Asymptotic Behaviour of Platoon Systems

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Mathematical Theory of Networks and Systems
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An Infinite Vehicle Platoon

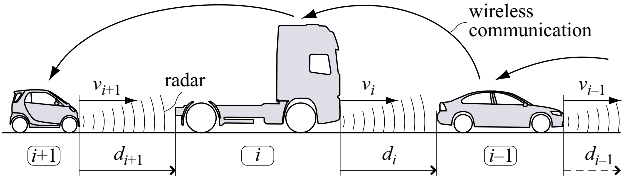
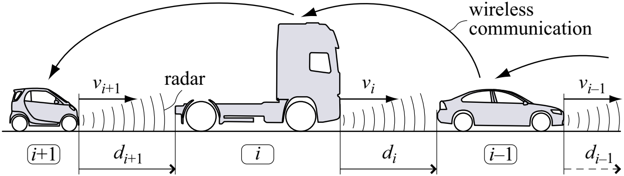


Figure: Source: Ploeg *et. al.*, IEEE, 2011.

An Infinite Vehicle Platoon



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An Infinite Vehicle Platoon

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$y_k(t) =$ displacement from ideal distance between k and $k-1$

$w_k(t) =$ velocity of kth vehicle (displacement from ideal)

$a_k(t) =$ acceleration of kth vehicle

Objective: Choose $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{R}$ so that $\sup_{k \in \mathbb{Z}} |y_k| \to 0$ as $t \to \infty$.

Aims and Main Results

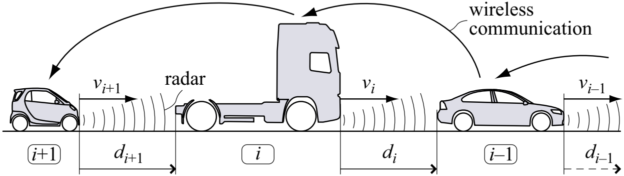

Figure: Source: Ploeg *et. al.*, IEEE, 2011.

We analyze convergence of the displacements to ideal distances in three different scenarios:

(1) Control employs state feedback (original situation).

(2) Control of vehicles require observer design.

(3) “Constant headway time” spacing, where ideal distance depends on velocity.

Structure

- Part I: Platoon Systems with State Feedback
  - Strong asymptotic convergence
  - "Nonuniform" subexponential rates of convergence
- Part II: Infinite Systems with Observers
  - Demonstrate that stability is unachievable
- Part III: Constant Headway Spacing Policy
  - Improved stability properties and simplified analysis
  - Subexponential rates of convergence

Part I:

Platoon Systems with State Feedback

Main Problem

Study asymptotics of infinite systems of the form

$$\dot{x}_k(t) = A_0 x_k(t) + A_1 x_{k-1}(t), \quad k \in \mathbb{Z},\ t \geq 0,$$

where $A_0, A_1 \in \mathbb{C}^{m \times m}$ do not depend on $k \in \mathbb{Z}$.

We want to study, e.g.,

$$\sup_{k \in \mathbb{Z}} \|x_k(t) - y_k\|_{\mathbb{C}^m} \to 0, \qquad \text{as} \quad t \to \infty$$

with rates.

Platoon Systems

Our system can be formulated as an abstract Cauchy problem

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0 \in X$$

on $X = \ell^p(\mathbb{C}^m)$ for $1 \leq p \leq \infty$ by choosing $x(t) = (x_k(t))_{k \in \mathbb{Z}}$ and

$$Ax = (A_0 x_k + A_1 x_{k-1})_{k \in \mathbb{Z}}.$$

i.e.

$$A = \begin{pmatrix} \ddots & \ddots & & & \\ & A_1 & A_0 & & \\ & & A_1 & A_0 & \\ & & & \ddots & \ddots \end{pmatrix}$$

Here $A \in \mathcal{L}(X)$ and our system belongs to the class of "Spatially invariant systems" (Bamieh et. al. and others).

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$$Ax = (A_0 x_k + A_1 x_{k-1})_{k\in\mathbb{Z}}.$$

The operator $A \in \mathcal{L}(X)$ generates a strongly continuous semigroup $T(t)$ (i.e., $T(t) = e^{At}$), and the solutions of the system are given by

$$x(t) = (x_k(t))_{k\in\mathbb{Z}} = T(t)x_0$$

Semigroup $T(t)$ does not admit a simple expression, but can be used in the analysis.

Earlier Work

Platoon systems have been studied extensively in the literature:

- Jovanovic & Bamieh 2005: Exponential stability is unachievable.
- Curtain, Iftime & Zwart 2009: Strong stability possible for ℓ^2 (Fourier methods).
- No analysis of convergence rates.
- Also analysis of so-called **string stability** by Swaroop & Hedrick (1996) and many others.

Our work: Analysis of stability and convergence rates for all ℓ^p, $1 \leq p \leq \infty$ (with emphasis on $p = \infty$) using semigroup methods.

The Characteristic Function

Assumption

Assume $A_1 \neq 0$, $\sigma(A_0) \subset \mathbb{C}_-$, and there exists $\phi : \mathbb{C} \to \mathbb{C}$ s.t.

$$A_1(\lambda - A_0)^{-1} A_1 = \phi(\lambda) A_1, \qquad \lambda \in \mathbb{C} \setminus \sigma(A_0).$$

$\phi(\cdot)$ is called *characteristic function* of the infinite system.

Lemma

Assumption holds whenever $\operatorname{rank} A_1 = 1$. *For the platoon system*

$$\phi(\lambda) = \frac{\alpha_0}{p(\lambda)} = \frac{\alpha_0}{\lambda^3 + \alpha_2 \lambda^2 + \alpha_1 \lambda + \alpha_0}.$$

Main Results

We will do the following: For solutions $x(t) = T(t)x_0$

(i) Characterize spectrum of A

(ii) Present conditions for boundedness $\sup_{t\geq 0}\|T(t)\| < \infty$

(iii) Study rates of convergence of $\|T(t)x_0 - y\| \to 0$ as $t \to \infty$.

For all these purposes use the characteristic function $\phi(\cdot)$:

$$A_1(\lambda - A_0)^{-1}A_1 = \phi(\lambda)A_1, \qquad \lambda \in \mathbb{C} \setminus \sigma(A_0).$$

Main idea: Existence of $\phi(\cdot)$ compensates for the lack of commutativity of A_0 and A_1.

Spectrum of the System

Characteristic function $\phi(\cdot)$ determines the spectrum of A:

Theorem

Let $X = \ell^p(\mathbb{C}^m)$ *with* $1 \le p \le \infty$. *Then for* $\lambda \in \mathbb{C} \setminus \sigma(A_0)$

$$\lambda \in \sigma(A) \qquad \text{if and only if} \qquad |\phi(\lambda)| = 1.$$

Moreover, $\sigma(A) \setminus \sigma(A_0)$ *is*

- *point spectrum if and only if* $p = \infty$
- *continuous spectrum if and only if* $1 < p < \infty$.

The type of spectrum depends on p, but the location does not.

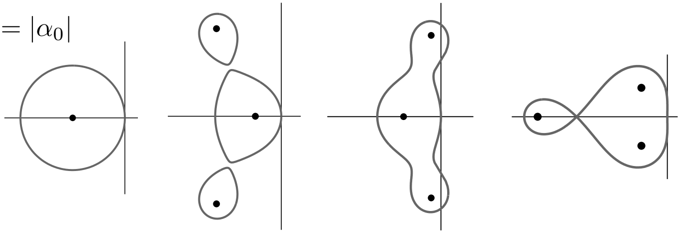
Spectrum of the Platoon System

$$\begin{pmatrix} \dot{y}_k \\ \dot{w}_k \\ \dot{a}_k \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 \end{pmatrix} \begin{pmatrix} y_k \\ w_k \\ a_k \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{k-1} \\ w_{k-1} \\ a_{k-1} \end{pmatrix}$$

Characteristic function: $\phi(\lambda) = \frac{\alpha_0}{p(\lambda)} = \frac{\alpha_0}{\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0}$.

Spectrum of the platoon system is determined by α_0, α_1, and α_2.

$|p(\lambda)| = |\alpha_0|$



Uniform Boundedness of the Semigroup

Theorem

Let $1 \le p \le \infty$. *If* $\sigma(A) \subset \mathbb{C}_- \cup \{0\}$,

$$\sup_{0<\lambda\le 1} \frac{\lambda}{1-|\phi(\lambda)|} < \infty \quad \textit{and} \quad \sup_{n\in\mathbb{N}} \sup_{\lambda>0} \frac{\lambda^{n+1}}{n!} \sum_{\ell=1}^{\infty} \left| \frac{d^n}{d\lambda^n} \phi(\lambda)^\ell \right| < \infty,$$

then the semigroup $T(t)$ *generated by* A *is uniformly bounded.*

Proof.

A fairly direct Hille–Yosida approach using a resolvent formula. □

Property: Systems for $m \ge 2$ are typically not contractive. In particular, the platoon system is <u>never</u> contractive.

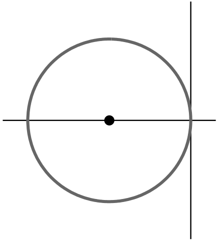
Uniform Boundedness for the Platoon System

Lemma

If $\phi(\cdot)$ is such that for some $\zeta > 0$,

$$\phi(\lambda) = \frac{\zeta^3}{(\lambda+\zeta)^3}, \qquad \lambda \neq -\zeta$$

then the semigroup $T(t)$ is uniformly bounded.



The characteristic function of the platoon system is of this form if parameters $\alpha_0, \alpha_1, \alpha_2$ are chosen so that $\sigma(A_0) = \{-\zeta\}$.

Then the platoon system is guaranteed to be uniformly bounded.

(Unquantified) Asymptotic Behaviour

Combining the results on spectrum and uniform boundedness:

Theorem

Let $X = \ell^p(\mathbb{C}^m)$ *with* $p = \infty$ *and for some* $\zeta > 0$

$$\phi(\lambda) = \frac{\zeta^3}{(\lambda+\zeta)^3}, \qquad \lambda \neq -\zeta.$$

If $x = x_0 + x_1 \in \overline{\operatorname{Ran}(A)} \oplus \operatorname{Ker}(A) \neq X$, *then*

$$T(t)x \to x_1 \qquad \text{as} \qquad t \to \infty.$$

Moreover,

- If $1 < p < \infty$, then $T(t)$ is strongly stable, i.e., $T(t)x \to 0$

The Null Space $\mathrm{Ker}(A)$ for Platoons

We can show that if $\alpha_0, \alpha_1, \alpha_2$ are chosen such that $\sigma(A_0) = \{-\zeta\}$ for $\zeta > 0$, then $x \in \mathrm{Ker}(A)$ if and only if

$$x = \left(\dots, \begin{pmatrix} c \\ -\zeta c/3 \\ 0 \end{pmatrix}, \begin{pmatrix} c \\ -\zeta c/3 \\ 0 \end{pmatrix}, \begin{pmatrix} c \\ -\zeta c/3 \\ 0 \end{pmatrix}, \dots \right).$$

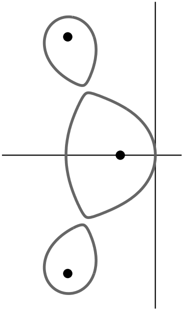
for some $c \in \mathbb{C}$.

Rates of Convergence

Next aim: Find rates of convergence for

$$\|T(t)x - y\| \to 0 \qquad \text{as} \quad t \to \infty$$

under the assumption $\sigma(A) \cap i\mathbb{R} = \{0\}$.



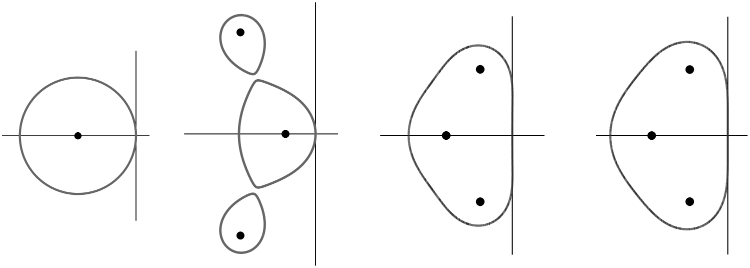
Key points:

- Martinez 2011: Convergence rate if $\|(is - A)^{-1}\| \leq M(1 + |s|^{-\alpha})$ near $s = 0$.
- For platoon systems

$$\|(is - A)^{-1}\| \sim \frac{1}{1 - |\phi(is)|} \sim \frac{1}{\mathrm{dist}(is, \sigma(A))}$$

Decay Rates for the Platoon System

Platoons: The possible growth rates are $|s|^{-n_\phi}$ with $n_\phi \in \{2, 4, 6\}$.



Corresponding rates are $\left(\frac{\log t}{t}\right)^{-\frac{1}{2}}$, $\left(\frac{\log t}{t}\right)^{-\frac{1}{4}}$ and $\left(\frac{\log t}{t}\right)^{-\frac{1}{6}}$ (though uniform boundedness was just shown for the first case).

Quantified Decay for the Platoon System

$$\begin{pmatrix} \dot{y}_k \\ \dot{w}_k \\ \dot{a}_k \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\zeta^3 & -3\zeta^2 & -3\zeta \end{pmatrix} \begin{pmatrix} y_k \\ w_k \\ a_k \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{k-1} \\ w_{k-1} \\ a_{k-1} \end{pmatrix}$$

Theorem

Let $X = \ell^\infty(\mathbb{C}^3)$. *If there exists* $c \in \mathbb{R}$

$$\sup_{k\in\mathbb{Z}} \left| c - \frac{1}{n} \sum_{j=1}^{n} y_{k-j}(0) \right| = O\left(\frac{1}{n}\right) \quad \textit{as} \quad n \to \infty,$$

then

$$\|T(t)x - x_1\| = O\left(\frac{1}{\sqrt{t}}\right)$$

where again $x_1 = ((c, -\zeta c/3, 0)^T)_{k\in\mathbb{Z}}$

Part II:

Platoon Systems with Observers

Long Story Short...

Theorem

If the control employs identical observers in all vehicles, the system is always unstable.

The main idea here is to demonstrate how fragile the stability of the platoon system can be.

Background

Consider

$$\begin{pmatrix}\dot{y}_k\\ \dot{w}_k\\ \dot{a}_k\end{pmatrix} = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & -1/\tau\end{pmatrix}\begin{pmatrix}y_k\\ w_k\\ a_k\end{pmatrix} + \begin{pmatrix}0 & -1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}y_{k-1}\\ w_{k-1}\\ a_{k-1}\end{pmatrix} + \begin{pmatrix}0\\ 0\\ 1\end{pmatrix} u_k(t)$$

with observation $y_k(t) = (1,0,0)x_k(t) = C_0 x_k(t)$.

For each $k \in \mathbb{Z}$, add a Luenberger observer to estimate $x_k(t)$:

$$\begin{aligned}\dot{z}_k(t) &= (A_0 + LC_0)z_k(t) + B_0 u_k(t) - L y_k(t),\\ u_k(t) &= K z_k(t)\end{aligned}$$

where $\sigma(A + B_0 K) \subset \mathbb{C}_-$ and $\sigma(A_0 + LC_0) \subset \mathbb{C}_-$.

The System of Closed-Loop Systems

The full system is of the form

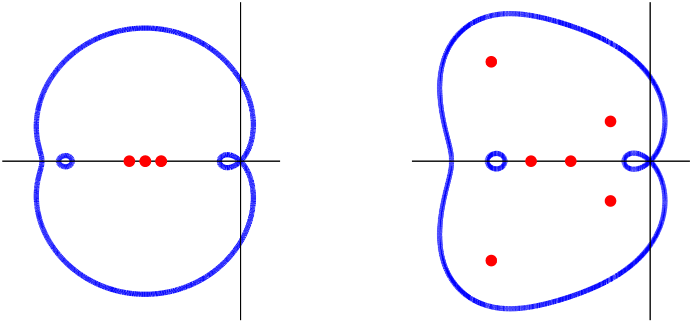
$$\begin{pmatrix} \dot{x}_k(t) \\ \dot{z}_k(t) \end{pmatrix} = A_0^e \begin{pmatrix} x_k(t) \\ z_k(t) \end{pmatrix} + A_1^e \begin{pmatrix} x_{k-1}(t) \\ z_{k-1}(t) \end{pmatrix}$$

where the pair (A_0^e, A_1^e) admits a characteristic function satisfying

$$A_1 R(\lambda, A + B_0 K)(\lambda - A_0 - B_0 K - LC_0) R(\lambda, A + LC_0) A_1 = \phi(\lambda) A_1$$

$\Rightarrow$ The same approach is applicable.

Spectrum of the Full System



Phenomenon: The “knot” at the origin makes the system unstable.

Part III:

"Constant Headway Time" Spacing

Constant Headway Time Spacing

Alternatively, we can consider a control objective, where the **ideal distance between vehicles depends on the velocity**. The idea is that we impose an ideal distance in **seconds** instead of meters.

The ideal separation is then the “constant headway time”.

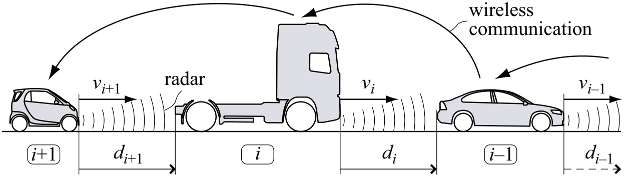


Figure: Source: Ploeg *et. al.*, IEEE, 2011.

Constant Headway Time Spacing

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Motivation:

- Constant headway time spacing has been observed to improve the string stability of the platoon system [Ploeg et. al. 2011].
- We show that it also leads to better stability properties of the semigroup.

The Full System

For ideal separation of form $c + hv_k(t)$, the system beocmes

$$\dot{x}(t) = A_0 x_k(t) + A_1 x_{k-1}(t),$$

with

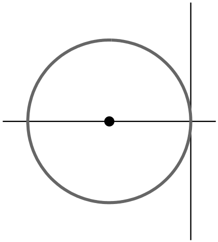
$$A_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta_0 & -\beta_1 & -\beta_2 - 1 & 0 \\ \beta_0/h & \beta_1/h & \beta_2/h & -1/h \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/h \end{pmatrix}.$$

where $\beta_0, \beta_1, \beta_2 \in \mathbb{R}$ are parameters of the feedback law.

The Characteristic Function

Property: For **any** $\beta_0, \beta_1, \beta_2 \in \mathbb{R}$ we have

$$\phi(\lambda) = \frac{1}{h\lambda + 1}, \qquad \lambda \neq -\frac{1}{h}.$$



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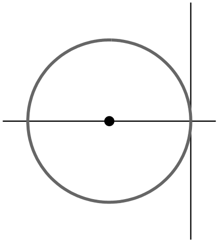

For **any** $\beta_0, \beta_1, \beta_2 \in \mathbb{R}$ we immediately get

- Stable spectrum
- Uniform boundedness
- Convergence with rate $\left(\frac{\log t}{t}\right)^{1/2}$ for $x_0 \in \mathrm{Ran}(A) \oplus \mathrm{Ker}(A)$

In the original problem this was possible only for **some** $\alpha_0, \alpha_1, \alpha_2$.

Decay for the Platoon System

Theorem

Let $X = \ell^\infty(\mathbb{C}^4)$. $T(t)x$ converges if and only if there exists $c \in \mathbb{R}$

$$\sup_{k\in\mathbb{Z}} \left| hc - \frac{1}{n}\sum_{j=1}^{n} \left[x_{k-j}^2(0) + x_{k-j}^3(0) + h x_{k-j}^4(0) \right] \right| \to 0, \quad n \to \infty,$$

and if this holds then the distances $d_k(t)$ converge as

$$\sup_{k\in\mathbb{Z}} |d_k(t) - (c + hv_k(t))| \to 0$$

and the main objective holds.

Decay for the Platoon System

Theorem

Let $X = \ell^\infty(\mathbb{C}^4)$. If there exists $c \in \mathbb{R}$ such that

$$\sup_{k\in\mathbb{Z}} \left| hc - \frac{1}{n} \sum_{j=1}^{n} \left[x^2_{k-j}(0) + x^3_{k-j}(0) + h x^4_{k-j}(0) \right] \right| = O\left(\frac{1}{n}\right)$$

then the distances $d_k(t)$ converge as

$$\sup_{k\in\mathbb{Z}} |d_k(t) - (c + h v_k(t))| = O\left(\frac{1}{\sqrt{t}}\right).$$

Conclusions

In this presentation:

- Study of the infinite platoon system using semigroup methods.
- Three variants: State feedback control, output feedback control, and constant headway time spacing.
- Study of spectrum, uniform boundedness and asymptotic convergence.

References (available at arxiv.org)

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Thank You!