# Nonuniform decay rates for systems of differential equations

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# Structure

- Part I: A Coupled Wave–Heat System in 1D
  - Resolvent estimates
  - Rational decay of energy
- Part II: Infinite Systems of Differential Equations
  - Platoon-type systems
  - Stability and rational decay rates
- In Closing: Thoughts on Infinite Systems of PDEs

# Part I:

## A Coupled Wave–Heat System in 1D

# Coupled Wave–Heat Systems

- Boundary coupled PDEs are used to describe, e.g., fluid–structure and heat–struture interactions.
- Coupling often leads to rational decay of energy.

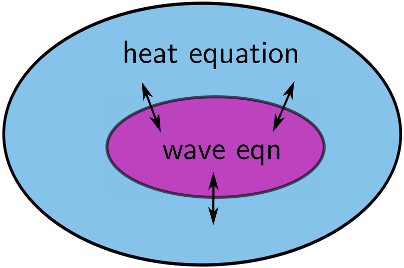
$$\frac{\partial^2 u}{\partial t^2}(x,t) = \Delta u(x,t)$$

$$\updownarrow$$

coupling BCs

$$\updownarrow$$

$$\frac{\partial w}{\partial t}(x,t) = \Delta w(x,t)$$



# Background and Motivation

- Our model based on coupled 2D and 3D systems studied by Avalos, Triggiani and others.
- Energy decay in related 1D models studied previously using Riesz basis methods
- Research also related to a more general field of coupled PDEs, by Mercier, Nicaise, Ammari, Zuazua, Guo and many others. (also nonuniform rates have been studied for many systems of this class)

Our aim was to use resolvent estimates to simplify analysis (Avalos and Triggiani use resolvent estimates, but for general geometries the analysis becomes involved).

# 1D Wave–Heat Model

$$
\begin{cases}
u_{tt}(\xi,t) = u_{\xi\xi}(\xi,t), & \xi \in (-1,0),\ t>0,\\
w_t(\xi,t) = w_{\xi\xi}(\xi,t), & \xi \in (0,1),\ t>0,\\
u_\xi(-1,t) = 0, \qquad w(1,t) = 0, & t>0,\\
u_\xi(0,t) = w_\xi(0,t), \quad u_t(0,t) = w(0,t), & t>0,
\end{cases}
$$

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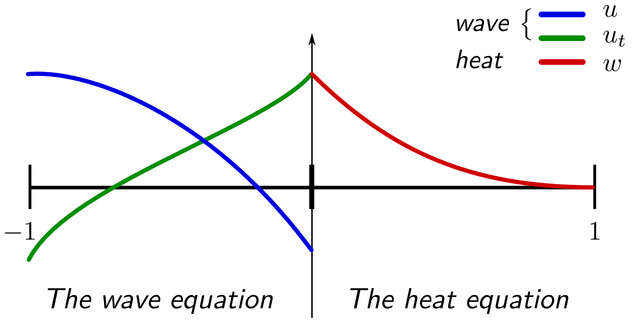
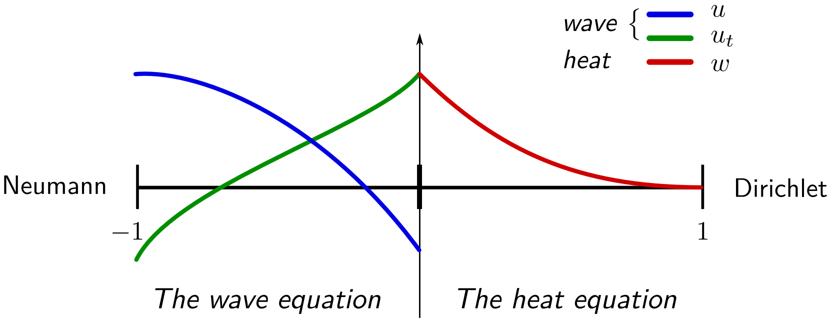


$$\begin{cases} u_{tt}(\xi,t) = u_{\xi\xi}(\xi,t), & \xi \in (-1,0),\ t>0,\\ w_t(\xi,t) = w_{\xi\xi}(\xi,t), & \xi \in (0,1),\ t>0,\\ {\color{blue} u_\xi(-1,t) = 0, \quad w(1,t) = 0,} & t>0,\\ u_\xi(0,t) = w_\xi(0,t), \quad u_t(0,t) = w(0,t), & t>0, \end{cases}$$



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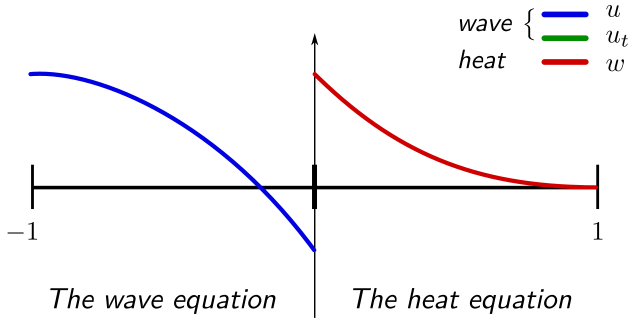
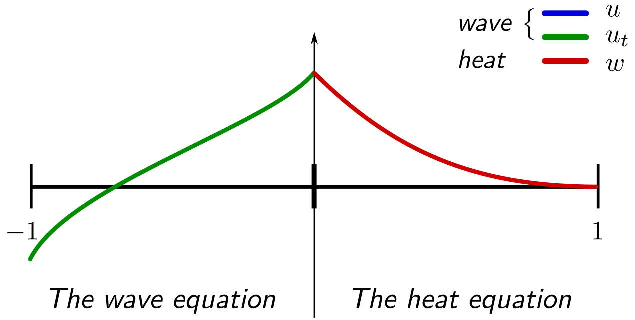


$$\begin{cases} u_{tt}(\xi,t) = u_{\xi\xi}(\xi,t), & & \xi \in (-1,0),\ t>0, \\ w_t(\xi,t) = w_{\xi\xi}(\xi,t), & & \xi \in (0,1),\ t>0, \\ u_\xi(-1,t)=0, & w(1,t)=0, & t>0, \\ u_\xi(0,t) = w_\xi(0,t), & \color{blue}{u_t(0,t) = w(0,t),} & t>0, \end{cases}$$



# 1D Wave–Heat Model

$$\begin{cases} u_{tt}(\xi,t) = u_{\xi\xi}(\xi,t), & & \xi \in (-1,0),\ t>0, \\ w_t(\xi,t) = w_{\xi\xi}(\xi,t), & & \xi \in (0,1),\ t>0, \\ u_\xi(-1,t) = 0, & w(1,t) = 0, & t>0, \\ u_\xi(0,t) = w_\xi(0,t), & u_t(0,t) = w(0,t), & t>0, \end{cases}$$

Total energy of the system:

$$E_{x_0}(t) = \frac{1}{2}\int_{-1}^{0} |u_\xi(\xi,t)|^2 + |u_t(\xi,t)|^2 \, d\xi + \frac{1}{2}\int_0^1 |w(\xi,t)|^2 \, d\xi$$

We are most interested in showing that $E_{x_0}(t)$ decays at a rational rate for all classical solutions of the system.

# Abstract Formulation

The system can be written as an abstract Cauchy problem

$$\begin{cases} x'(t) = Ax(t), \quad t \geq 0, \\ x(0) = x_0, \end{cases}$$

on $X = H^1(-1,0) \times L^2(-1,0) \times L^2(0,1)$ with operator

$$A = \left( \begin{array}{cc|c} 0 & I & 0 \\ \Delta_{\text{wave}} & 0 & 0 \\ \hline 0 & 0 & \Delta_{\text{heat}} \end{array} \right)$$

$$D(A) = \Big\{ (u,v,w) \in H^2(-1,0) \times H^1(-1,0) \times H^2(0,1) \ \Big| \ u'(-1) = w(1) = 0, \ v(0) = w(0), \ u'(0) = w'(0) \Big\}$$

# Semigroup Generation and Properties

**Theorem**

*The operator $A$ generates a uniformly bounded $C_0$-semigroup $(T(t))_{t\geq 0}$ on $X$, $\sigma(A) \cap i\mathbb{R} = \{0\}$ and $A$ has compact resolvent.*

**Proof.**

We have $X = \mathrm{Ran}(A) \oplus \mathrm{Ker}(A)$ where $\dim \mathrm{Ker}(A) = 1$ and $\mathrm{Ran}(A)$ is closed. The restriction $A|_{\mathrm{Ran}(A)}$ generates a bounded semigroup due to Lumer–Phillips. □

Relation to the total energy of the system: For initial state $x_0 = x_{01} + x_{02} \in \mathrm{Ran}(A) \oplus \mathrm{Ker}(A)$ the energy of $x(t)$ satisfies

$$E_{x_0}(t) \asymp \|T(t)|_{\mathrm{Ran}(A)} x_{01}\|^2.$$

# Main Result: Rational Decay of Energy

**Theorem**

*If $x_0 \in D(A)$, then $E_{x_0}(t) = o(t^{-4})$ as $t \to \infty$.*

Due to relation between energy and norms of orbits, the decay is equivalent to

$$\|T_0(t)A_0^{-1}\| = O\left(\frac{1}{t^2}\right)$$

where $A_0 = A|_{\mathrm{Ran}(A)}$ and $T_0(t) = T(t)|_{\mathrm{Ran}(A)}$

# Decay of Energy

Decay of energy can be deduced from the resolvent estimates.

**Theorem (Borichev & Tomilov 2010)**

*Let $(T(t))_{t\geq 0}$ be a uniformly bounded $C_0$-semigroup on a Hilbert space $X$. Let $A$ be the generator of $(T(t))_{t\geq 0}$ and suppose that $\sigma(A) \cap i\mathbb{R} = \emptyset$.*

*For any constant $\alpha > 0$, the following conditions are equivalent:*

(i) $\|R(is, A)\| = O(|s|^\alpha)$ *as* $|s| \to \infty$;

(ii) $\|T(t)A^{-1}\| = O(t^{-1/\alpha})$ *as* $t \to \infty$;

(iii) $\|T(t)x\| = o(t^{-1/\alpha})$ *as* $t \to \infty$ *for all* $x \in D(A)$.

# Resolvent Estimates

**Theorem**

$\|R(is, A_0)\| = O(|s|^{1/2})$ *as* $|s| \to \infty$, *and* $\|T_0(t)A_0^{-1}\| = O(t^{-2})$.

Here we again denote $A_0 = A|_{\mathrm{Ran}(A)}$ and $T_0(t) = T(t)|_{\mathrm{Ran}(A)}$.

# Comments on the Proof

For $x = (u, v, w)$ and $y = (f, g, h)$ the equation

$$(is - A)x = y$$

is equivalent to the boundary value problems

$$\begin{aligned} u''(\xi) &= -s^2 u(\xi) - isf(\xi) - g(\xi), && \xi \in (-1, 0), \\ v(\xi) &= isu(\xi) - f(\xi), && \xi \in (-1, 0), \\ w''(\xi) &= isw(\xi) - h(\xi), && \xi \in (0, 1), \\ u'(-1) &= w(1) = 0,\ v(0) = w(0),\ u'(0) = w'(0). \end{aligned}$$

Estimate $\|R(is, A_0)y\| \lesssim \sqrt{|s|}\|y\|$ follows from solving for $(u, v, w)$ and using some convenient tricks in the estimates.

# Main Result: Rational Decay of Energy

**Theorem**

*If $x_0 \in D(A)$, then $E_{x_0}(t) = o(t^{-4})$ as $t \to \infty$.*

# Optimality of the Decay Rates

Locations of eigenvalues imply that $E_{x_0}(t) = o(t^{-4})$ is optimal.

**Lemma**

$$\sigma(A) = \left\{ \lambda \in \mathbb{C} \;\middle|\; \sqrt{\lambda}\cosh(\lambda)\cosh(\sqrt{\lambda}) + \sinh(\lambda)\sinh(\sqrt{\lambda}) = 0 \right\}.$$

**Proof.**

Solve the boundary value problems equivalent to $(\lambda - A)x = 0$ for $x = (u, v, w)$. □

# Optimality of the Decay Rates

A Rouché argument implies:

**Theorem**

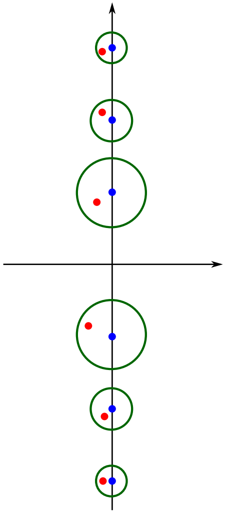
*There exists* $(\lambda_n) \subset \sigma(A)$ *such that* $\operatorname{Im} \lambda_n \sim n\pi$ *as* $n \to \infty$ *and*

$$-\frac{1}{|\operatorname{Im} \lambda_n|^{1/2}} \lesssim \operatorname{Re} \lambda_n < 0, \quad n \geq 0.$$

In particular,

$$\limsup_{|s|\to\infty} |s|^{-1/2} \|R(is, A)\| > 0$$

$E_{x_0}(t) = o(t^{-4})$ is optimal for $x_0 \in D(A)$.



# Part II:

## Infinite Systems of Differential Equations

# Main Problem

Study asymptotics of infinite systems of the form

$$\dot{x}_k(t) = A_0 x_k(t) + A_1 x_{k-1}(t), \quad k \in \mathbb{Z},\ t \geq 0,$$

where $A_0, A_1 \in \mathbb{C}^{m\times m}$ do not depend on $k \in \mathbb{Z}$.

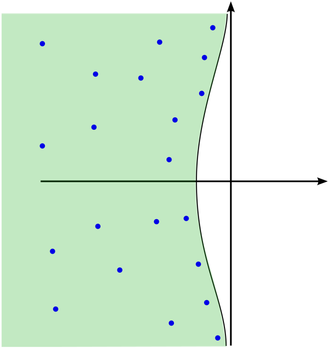
We want to study, e.g.,

$$\sup_{k\in\mathbb{Z}} \|x_k(t) - y_k\|_{\mathbb{C}^m} \to 0, \qquad \text{as} \quad t \to \infty$$

with rates.

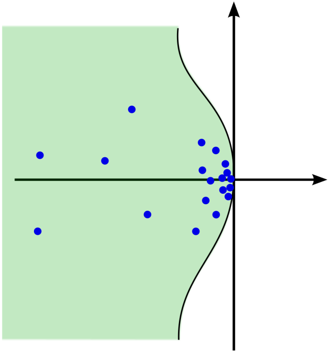
For $x \in D(A)$

$$\|T(t)x\| \leq \frac{M}{t^{1/\alpha}} \|Ax\|,$$



For $x \in \mathrm{Ran}(A)$

$$\|T(t)x\| \leq \frac{M}{t^{1/\alpha}} \|A^{-1}x\|,$$



Martinez '11, Batty, Chill & Tomilov '13 ('16), Chill & Seifert '16

# Infinite Systems of Differential Equations

Our system can be formulated as an abstract Cauchy problem

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0 \in X$$

on $X = \ell^p(\mathbb{C}^m)$ for $1 \leq p \leq \infty$ by choosing $x(t) = (x_k(t))_{k\in\mathbb{Z}}$ and

$$Ax = (A_0 x_k + A_1 x_{k-1})_{k\in\mathbb{Z}}.$$

i.e.

$$A = \begin{pmatrix} \ddots & \ddots & & & \\ & A_1 & A_0 & & \\ & & A_1 & A_0 & \\ & & & \ddots & \ddots \end{pmatrix}$$

Here $A \in \mathcal{L}(X)$ and our system belongs to the class of "Spatially invariant systems" (Bamieh et. al. and others).

# A Concrete Model: An Infinite Vehicle Platoon

$$\begin{pmatrix} \dot{y}_k \\ \dot{w}_k \\ \dot{a}_k \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 \end{pmatrix} \begin{pmatrix} y_k \\ w_k \\ a_k \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{k-1} \\ w_{k-1} \\ a_{k-1} \end{pmatrix}$$

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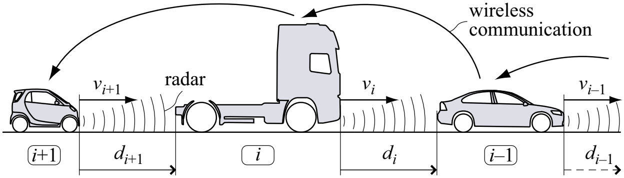



Figure: Source: Ploeg *et. al.*, IEEE, 2011.

# A Concrete Model: An Infinite Vehicle Platoon

$$\begin{pmatrix} \dot{y}_k \\ \dot{w}_k \\ \dot{a}_k \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 \end{pmatrix}}_{= A_0} \begin{pmatrix} y_k \\ w_k \\ a_k \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{= A_1} \begin{pmatrix} y_{k-1} \\ w_{k-1} \\ a_{k-1} \end{pmatrix}$$

$y_k(t) =$ displacement from ideal distance between $k$ and $k-1$

$w_k(t) =$ velocity of $k$th vehicle

$a_k(t) =$ acceleration of $k$th vehicle

Objective: Choose $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{R}$ so that $\sup_{k\in\mathbb{Z}} |y_k| \to 0$ as $t \to \infty$.

Challenge: The matrices $A_0$ and $A_1$ do not commute.

# A More Abstract Class

## Assumption

Assume $A_1 \neq 0$, $\sigma(A_0) \subset \mathbb{C}_-$, and there exists $\phi : \mathbb{C} \to \mathbb{C}$ s.t.

$$A_1(\lambda - A_0)^{-1}A_1 = \phi(\lambda)A_1, \qquad \lambda \in \mathbb{C} \setminus \sigma(A_0).$$

$\phi(\cdot)$ is called *characteristic function* of the infinite system.

## Lemma

*If $A_1$ is of rank one such that $A_1 = ab^* \in \mathbb{C}^{m\times m}$, then*

$$\phi(\lambda) = b^*(\lambda - A_0)^{-1}a.$$

Clearly $\phi(\lambda) = \frac{n(\lambda)}{d(\lambda)}$ where $\deg n(\cdot) \leq \deg d(\cdot)$.

# Main Results

In the following we will do the following:

(i) Characterize spectrum of $A$

(ii) Present conditions for uniform boundedness of $T(t)$

(iii) Study rates of convergence of $\|T(t)x_0 - y\| \to 0$ as $t \to \infty$.

For all these purposes use the characteristic function $\phi(\cdot)$:

$$A_1(\lambda - A_0)^{-1}A_1 = \phi(\lambda)A_1, \qquad \lambda \in \mathbb{C} \setminus \sigma(A_0).$$

Main idea: Existence of $\phi(\cdot)$ compensates for the lack of commutativity of $A_0$ and $A_1$.

# Spectrum of the System

Characteristic function $\phi(\cdot)$ determines the spectrum of $A$:

### Theorem

*Let* $X = \ell^p(\mathbb{C}^m)$ *with* $1 \leq p \leq \infty$*. Then for* $\lambda \in \mathbb{C} \setminus \sigma(A_0)$

$$\lambda \in \sigma(A) \qquad \textit{if and only if} \qquad |\phi(\lambda)| = 1.$$

*Moreover, for* $\lambda \in \sigma(A) \setminus \sigma(A_0)$ *we have*

- $\mathrm{Ker}(\lambda - A) \neq \{0\}$ *if and only if* $p = \infty$
- $\overline{\mathrm{Ran}(\lambda - A)} = X$ *if and only if* $1 < p < \infty$*.*

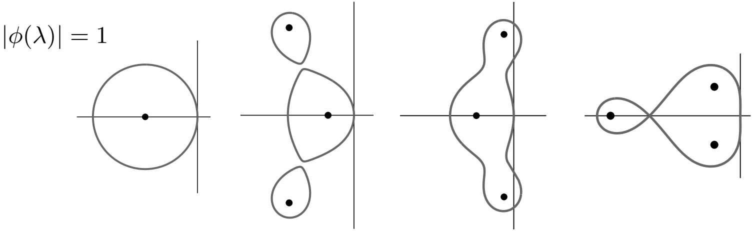
The type of spectrum depends on $p$, but the location does not.

# Spectrum of the Platoon System

$$\begin{pmatrix} \dot{y}_k \\ \dot{w}_k \\ \dot{a}_k \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 \end{pmatrix} \begin{pmatrix} y_k \\ w_k \\ a_k \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{k-1} \\ w_{k-1} \\ a_{k-1} \end{pmatrix}$$

Characteristic function: $\phi(\lambda) = \dfrac{\alpha_0}{p(\lambda)} = \dfrac{\alpha_0}{\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0}$.

Spectrum of the platoon system is determined by $\alpha_0$, $\alpha_1$, and $\alpha_2$.



# Uniform Boundedness of the Semigroup

**Theorem**

*Let* $1 \le p \le \infty$. *If* $\sigma(A) \subset \mathbb{C}_- \cup \{0\}$,

$$\sup_{0<\lambda\le 1} \frac{\lambda}{1-|\phi(\lambda)|} < \infty \quad \textit{and} \quad \sup_{n\in\mathbb{N}} \sup_{\lambda>0} \frac{\lambda^{n+1}}{n!} \sum_{\ell=1}^{\infty} \left| \frac{d^n}{d\lambda^n} \phi(\lambda)^\ell \right| < \infty,$$

*then the semigroup generated by* $A$ *is uniformly bounded.*

**Proof.**

A fairly direct Hille–Yosida approach using a resolvent formula. □

Property: Systems for $m \ge 2$ are typically not contractive. In particular, the platoon system is <u>never</u> contractive.

# Uniform Boundedness for the Platoon System

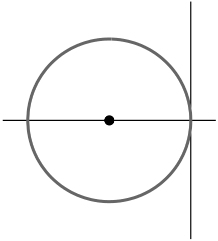
## Lemma

*If* $\phi(\cdot)$ *is such that for some* $\zeta > 0$, $q \in \mathbb{N}$,

$$\phi(\lambda) = \frac{\zeta^q}{(\lambda+\zeta)^q}, \qquad \lambda \neq -\zeta$$

*then*

$$\sup_{n\in\mathbb{N}} \sup_{\lambda>0} \frac{\lambda^{n+1}}{n!} \sum_{\ell=1}^{\infty} \left| \frac{d^n}{d\lambda^n} \phi(\lambda)^\ell \right| < \infty.$$



The characteristic function of the platoon system is of this form if parameters $\alpha_0, \alpha_1, \alpha_2$ are chosen so that $\sigma(A_0) = \{-\zeta\}$.

Then the platoon system is guaranteed to be uniformly bounded.

# (Unquantified) Asymptotic Behaviour

Combining the results on spectrum and uniform boundedness:

## Theorem

*Let* $X = \ell^p(\mathbb{C}^m)$ *and*

$$\phi(\lambda) = \frac{\zeta^q}{(\lambda+\zeta)^q}, \qquad \lambda \neq -\zeta.$$

- *If* $1 < p < \infty$, *then* $T(t)$ *is strongly stable, i.e.,* $T(t)x \to 0$
- *If* $p = 1$, *then* $T(t)x \to 0$ *as* $t \to \infty$ *for all* $x \in \overline{\mathrm{Ran}(A)} \neq X$.
- *If* $p = \infty$ *and* $x = x_0 + x_1 \in \overline{\mathrm{Ran}(A)} \oplus \mathrm{Ker}(A) \neq X$, *then*

$$T(t)x \to x_1 \qquad \textit{as} \qquad t \to \infty.$$

# Rates of Convergence

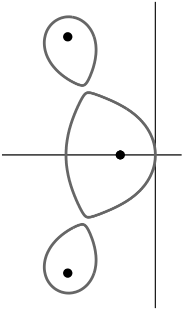
Next aim: Find rates of convergence for

$$\|T(t)x\| \to 0 \qquad \text{as} \quad t \to \infty$$

and

$$\|T(t)x - y\| \to 0 \qquad \text{as} \quad t \to \infty$$

under the assumption $\sigma(A) \cap i\mathbb{R} = \{0\}$.



# From Resolvent Estimates Near $0$ to Rates

### Theorem (Martinez, Chill–Seifert)

*Let $T(t)$ generated by $A \in \mathcal{L}(X)$ be uniformly bounded and $\sigma(A) \cap i\mathbb{R} = \{0\}$. If for some $\alpha \geq 1$ we have*

$$\|R(is, A)\| = O(|s|^{-\alpha}) \qquad \text{as} \qquad |s| \to 0$$

*then*

$$\|T(t)Ax\| = O\left( \left( \frac{\log t}{t} \right)^{1/\alpha} \right), \quad t \to \infty.$$

By Batty–Chill–Tomilov '13, logarithm can be omitted if $X$ Hilbert.

# Convergence Rates via Resolvent Estimates

**Theorem**

*Let* $1 \le p \le \infty$. *If* $0 \in \sigma(A) \subset \mathbb{C}_- \cup \{0\}$, *then for* $0 < s \le 1$

$$\|R(is, A)\| \asymp \frac{1}{|1 - |\phi(is)||} = O(|s|^{-n_\phi})$$

*for an even integer* $n_\phi \ge 2$. *Thus if* $T(t)$ *is uniformly bounded and* $x = x_0 + x_1 \in \mathrm{Ran}(A) \oplus \mathrm{Ker}(A)$, *then*

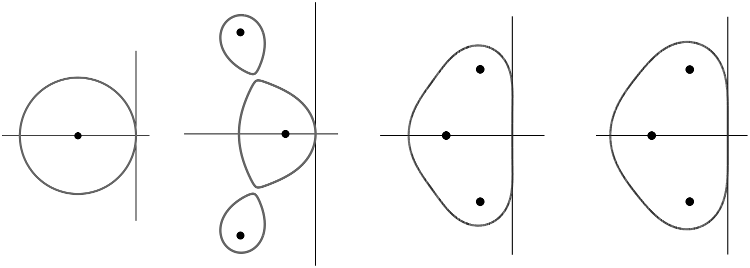
$$\|T(t)x - x_1\| = O\left(\left(\frac{\log t}{t}\right)^{1/n_\phi}\right)$$

Batty, Chill & Tomilov '13: Logarithm can be omitted if $p = 2$.

(Note that $x_1 = 0$ if $1 \le p < \infty$)

# Decay Rates for the Platoon System

For the platoon system, the possible exponents $n_\phi$ are 2, 4 and 6.



Corresponding rates are $\left(\frac{\log t}{t}\right)^{-\frac{1}{2}}$, $\left(\frac{\log t}{t}\right)^{-\frac{1}{4}}$ and $\left(\frac{\log t}{t}\right)^{-\frac{1}{6}}$ (though uniform boundedness was just shown for the first case).

# Characterizing Initial States Leading to Convergence

## Problem

*Characterize elements* $x \in \mathrm{Ran}(A)$ *and* $x \in \overline{\mathrm{Ran}(A)}$.

Motivation: Initial states $x_0 \in \mathrm{Ran}(A) \oplus \mathrm{Ker}(A)$ lead to convergence with rate

$$\|T(t)x_0\| \leq M \left(\frac{\log t}{t}\right)^{1/n_\phi} \|A^{-1}x_0\|.$$

In the cases $X = \ell^1(\mathbb{C}^m)$ and $X = \ell^\infty(\mathbb{C}^m)$

$$t \mapsto T(t)x$$

converges to some $y \in X$ if and only if $x \in \overline{\mathrm{Ran}(A)} \oplus \mathrm{Ker}(A)$.

## Theorem

*Let* $X = \ell^\infty(\mathbb{C}^m)$, $0 \in \sigma(A) \subset \mathbb{C}_- \cup \{0\}$, $T(t)$ *is bdd,* $\phi'(0) \neq 0$.

$x \in \overline{\mathrm{Ran}(A)} \oplus \mathrm{Ker}(A)$ *if and only if there exists* $y_0 \in \mathrm{Ran}(A_1)$

$$\sup_{k\in\mathbb{Z}} \left\| \frac{1}{n} \sum_{j=1}^{n} \phi(0)^{j-k} A_1 A_0^{-1} x_{k-j} - y_0 \right\|_{\mathbb{C}^m} \to 0, \quad n \to \infty, \qquad (1)$$

## Theorem

*Let* $X = \ell^\infty(\mathbb{C}^m)$, $0 \in \sigma(A) \subset \mathbb{C}_- \cup \{0\}$, $T(t)$ *is bdd,* $\phi'(0) \neq 0$.

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*Moreover, if the decay in* (1) *is like* $O(n^{-1})$ *as* $n \to \infty$ *then* $x = x_0 + x_1 \in \mathrm{Ran}(A) \oplus \mathrm{Ker}(A)$ *and*

$$\|T(t)x - x_1\| = O\left( \left( \frac{\log t}{t} \right)^{1/n_\phi} \right), \quad t \to \infty.$$

# Decay for the Platoon System

$$\begin{pmatrix} \dot{y}_k \\ \dot{w}_k \\ \dot{a}_k \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\zeta^3 & -3\zeta^2 & -3\zeta \end{pmatrix} \begin{pmatrix} y_k \\ w_k \\ a_k \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{k-1} \\ w_{k-1} \\ a_{k-1} \end{pmatrix}$$

## Theorem

*Let* $X = \ell^\infty(\mathbb{C}^3)$*.* $T(t)x$ *converges if and only if there exists* $c \in \mathbb{R}$

$$\sup_{k\in\mathbb{Z}} \left| c - \frac{1}{n}\sum_{j=1}^{n} y_{k-j}(0) \right| \to 0, \quad n \to \infty,$$

*and if this holds then* $T(t)x \to x_1$*, where*

$$x_1 = \left( \dots, \begin{pmatrix} c \\ -\zeta c/3 \\ 0 \end{pmatrix}, \begin{pmatrix} c \\ -\zeta c/3 \\ 0 \end{pmatrix}, \begin{pmatrix} c \\ -\zeta c/3 \\ 0 \end{pmatrix}, \dots \right).$$

# Quantified Decay for the Platoon System

$$\begin{pmatrix} \dot{y}_k \\ \dot{w}_k \\ \dot{a}_k \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\zeta^3 & -3\zeta^2 & -3\zeta \end{pmatrix} \begin{pmatrix} y_k \\ w_k \\ a_k \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{k-1} \\ w_{k-1} \\ a_{k-1} \end{pmatrix}$$

## Theorem

*Let $X = \ell^\infty(\mathbb{C}^3)$. If there exists $c \in \mathbb{R}$*

$$\sup_{k \in \mathbb{Z}} \left| c - \frac{1}{n} \sum_{j=1}^{n} y_{k-j}(0) \right| = O\left(\frac{1}{n}\right) \quad \text{as} \quad n \to \infty,$$

*then $\|T(t)x - x_1\| = O(\frac{1}{\sqrt{t}})$ where again $x_1 = ((c, -\zeta c/3, 0)^T)_{k \in \mathbb{Z}}$*

Here the logarithm was removed using N. Dungey 2008.

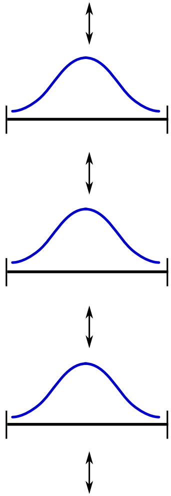
# Infinite Systems of PDEs

Ideas for systems consisting of infinite coupled PDEs.

Two types of couplings:

- Inside the domain
- Through shared boundaries

# An Infinite System of Heat Equations



One-dimensional systems on $(0,1)$:

$$\begin{aligned} u_t^k(\xi,t) &= u_{\xi\xi}^k(\xi,t) + \gamma u^{k-1}(\xi,t) \\ u^k(0,t) &= u^k(1,t) = 0 \\ u^k(\xi,0) &= u_0^k(\xi) \end{aligned}$$

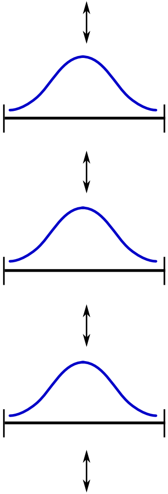
for $k \in \mathbb{Z}$.

Full system $x(t) = (u^k(\cdot,t))_{k\in\mathbb{Z}} \in \ell^2(L^2(0,1))$

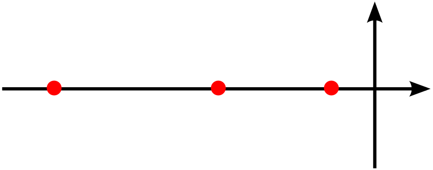
$$Ax = (x_k'' + \gamma x_{k-1})_{k\in\mathbb{Z}}$$

for $x = (x_k)_{k\in\mathbb{Z}} \in \ell^2(H^2 \cap H_0^1)$.

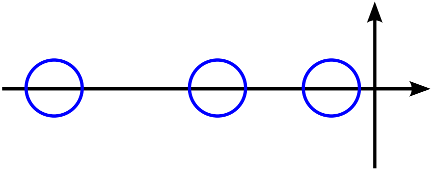
# An Infinite System of Heat Equations



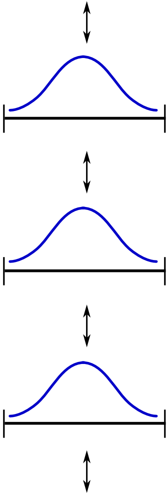
Spectrum $\sigma(A_0)$ of the subsystem $A_0 = \frac{d^2}{d\xi^2}$:



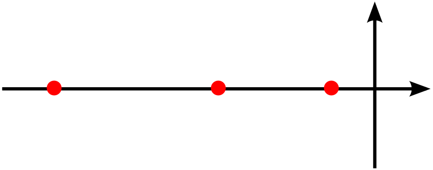
Spectrum $\sigma(A)$ of the full system (Radii $= \gamma > 0$):



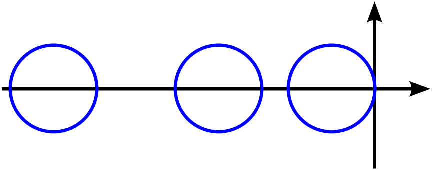
# An Infinite System of Heat Equations



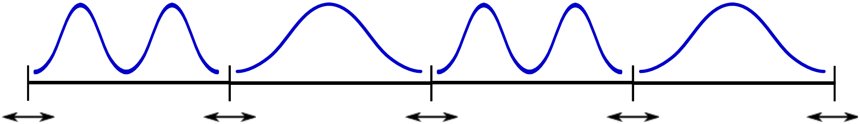
Spectrum $\sigma(A_0)$ of the subsystem $A_0 = \frac{d^2}{d\xi^2}$:



Spectrum $\sigma(A)$ of the full system (Ch. $\gamma = \pi^2$):

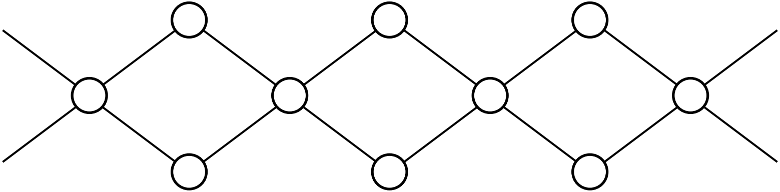


# Infinite Wave–Heat System with Boundary Coupling



$$A = \begin{pmatrix} \ddots & \ddots & \ddots & & & \\ & A_{-1} & A_0 & A_1 & & \\ & & A_{-1} & A_0 & A_1 & \\ & & & A_{-1} & A_0 & A_1 \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}$$

# Infinite Network of PDEs



A doubly infinite block operator matrix with finite bandwidth.

$$A = \begin{pmatrix} \ddots & \ddots & \ddots & & & \\ & * & * & * & & \\ & & * & * & * & \\ & & & \ddots & \ddots & \ddots \end{pmatrix}$$

# References

Batty, C.J.K, Paunonen, L. and Seifert, D. *Optimal energy decay in a one-dimensional coupled wave–heat system*, Journal of Evolution Equations, 2016.

Paunonen, L. and Seifert, D. *Asymptotics for infinite systems of differential equations*, submitted (preprint on Arxiv Nov 2015)

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Thank You!