

A Control-Theoretic Approach to Stability of PDE Networks

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Main Objectives

Problem

*Study dynamics of networks consisting of Partial Differential Equations (PDEs), with focus on **long-time behaviour of the energy**.*

Motivation:

- Dynamics of networked systems are interesting!
- Both network structure and component dynamics contribute
- We focus on what happens when $t \rightarrow \infty$

What are PDE networks?

Throughout the talk a “**PDE network**” refers to

*A collection of Partial Differential Equation models
whose the connectivity can be described by a graph.*

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Perhaps some examples will illustrate the concept. . .

Beam structures



- A truss bridge — beams which are joined at their ends.
- Deflection of each beam described by a PDE — **beam equation**
- Can study either dynamic behaviour or the steady state.

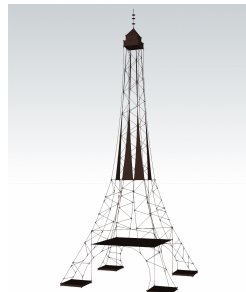
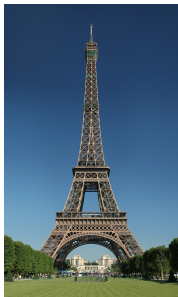
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Truss structures

Structure determines a graph where **the beams are the edges** and **the meeting points are the nodes**.



We focus on such PDE networks, i.e., where

PDE $\leftrightarrow$ edge

connection $\leftrightarrow$ node/vertex

Pipe Networks

Pipe delivery networks for gas, water, and petrol can be modelled as PDE networks. PDEs are **transport equations**.



- Waterflow in channels: **wave equations**
- Heat or material diffusion in structures: **heat or convection-diffusion equations**

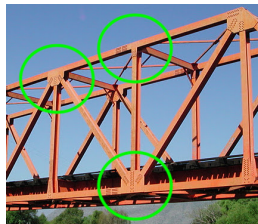
Couplings

Question

How do the PDEs in the network communicate?

Each PDE in the network involves:

- The actual differential equation (deflection along the beam)
- **Boundary conditions** (deflection, slope, strain etc **at the endpoints**)



↪ The beam models interact with each other at the network nodes via their **boundary conditions**.

Examples of Boundary Couplings

Physical system: Gas pipe network

- PDE type: Transport equation
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Physical system: Truss bridge

- PDE type: Beam equation
- Coupling: Deflections are equal **and** forces sum to 0 **and**

All expressed in terms of boundary conditions, and the number depends on the PDE type.

Dynamics and Stability

Definition (Rough, but close enough)

We say the PDE network is **stable** if all its solutions converge to a steady state as $t \rightarrow \infty$.

- **Linear case** $\leadsto$ zero is the unique steady state
- Stability is often equivalent to the property that the **energy** of the solutions satisfies $E(t) \rightarrow 0$ as $t \rightarrow \infty$.

Goal

*We are especially interested in the **rate** of the convergence.*

Mixed-Type PDE Networks

Goal

*We focus on studying **stability** of networks which involve **two different types of PDEs**.*

- Beams and strings (suspension bridge).
- Wave and heat equations (fluid-structure interactions?)
- Damped and undamped strings or beams.
- (PDE networks with dynamic couplings)

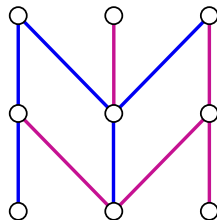


Stability of Mixed-Type PDE Networks

Goal

We focus on studying **stability** of networks which involve **two different types of PDEs**.

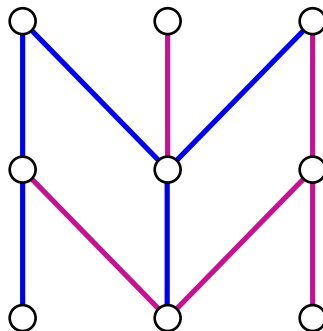
- Stability of mixed-type networks is particularly interesting!
- Convergence of solutions often *slower than exponential* $\leadsto$ “**polynomial stability**”
- Motivates deriving accurate (rational) convergence rates for $E(t) \rightarrow 0$



Part II

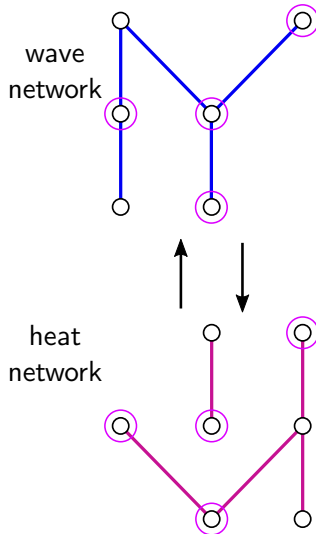
Stability Analysis

Dynamics of Mixed-Type Networks

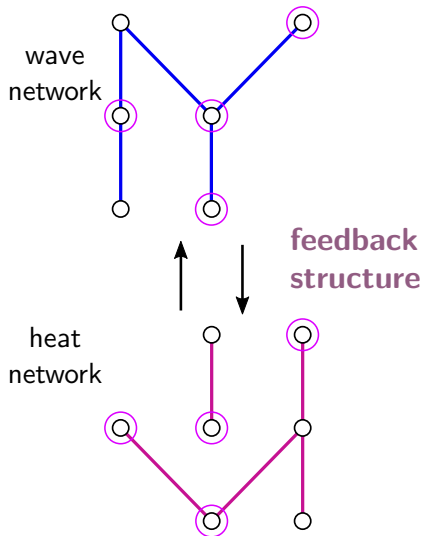


Wave equations and heat equations

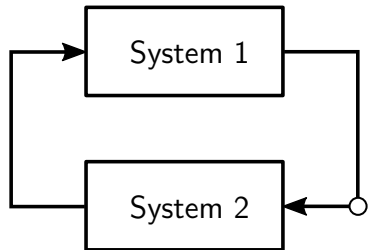
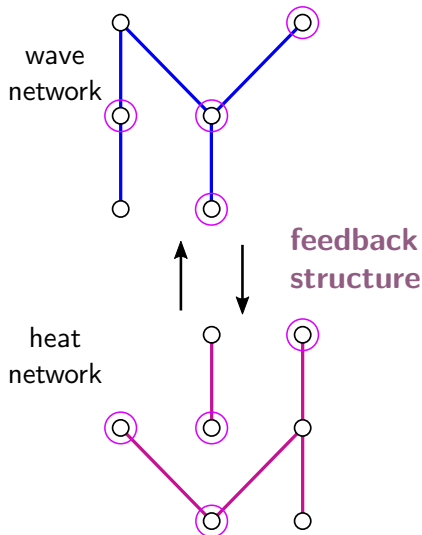
Wave-Heat Networks – Decoupled



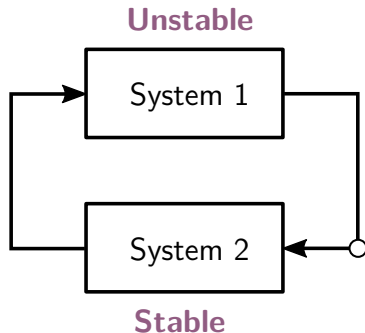
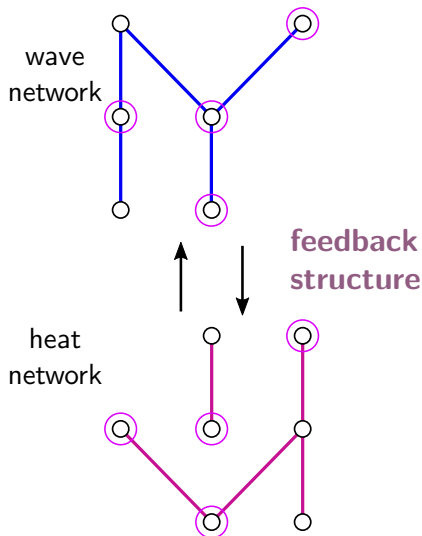
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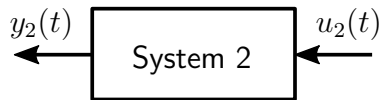
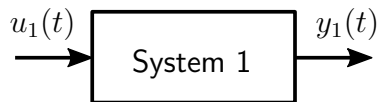
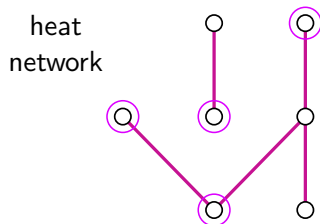
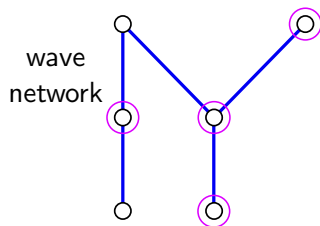
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Wave-Heat Networks – Decoupled

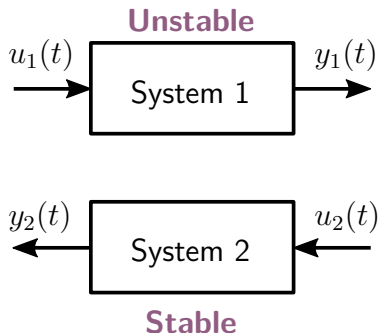


Inputs and Outputs



Problem

Use the properties of the two systems to deduce stability of the coupled PDE.

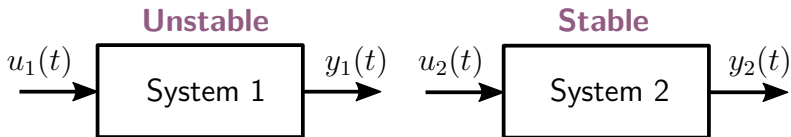


Benefits:

- “*Divide and conquer*”
- Reduce to well-known parts
- Modularity

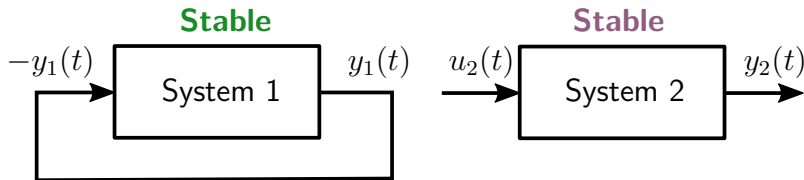
Assumption

- The two systems are **impedance passive**
 - They have no “internal sources of energy”



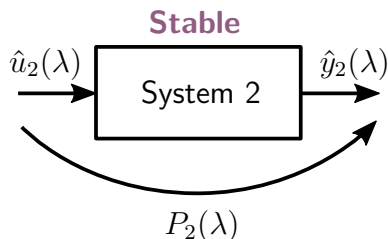
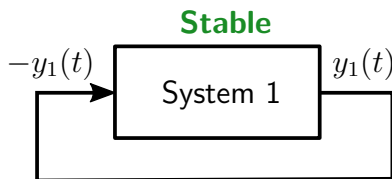
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- The two systems are **impedance passive**
 - They have no “internal sources of energy”
- **System 1** is stable under (**neg**) **output feedback**
 - This often represents addition of **damping** to System 1.
- Knowledge of the **transfer function** $P_2(\lambda)$ of **System 2**
 - $P_2(\lambda)$ describes how “Laplace transform of u_2 is mapped to Laplace transform of y_2 ”, i.e., $\hat{y}_2(\lambda) = P_2(\lambda)\hat{u}_2(\lambda)$.



Main Stability Result

Theorem

Assume the following:

- *The two systems are **impedance passive***
- *System 1 becomes stable under **(neg) output feedback** and we know the energy decay rate $E_1(t) \rightarrow 0$*
- *We have an estimate for the **transfer function** $P_2(\lambda)$ of System 2 on the imaginary axis*

Then:

*The network is stable and we have
a very good estimate for the decay rate of $E(t) \rightarrow 0$.*

A Quick Mathematical Overview

- The abstract model for the mixed-type networks is a coupled system of two **boundary control systems**,

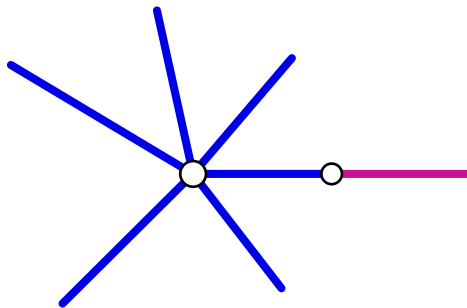
$$\begin{aligned}\dot{z}_1(t) &= L_1 z_1(t), & t \geq 0, \\ \dot{z}_2(t) &= L_2 z_2(t), & t \geq 0, \\ G_1 z_1(t) &= K_2 z_2(t), & t \geq 0, \\ G_2 z_2(t) &= -K_1 z_1(t), & t \geq 0\end{aligned}$$

- Has a contraction semigroup [Aalto–Malinen 2013]
- Our main results establish **resolvent estimates**

$$\|(is - A)^{-1}\| \leq M_1(s)M_2(s), \quad \forall s \in \mathbb{R},$$

and these lead to **polynomial** or **semi-uniform stability**.

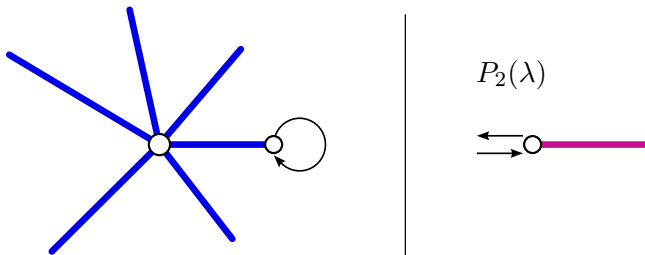
Application to Wave-Heat Networks



Question

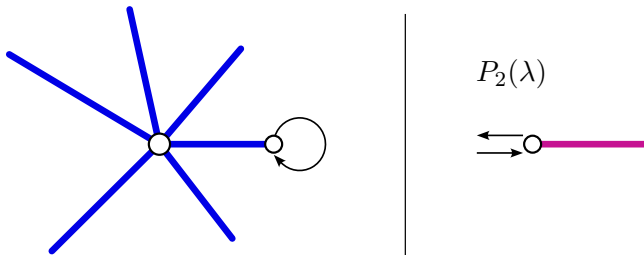
Decay rate for energy $E(t)$ in a network of N wave equations and 1 heat equation?

Wave/Heat Decoupling



- In the wave network (System 1) the “output feedback” is equivalent to damping at a single exterior node.
 $\leadsto$ Stability results in the literature
- Transfer function $P_2(\lambda)$ of the heat equation (System 2) can be computed explicitly.

Wave/Heat Decoupling

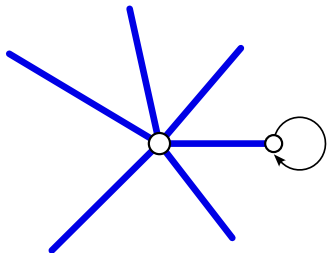


Theorem

If decay rate for classical solutions in the **decoupled** wave-network is $E_1(t) \sim t^{-\alpha}$, $\alpha > 0$, then the energy $E(t)$ of classical solutions of the **wave-heat network** has decay

$$E(t) \sim t^{-\frac{4\alpha}{4+\alpha}}, \quad t \rightarrow \infty. \quad (\text{slower})$$

Wave/Heat Decoupling



$$P_2(\lambda)$$



Theorem (Using Valein–Zuazua '09)

Assume that **decoupled** wave-network is **star-shaped** with same wave speeds, and ratios of lengths of the waves are irrational numbers “of constant type”. Then energy of classical solutions has decay

$$E(t) \sim t^{-\frac{4}{4N-3}}, \quad t \rightarrow \infty.$$

A Quick Literature Survey

- PDE networks — single-type
 - Valein–Zuazua, Ammari–Jellouli, Ammari–Shel, Nicaise–Valein, Kramar-Fijavž et. al., Augner, ...
- Coupled PDEs — two equations or simple network structures (3–4 PDEs)
 - Zhang–Zuazua, Duyackerts, Li–Wang, Augner, Avalos–Lasiecka–Triggiani, Rao–Zhang, Avalos–Geredeli, ...
- Coupled abstract systems
 - Ben Ait Hassi et. al., Ammari et. al., Feng–Guo, LP, Boulouz–Bounit–Hadd, Dell’Oro–LP–Seifert, ...

Conclusions

In this talk:

- Study of dynamics and stability of networks of partial differential equations of mixed type.
- Main result on energy decay rate for the full network in terms of the properties of simpler networks.



S. Nicaise, LP, D. Seifert “Stability of Abstract Boundary-Coupled Systems”, Journal of Functional Analysis, 2025.



LP, “On polynomial stability of coupled partial differential equations in 1D” Proceedings of SOTA 2018, arXiv:1911.06715

Thank You!