On Polynomial Stability of Linear Systems

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Introduction

On a Hilbert space X, consider

$$\begin{aligned} \dot{x} &= Ax + Bu \qquad x(0) = x_0 \in X \\ y &= Cx \end{aligned}$$

with bounded B and C, and A generates a strongly stable semigroup $T(t)$.

Problem

Consider and compare stability concepts related to the system:

- $T(t)$ *not exponentially stabilizable*
- *Stability in terms of signals: Input-to-output stability*
- *Stability in the frequency domain vs. the time domain*

Motivation

In robust output regulation with nonsmooth periodic reference and disturbance signals, the closed-loop system is not exponentially stabilizable.

- *A demand for solvability conditions etc.*
- *Polynomial semigroup stability often achievable.*

References

LP & S. Pohjolainen: *The Internal Model Principle for Systems with Unbounded Control and Observation*, SICON, in review.

LP & S. Pohjolainen: *Robust Output Regulation and the Preservation of Polynomial Closed-Loop Stability*, IJRNC, 2013.

PART I: Stability types in the literature.

- Input-Output Stability, H^∞-Stability
- Strong Stability of a Linear System
- RESULTS: Conditions for I/O-stability

PART II: Newer stability types.

- P-Stability (frequency domain)
- Polynomial Input-Output Stability (time domain)
- RESULTS: Relationship between P-stability and polynomial I/O-stability

Input-Output Stability

Definition (Input-Output Map)

Consider a map $\mathbb{F}$ such that $\mathbb{F}u = y$ for $u \in L^2(0,\infty;U)$ where

$$y(t) = C\int_0^t T(t-s)Bu(s)ds$$

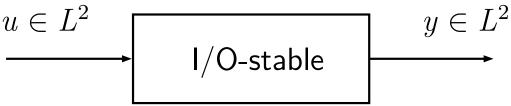
$y(t)$ is the output corresponding to $x(0) = 0$ and the input $u(t)$.

Definition (Input-Output Stability)

The linear system is *input-output stable*, if $\mathbb{F}$ exists as a bounded linear operator from $L^2(0,\infty;U)$ to $L^2(0,\infty;Y)$.

Input-Output Stability

Roughly: An L^2-input produces an L^2-output



and for some $M \geq 1$

$$\|y\|_{L^2} \leq M\|u\|_{L^2} \qquad \forall u \in L^2$$

H^∞-Stability

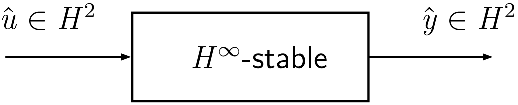
Definition

The plant is called H^∞*-stable* if its transfer function $P(\lambda) = C(\lambda - A)^{-1}B$ satisfies $P(\cdot) \in H^\infty(\mathbb{C}^+; \mathcal{L}(U, Y))$, i.e., if

$$\sup_{\operatorname{Re}\lambda > 0} \|P(\lambda)\| < \infty.$$

Recall: $T(t)$ was assumed to be strongly stable $\Rightarrow \mathbb{C}^+ \subset \rho(A)$.

I/O Stability vs. H^∞-Stability



Input-output stability corresponds to the H^∞-stability.

Theorem

The system is input-output stable if and only if it is H^∞-stable, i.e., if $P(\cdot) \in H^\infty(\mathbb{C}^+; \mathcal{L}(U, Y))$.

Strong Stability of a System

I/O-stability a part of the definition of a *strongly stable system*:

Definition

Strongly stable system =
I/O-stab. + input stab. + output stab. + semigroup stability.

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References

O. Staffans, *Quadratic optimal control of stable well-posed linear systems*, 1997.

R. Curtain and J. Oostveen, *Necessary and sufficient conditions for strong stability of distributed parameter systems*, 1999.

J. Oostveen, "Strongly stabilizable infinite-dimensional systems," Ph.D. thesis, 1999.

PART I: Conditions for Input-Output Stability

Conditions for Input-Output Stability

Problem

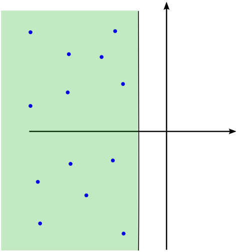
Find sufficient conditions for input-output stability of a plant.

Assumption: $T(t)$ generated by A is *polynomially stable*.

Detour: Stability of Semigroups

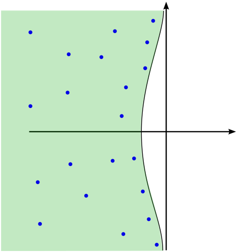
- X is Hilbert, $A : \mathcal{D}(A) \subset X \to X$
- A generates a uniformly bounded semigroup $T(t)$

Exponential:

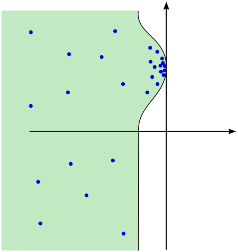


$\|T(t)\| \leq Me^{-\omega t}$

Polynomial:

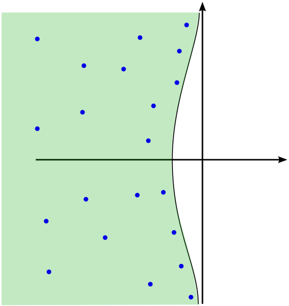


Strong:



$\|T(t)x\| \longrightarrow 0$

Polynomial Stability



Definition

$T(t)$ is called *polynomially stable* if

- $T(t)$ is uniformly bounded,
- $i\mathbb{R} \subset \rho(A)$,
- There exist $\alpha > 0$ and $M_A > 0$ s.t.

$$\| T(t)A^{-1} \| \leq \frac{M_A}{t^{1/\alpha}} \qquad \forall t > 0.$$

Consequence: If $x \in \mathcal{D}(A)$, then $\| T(t)x \| = \mathcal{O}(t^{-1/\alpha})$.

Characterization on a Hilbert Space

Theorem

If $T(t)$ is a uniformly bounded semigroup and $i\mathbb{R} \subset \rho(A)$. For a fixed $\alpha > 0$ the following are equivalent.

$$\text{(a)} \quad \|T_A(t)A^{-1}\| \leq \frac{M_A}{t^{1/\alpha}}, \qquad \forall t > 0$$

$$\text{(b)} \quad \|R(i\omega, A)\| \leq M(1 + |\omega|^\alpha)$$

Batty, Chill & Tomilov (2013), Borichev & Tomilov (2010), and Batty & Duyckaerts (2008).

Main Results: Conditions for Input-Output Stability

Question

So..? When is a system input-output stable?

Assume A is polynomially stable with $\alpha > 0$, and B and C satisfy

$$\mathcal{R}(B) \subset \mathcal{D}((-A)^n) \qquad \text{and} \qquad \mathcal{R}(C^*) \subset \mathcal{D}((-A^*)^m)$$

for some $n, m \geq 0$ (**not** necessarily integers).

"Smoothness conditions" for bounded operators B and C.

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"Smoothness conditions" for bounded operators B and C.

Theorem

If $n + m \geq \alpha$, then system is input-output stable (and H^∞-stable).

PART II: P-Stability and Polynomial I/O-Stability

P-Stability: Introduction

If $T(t)$ generated by A is polynomially stable and B and C do not possess sufficient level of smoothness, the transfer function $P(\lambda)$ may become unbounded on $i\mathbb{R}$.

Motivates defining *P-stable systems* [Laakkonen, PhD thesis 2013]. (Used in frequency domain theory for robust output regulation)

P-Stability (Frequency Domain)

Definition (P-Stability)

The plant is called *P-stable (with $\alpha > 0$)* if the following conditions are satisfied.

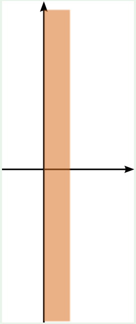
(a) $P(\cdot)$ is analytic in a domain containing $\overline{\mathbb{C}^+}$

(b) $P(\cdot) \in H^\infty(\mathbb{C}_\beta^+; \mathcal{L}(U, Y))$ for every $\beta > 0$.

(c) There exist $\varepsilon > 0$ and $M_P \geq 1$ such that

$$\|P(\lambda)\| \leq M_P(1 + |\lambda|^\alpha)$$

in a strip $0 \leq \operatorname{Re}\lambda < \varepsilon$.

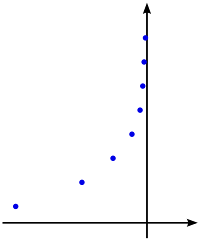


Example

The transfer function

$$P(\lambda) = \sum_{k=1}^{\infty} \frac{1}{k^{4/3}(\lambda + 1/k^2 - ik)},$$

is P-stable with $\alpha = 2/3$ (and $P(\cdot) \notin H^{\infty}$).



From a diagonal system on $X = \ell^2(\mathbb{C})$, with a polynomial semigroup:

$$A = \operatorname{diag}\Big(-\frac{1}{k^2} + ik\Big)_{k=1}^{\infty}$$

$$B = C^* = \Big(\frac{1}{k^{2/3}}\Big)_{k=1}^{\infty} \in \ell^2$$

Question:

Question

What is the time-domain stability concept corresponding to P-stability?

$$
\begin{array}{ccc}
H^\infty\text{-stability} & \longleftrightarrow & \text{input-output stability} \\
\text{P-stability} & \longleftrightarrow & ???
\end{array}
$$

Polynomial Input-Output Stability

Definition (Polynomially input-output stable system)

Assume $\alpha \in \mathbb{N}_0$. The plant is called *polynomially input-output stable (with α)* if the input-output map $y = \mathbb{F}u$ satisfies

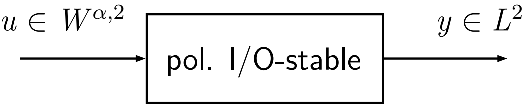
$$\mathbb{F} \in \mathcal{L}(W^{\alpha,2}(0,\infty;U), L^2(0,\infty;Y))$$

Here $W^{\alpha,2}$ is the Sobolev space of order α,

$$W^{\alpha,2} = \{ u \in L^2 \mid u^{(j)} \in L^2 \text{ for } 0 \leq j \leq \alpha \}.$$

Polynomial Input-Output Stability

Roughly: A $W^{\alpha,2}$-input produces an L^2-output



and for some $M \geq 1$

$$\|y\|_{L^2} \leq M\|u\|_{W^{\alpha,2}} \qquad \forall u \in W^{\alpha,2}$$

Main Results: P-Stability $\longleftrightarrow$ Polynomial I/O-Stablity

Theorem

Assume $\alpha \in \mathbb{N}$.

(i) *If the plant is P-stable with* α*, then it is polynomially input-output stable with* α.

I.e. (roughly)

$$\|P(\lambda)\| \leq M_P(1 + |\lambda|^\alpha), \qquad \text{near} \quad i\mathbb{R}$$

implies

$$\mathbb{F} \in \mathcal{L}(W^{\alpha,2}(0,\infty;U), L^2(0,\infty;Y))$$

Main Results: P-Stability $\longleftrightarrow$ Polynomial I/O-Stablity

Theorem

Assume $\alpha \in \mathbb{N}$, $T(t)$ *is uniformly bounded, and* $i\mathbb{R} \subset \rho(A)$.

(ii) *If the plant is polynomially input-output stable with* α, *then it is P-stable with* α.

I.e. (roughly)

$$\mathbb{F} \in \mathcal{L}(W^{\alpha,2}(0,\infty;U), L^2(0,\infty;Y))$$

implies

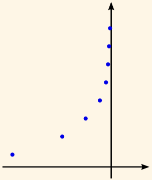
$$\|P(\lambda)\| \leq M_P(1+|\lambda|^\alpha), \qquad \textit{near} \quad i\mathbb{R}$$

Example

In our earlier diagonal example on $X = \ell^2(\mathbb{C})$ we had $\alpha = 2/3$.

$$A = \operatorname{diag}\left(-\frac{1}{k^2} + ik\right)_{k=1}^{\infty}$$

$$B = C^* = \left(\frac{1}{k^{2/3}}\right)_{k=1}^{\infty} \in \ell^2$$



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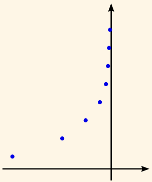
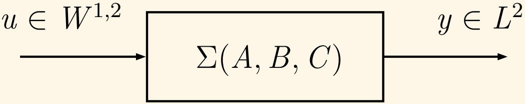


Conclusion: The input-output map has the property:

$$\mathbb{F} \in \mathcal{L}(W^{1,2}(0,\infty;U), L^2(0,\infty;Y))$$



Conclusions

- Review of selected stability types for systems.
- P-stability and polynomial input-output stability.
- Comparison of concepts.

Thank You!