On Infinite-Dimensional Sylvester Equation and The Internal Model Principle

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Introduction

We study the Sylvester equations

$$\begin{aligned} \Pi S &= A\Pi + BK\Gamma + E \\ \Gamma S &= \mathcal{G}_1\Gamma + \mathcal{G}_2(C\Pi + DK\Gamma + F) \end{aligned} \qquad \text{(S1)}$$

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and their decomposing into

$$\begin{aligned} \Pi S &= A\Pi + BK\Gamma + E \\ \Gamma S &= \mathcal{G}_1\Gamma \\ 0 &= C\Pi + DK\Gamma + F \end{aligned} \quad \text{(S2)}$$

Motivation

The Sylvester equations are closely related to robust output regulation for a system $\Sigma(A, B, C, D, E, F)$ on X

$$\begin{aligned} \dot{x} &= Ax + Bu + Ev, \qquad x(0) = x_0 \\ e &= Cx + Du + Fv \end{aligned}$$

with a signal generator on W

$$\dot{v} = Sv, \qquad v(0) = v_0,$$

and an error feedback controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ on Z

$$\begin{aligned} \dot{z} &= \mathcal{G}_1 z + \mathcal{G}_2 e, \qquad z(0) = z_0 \\ u &= Kz \end{aligned}$$

The basic idea

If the closed-loop system with state $(x, z)^T$ on $X \times Z$ is stable, then the equations

$$\begin{aligned} \Pi S &= A\Pi + BK\Gamma + E \\ \Gamma S &= \mathcal{G}_1\Gamma + \mathcal{G}_2(C\Pi + DK\Gamma + F) \end{aligned} \qquad \text{(S1)}$$

have a unique solution (Π, Γ) and

$$\| \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} - \begin{bmatrix} \Pi \\ \Gamma \end{bmatrix} v(t) \| \to 0 \qquad \text{as } t \to \infty,$$

The basic idea

If the closed-loop system with state $(x,z)^T$ on $X \times Z$ is stable, then the equations

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have a unique solution (Π,Γ) and

$$\left\| \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} - \begin{bmatrix} \Pi \\ \Gamma \end{bmatrix} v(t) \right\| \to 0 \qquad \text{as } t \to \infty,$$

Further, $e(t) \to 0$ as $t \to \infty$ if $C\Pi + DK\Gamma + F = 0$.

Regulation: Choose controller parameters $(\mathcal{G}_1, \mathcal{G}_2, K)$ such that the closed-loop system is stable and the equations

$$\begin{aligned} \Pi S &= A\Pi + BK\Gamma + E \\ \Gamma S &= \mathcal{G}_1\Gamma + \mathcal{G}_2(C\Pi + DK\Gamma + F) \end{aligned} \tag{S1}$$

decompose into

$$\begin{aligned} \Pi S &= A\Pi + BK\Gamma + E \\ \Gamma S &= \mathcal{G}_1\Gamma \\ 0 &= C\Pi + DK\Gamma + F \end{aligned} \tag{S2}$$

Then $e(t) \to 0$ as $t \to \infty$ since the closed-loop system is stable and $C\Pi + DK\Gamma + F = 0$ is satisfied.

Robust Regulation: Choose controller parameters $(\mathcal{G}_1, \mathcal{G}_2, K)$ such that the closed-loop system is stable and the equations

$$\begin{aligned} \Pi_p S &= A_p \Pi_p + B_p K \Gamma_p + E_p \\ \Gamma_p S &= \mathcal{G}_1 \Gamma_p + \mathcal{G}_2 (C_p \Pi_p + D_p K \Gamma_p + F_p) \end{aligned} \qquad \text{(S1)}$$

decompose into

$$\begin{aligned} \Pi_p S &= A_p \Pi_p + B_p K \Gamma_p + E_p \\ \Gamma_p S &= \mathcal{G}_1 \Gamma_p \\ 0 &= C_p \Pi_p + D_p K \Gamma_p + F_p \end{aligned} \qquad \text{(S2)}$$

for all A_p, B_p, C_p, D_p, E_p, F_p, Π_p and Γ_p.

Then $e(t) \to 0$ as $t \to \infty$ for all perturbations p preserving closed-loop stability.

Main Problem

Problem

Find necessary and sufficient conditions for $(\mathcal{G}_1, \mathcal{G}_2, K)$ *such that*

$$\begin{cases} \Pi S &= A\Pi + BK\Gamma + E \\ \Gamma S &= \mathcal{G}_1\Gamma + \mathcal{G}_2(C\Pi + DK\Gamma + F) \end{cases} \tag{S1}$$

$$\Leftrightarrow \quad \begin{cases} \Pi S &= A\Pi + BK\Gamma + E \\ \Gamma S &= \mathcal{G}_1\Gamma \\ 0 &= C\Pi + DK\Gamma + F \end{cases} \tag{S2}$$

for all operators (A, B, C, D, E, F), $\Pi \in \mathcal{L}(W, Z)$ *and* $\Gamma \in \mathcal{L}(W, Z)$ *for which* $\Pi(\mathcal{D}(S)) \subset \mathcal{D}(A)$ *and* $\Gamma(\mathcal{D}(S)) \subset \mathcal{D}(\mathcal{G}_1)$.

Earlier Research

This problem is related to the Internal Model Principle.

The Internal Model Principle for infinite-dimensional systems:

- Bhat, 1976
- Immonen, 2006

Assumptions

- The exosystem:

$$\begin{aligned} Sv &= \sum_{k\in\mathbb{Z}} i\omega_k \langle v, \phi_k\rangle \phi_k, \\ \mathcal{D}(S) &= \Big\{ v \in W \;\Big|\; \sum_{k\in\mathbb{Z}} \omega_k^2 |\langle v, \phi_k\rangle|^2 < \infty \Big\}, \end{aligned}$$

  where $(\omega_k)_{k\in\mathbb{Z}} \subset \mathbb{R}$ has no finite accumulation points, $\omega_k \neq \omega_l$ for $k \neq l$ and $\{\phi_k\}_{k\in\mathbb{Z}}$ is an orthonormal basis of W.

- For all considered operators:
  - A generates a C_0-semigroup on X and $\sigma(A) \cap \sigma(S) = \varnothing$
  - (B, C, D, E, F) bounded

Main Results

Theorem

The Sylvester equations (S1) *and* (S2) *are equivalent for all operators* (A, B, C, D, E, F) *if and only if*

$$\begin{aligned} \mathcal{R}(sI - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) &= \{0\} \qquad \forall s \in \sigma(S) \\ \mathcal{N}(\mathcal{G}_2) &= \{0\} \end{aligned} \tag{GC}$$

Sufficiency: Hämäläinen & Pohjolainen '08

Necessity:

Proof.

Show that

$$\text{(S1)} \quad \Leftrightarrow \quad \text{(S2)}$$

implies

$$\begin{aligned} \mathcal{R}(sI - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) &= \{0\} \qquad \forall s \in \sigma(S) \\ \mathcal{N}(\mathcal{G}_2) &= \{0\} \end{aligned} \tag{GC}$$

Proof of "only if", $\mathcal{N}(\mathcal{G}_2) = \{0\}$.

1. Let $y \in \mathcal{N}(\mathcal{G}_2)$ and $\phi \in W$ with $\|\phi\| = 1$.
2. Choose $E = 0$, $\Pi = 0$, $\Gamma = 0$ and $F = \langle \cdot, \phi \rangle y$
3. The equations

$$\begin{aligned} \Pi S &= A\Pi + BK\Gamma + E \\ \Gamma S &= \mathcal{G}_1\Gamma + \mathcal{G}_2(C\Pi + DK\Gamma + F) \end{aligned} \quad \text{(S1)}$$

   are satisfied
4. The decomposing of the equations implies

$$0 = C\Pi + DK\Gamma + F = F$$

   and thus $y = 0$.

□

Necessity:

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Show that

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Proof of "only if", $\mathcal{R}(sI - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) = \{0\}$.

1. Let $i\omega_k \in \sigma(S)$ and $v \in \mathcal{R}(i\omega_k I - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2)$.
2. Then $v = (i\omega_k I - \mathcal{G}_1)z = \mathcal{G}_2 y$ for some z, y.
3. Use this to choose E, Π, Γ and F such that the equations

$$\begin{aligned} \Pi S &= A\Pi + BK\Gamma + E \\ \Gamma S &= \mathcal{G}_1\Gamma + \mathcal{G}_2(C\Pi + DK\Gamma + F) \end{aligned} \qquad \text{(S1)}$$

   are satisfied and $(\Gamma S - \mathcal{G}_1\Gamma)\phi_k = (i\omega_k I - \mathcal{G}_1)z = v$.
4. The decomposing of the equations implies $\Gamma S = \mathcal{G}_1\Gamma$ and thus

$$0 = (\Gamma S - \mathcal{G}_1\Gamma)\phi_k = v.$$

□

Sufficiency:

Proof.

Show that

$$\begin{aligned} \mathcal{R}(sI - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) &= \{0\} \qquad \forall s \in \sigma(S) \\ \mathcal{N}(\mathcal{G}_2) &= \{0\} \end{aligned} \tag{GC}$$

imply

$$\text{(S1)} \quad \Leftrightarrow \quad \text{(S2)}$$

Proof of "if", Hämäläinen & Pohjolainen, 2008.

1. It is sufficient to show that

$$\Gamma S = \mathcal{G}_1 \Gamma + \mathcal{G}_2 (C\Pi + DK\Gamma + F)$$

   implies $C\Pi + DK\Gamma + F = 0$.

2. For all $k \in \mathbb{Z}$ apply both sides of this equation to ϕ_k to obtain

$$(i\omega_k I - \mathcal{G}_1)\Gamma \phi_k = \mathcal{G}_2 (C\Pi + DK\Gamma + F)\phi_k$$

3. Since $\mathcal{R}(i\omega_k I - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) = \{0\}$, we have $\mathcal{G}_2(C\Pi + DK\Gamma + F)\phi_k = 0$.
4. Further, $(C\Pi + DK\Gamma + F)\phi_k = 0$ since $\mathcal{N}(\mathcal{G}_2) = \{0\}$.
5. Since $\{\phi_k\}_{k\in\mathbb{Z}}$ is a basis of W, we have $C\Pi + DK\Gamma + F = 0$.

□

Connection to The Internal Model Principle

Problem

Study the conditions

$$\begin{aligned} \mathcal{R}(sI - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) &= \{0\} \qquad \forall s \in \sigma(S) \\ \mathcal{N}(\mathcal{G}_2) &= \{0\} \end{aligned} \tag{GC}$$

in greater detail.

In particular, we want to compare them to the Internal Model of finite-dimensional control theory.

Internal Model

In finite-dimensional control theory the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ *incorporates an internal model of* S if the following holds:

If $s \in \sigma(S)$ *is an eigenvalue of* S *such that* $d(s)$ *is the dimension of the largest Jordan block associated to* s*, then* $s \in \sigma(\mathcal{G}_1)$ *and* $\mathcal{G}_1$ *has at least* $\dim Y$ *Jordan blocks of dimension* $\geq d(s)$ *associated to* s*.*

Internal Model

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For our operator S this reduces to

$$\dim \mathcal{N}(sI - \mathcal{G}_1) \geq \dim Y \qquad \forall s \in \sigma(S).$$

Notation

- Define
$$A_e = \begin{bmatrix} A & BK \\ \mathcal{G}_2 C & \mathcal{G}_1 + \mathcal{G}_2 DK \end{bmatrix}.$$
(System operator of the closed-loop system on $X \times Z$ with state $x_e(t) = (x(t), z(t))^T$)

- For $\lambda \in \rho(A)$ define
$$P(\lambda) = CR(\lambda, A)B + D \in \mathcal{L}(U, Y),$$
(Transfer function of the plant)

Theorem

If $\sigma(A_e) \cap \sigma(S) = \varnothing$ *and if*

$$\begin{array}{rcll} \mathcal{R}(sI - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) & = & \{0\} & \forall s \in \sigma(S) \\ \mathcal{N}(\mathcal{G}_2) & = & \{0\} & \end{array} \tag{GC}$$

then

$$\dim \mathcal{N}(sI - \mathcal{G}_1) = \dim Y \qquad \forall s \in \sigma(S).$$

Theorem

If $\sigma(A_e) \cap \sigma(S) = \varnothing$ *and if*

$$\begin{aligned} \mathcal{R}(sI - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) &= \{0\} \qquad \forall s \in \sigma(S) \\ \mathcal{N}(\mathcal{G}_2) &= \{0\} \end{aligned} \tag{GC}$$

then

$$\dim \mathcal{N}(sI - \mathcal{G}_1) = \dim Y \qquad \forall s \in \sigma(S).$$

In particular, operator $(P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)}$ is an isomorphism between $\mathcal{N}(sI - \mathcal{G}_1)$ and Y for all $s \in \sigma(S)$.

Proof.

1. Let $s \in \sigma(S)$. Then $s \in \rho(A_e)$.
2. $sI - A_e$ is injective $\Rightarrow (P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)}$ is injective
3. $sI - A_e$ is surjective & (GC) $\Rightarrow (P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)}$ is surjective
4. This concludes that $(P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)}$ is an isomorphism between $\mathcal{N}(sI - \mathcal{G}_1)$ and Y.

□

Theorem

If $\sigma_p(A_e) \cap \sigma(S) = \varnothing$, $\dim Y < \infty$ *and if*

$$\dim \mathcal{N}(sI - \mathcal{G}_1) = \dim Y \qquad \forall s \in \sigma(S).$$

then

$$\begin{aligned} \mathcal{R}(sI - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) &= \{0\} \qquad \forall s \in \sigma(S) \\ \mathcal{N}(\mathcal{G}_2) &= \{0\} \end{aligned} \qquad \text{(GC)}$$

Proof of $\mathcal{N}(\mathcal{G}_2) = \{0\}$.

1. Let $y \in \mathcal{N}(\mathcal{G}_2)$ and $s \in \sigma(S)$.
2. $sI - A_e$ is injective $\Rightarrow$ $(P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)}$ is injective
3. Since $\dim \mathcal{N}(sI - \mathcal{G}_1) = \dim Y < \infty$, the operator $(P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)}$ is invertible
4. There exists $z_1 \in \mathcal{N}(sI - \mathcal{G}_1)$ such that

$$y = P(s)Kz_1$$

5. The injectivity of $sI - A_e$ can be used to show that $z_1 = 0$ and thus $y = 0$.

□

Proof of $\mathcal{R}(sI - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) = \{0\}$.

1. Let $s \in \sigma(S)$ and $v \in \mathcal{R}(sI - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2)$
2. Then $v = (sI - \mathcal{G}_1)z = \mathcal{G}_2 y$ for some y, z.
3. Similarly as before, the operator $(P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)}$ is invertible
4. There exists $z_0 \in \mathcal{N}(sI - \mathcal{G}_1)$ such that

$$y = P(s)K(z + z_0)$$

and thus

$$v = (sI - \mathcal{G}_1)(z + z_0) = \mathcal{G}_2 P(s)K(z + z_0)$$

5. The injectivity of $sI - A_e$ can be used to show that $z + z_0 = 0$ and thus $v = 0$.

□

Lemma

If $\sigma_p(A_e) \cap \sigma(S) = \varnothing$, *then*

$$\dim \mathcal{N}(sI - \mathcal{G}_1) \leq \dim Y \qquad \forall s \in \sigma(S).$$

Proof.

Injectivity of $(P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)}$ and the Rank-Nullity Theorem. □

Corollary

Let $\sigma(A_e) \cap \sigma(S) = \varnothing$ *and* $\dim Y < \infty$. *Then*

$$\begin{aligned} \mathcal{R}(sI - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) &= \{0\} \qquad \forall s \in \sigma(S) \\ \mathcal{N}(\mathcal{G}_2) &= \{0\} \end{aligned} \tag{GC}$$

hold if and only if

$$\dim \mathcal{N}(sI - \mathcal{G}_1) \geq \dim Y \qquad \forall s \in \sigma(S).$$

Conclusions

In this presentation:

- Necessary and sufficient conditions for decomposing of certain infinite-dimensional Sylvester equations
- Under certain assumptions these conditions are equivalent to the Internal Model of finite-dimensional control theory

Future research:

- More general signal generator S.