ and $f_\partial(t)$ are linear combinations of $(\mathcal{H}x)(\cdot)$ and its $N-1$ derivatives evaluated at the boundaries $\zeta = a$ and $\zeta = b$.
- Port–Hamiltonian system are in particular boundary control systems whenever $W_B \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right) W_B^* \geq 0$ (Le Gorrec et al. 2005)

Background on Controller Design

- Simple internal model based controllers available for **stable** boundary control systems (Hämäläinen & Pohjolainen, 2002)
- Many interesting port-Hamiltonian system are **unstable**, but can be stabilized with negative output feedback (and the closed-loop systems are port–Hamiltonian).

The general approach is to combine the feedback stabilization and simple controller design.

Assumptions

Consider a port–Hamiltonian system

$$\begin{aligned} \dot{x}(t) &= \mathcal{A}x(t), \qquad x(0) = x_0, \\ \mathcal{B}x(t) &= u(t), \\ \mathcal{C}x(t) &= y(t), \end{aligned}$$

that is *impedance energy-preserving*, meaning that

$$\frac{1}{2}\frac{d}{dt}||x(t)||_X^2 = u^*(t)y(t).$$

- Plant is a boundary control system with a unitary group.
- Characterisation via matrices W_B and W_C.
- Denote transfer function by $P(\lambda)$.

Results on Stabilization: First Order

Theorem (Villegas et. al. 2005)

*A **first order** impedance energy preserving port–Hamiltonian system can be stabilized exponentially with negative output feedback $u(t) = -\kappa y(t)$.*

Results on Stabilization: Even Order

Lemma

*An **even order** impedance energy preserving port–Hamiltonian system can be stabilized exponentially with negative output feedback $u(t) = -\kappa y(t)$.*

Proof.

Using a result by Augner & Jacob, 2014, that the system is exponentially stable if

$$\operatorname{Re}\langle Ax, x\rangle_X \leq -\gamma \sum_{\zeta=a,b} \sum_{k=0}^{N-1} \alpha_{\zeta,k} \left\| (\mathcal{H}x)^{(k)}(\zeta) \right\|^2$$

for some $\gamma > 0$ and certain $\alpha_{\zeta,k} \geq 0$. □

The Robust Output Regulation Problem

Problem (The Robust Output Regulation Problem)

Choose $(\mathcal{G}_1, \mathcal{G}_2, \varepsilon K_0, \kappa)$ in such a way that

1. *The closed loop system is exponentially stable.*
2. *The controller asymptotically tracks the reference signal y_{ref},*

$$\|y(t) - y_{ref}(t)\| \to 0, \qquad \text{as} \quad t \to \infty \tag{1}$$

   at an exponential rate.
3. *If $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ are perturbed to $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}})$ in such a way that the closed-loop stability is preserved, then (1) continues to hold.*

$$y_{ref}(t) = \sum_{k=1}^{q} a_k e^{i\omega_k t}, \qquad a_k \in \mathbb{C}^p.$$

Stabilizing Output Feedback + Robust Controller

Consider a a dynamic error feedback controller of the form

$$\dot{z}(t) = \mathcal{G}_1 z(t) + \mathcal{G}_2(y(t) - y_{ref}(t)), \qquad z(0) = z_0,$$
$$u(t) = \varepsilon K_0 z(t) - \kappa y(t),$$

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$$\mathcal{G}_1 = \begin{bmatrix} i\omega_1 I_{p\times p} & & \\ & \ddots & \\ & & i\omega_q I_{p\times p} \end{bmatrix}, \qquad \begin{aligned} K_0 &= \left[K_0^1, K_0^2, \ldots, K_0^q\right] \\ \mathcal{G}_2 &= -\left[(P_\kappa(i\omega_k)K_0^k)^*\right]_{k=1}^q \end{aligned}$$

$P_\kappa(i\omega_k) = P(i\omega_k)(I + \kappa P(i\omega_k))^{-1}$, $P(i\omega_k)K_0^k$ invertible, $\varepsilon, \kappa > 0$.

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Theorem

For every $\kappa > 0$ there exists $\varepsilon_\kappa > 0$ such that for all $0 < \varepsilon \leq \varepsilon_\kappa$ the controller solves the robust output regulation problem.

An Example: A Schrödinger Equation

- The Schrödinger equation on the interval $\zeta \in [0,1]$:

$$\frac{\partial}{\partial t} w(\zeta,t) = i \frac{\partial^2}{\partial \zeta^2} w(\zeta,t), \quad t \geq 0$$

is a second-order impedance energy preserving port-Hamiltonian system for inputs and outputs

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} x'(0,t) \\ x(1,t) \end{bmatrix},$$

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} ix(0,t) \\ ix'(1,t) \end{bmatrix}.$$

Example: Schrödinger Equation

Consider reference signals of the form

$$y_{ref}(t) = \sum_{k=-N}^{N} a_k e^{ikt}, \qquad a_k \in \mathbb{C}^2, \quad 2\pi\text{-periodic.}$$

Then the controller parameters $(\mathcal{G}_1, \mathcal{G}_2, K, \kappa)$ are chosen as

$$\mathcal{G}_1 = \begin{bmatrix} -iN & & & & \\ & -iN & & & \\ & & \ddots & & \\ & & & iN & \\ & & & & iN \end{bmatrix} \qquad \begin{aligned} K &= \varepsilon \left[P_\kappa(-iN)^{-1}, \ldots P_\kappa(iN)^{-1} \right] \\ \mathcal{G}_2 &= -\Big[I_{2\times 2} \Big]_{k=1}^{q} \end{aligned}$$

where $P_\kappa(ik)$ can be computed explicitly, or measured from the system's response.

Conclusions

- We presented a simple robust regulating controller for an unstable, impedance energy-preserving port-Hamiltonian system of even order.
- Output feedback was added to the usual controller structure in order to exponentially stabilize the plant.
- When the plant was exponentially stabilized, we could utilize the robust output regulation results for exponentially stable systems in choosing the controller parameters.

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