Non-Uniform Stability of Damped Contraction Semigroups

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IWOTA

Helsinki, Finland

August, 2023

Goal of the Talk

*Introduce general conditions for non-uniform stability of **damped** hyperbolic Cauchy problems (and PDEs).*

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$$\begin{cases} \dot{x}(t) = (A - BB^*)x(t) \\ x(0) = x_0 \end{cases}$$

and

$$\begin{cases} \ddot{w}(t) + Lw(t) + DD^*\dot{w}(t) = 0 \\ w(0) = w_0, \quad \dot{w}(0) = w_1 \end{cases}$$

Problem

Formulate conditions on (A, B) and (L, D) such that

$$\|x(t)\| \to 0, \qquad \textit{or} \qquad \|w(t)\| \to 0 \qquad \textit{as} \quad t \to \infty$$

*and especially study the **rate** of the convergence.*

Main Assumptions (roughly, to keep things simple)

- A generates a contraction semigroup e^{At} on a Hilbert space X, i.e., $\|e^{At}\| \leq 1$.
- Either $B \in \mathcal{L}(U, X)$, or (A, B, B^*) is a "well-posed system".
- $\Rightarrow A - BB^*$ generates a contraction semigroup $e^{(A-BB^*)t}$

Polynomial and Non-Uniform Stability

Definition

$e^{(A-BB^*)t}$ generated by $A - BB^*$ is **non-uniformly stable** if there exist an unbounded increasing $N_T \colon [t_0, \infty) \to \mathbb{R}_+$ and $C > 0$ s.t.

$$\|e^{(A-BB^*)t}x_0\| \leq \frac{C}{N_T(t)}\|(A-BB^*)x_0\| \qquad x_0 \in D(A-BB^*)$$

[. . . , Liu–Rao ’05, Batty–Duyckaerts ’08, Borichev–Tomilov ’10, Rozendaal–Seifert–Stahn ’19]

Application: $E(t) \sim \|e^{(A-BB^*)t}x_0\|^2$ for many PDE systems.

Terminology: “Non-uniform stability” = “Semi-uniform stability”

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Theorem (BT'10, RSS'19)

Assume $e^{(A-BB^*)t}$ *is contractive,* $i\mathbb{R} \subset \rho(A-BB^*)$*, and*

$$\|(is - A + BB^*)^{-1}\| \le N(|s|), \qquad N \text{ non-decreasing.}$$

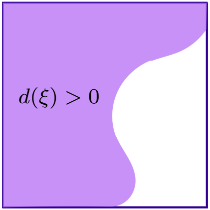
- *If* $N(s) \lesssim 1 + s^\alpha$*, then* $N_T(t) = t^{1/\alpha}$
- *If* N *has "positive increase", then* $N_T(t) = N^{-1}(t)$*.*

Damped Wave Equations

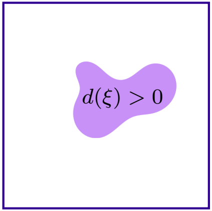
Non-uniform stability is encoutered in wave/beam/plate equations with **partial** or **weak** dampings. In the 2D wave equation

$$\begin{aligned} \ddot{w}(\xi,t) - \Delta w(\xi,t) + d(\xi)\dot{w}(\xi,t) &= 0, && \xi \in \Omega, \quad t > 0 \\ w(\xi,t) &= 0 && \xi \in \partial\Omega \end{aligned}$$

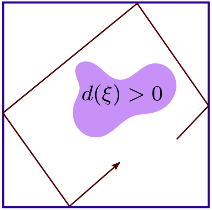
stability depends on geometry of Ω and $\omega := \{\, \xi \in \Omega \mid d(\xi) > 0 \,\}$:



Exponential stability



Non-uniform stability



Geometric Control Condition

Goal of the Talk

Introduce general conditions for non-uniform stability of ***damped*** *hyperbolic Cauchy problems and PDEs.*

Damped systems of the form

$$\dot{x}(t) = (A - BB^*)x(t) \quad \text{and} \quad \ddot{w}(t) + Lw(t) + DD^*\dot{w}(t) = 0$$

Motivation:

- So-called "polynomial" and "non-uniform" stability often arise in wave/beam/plate equations with weak or partial dampings
- Most of the current literature based on case-by-case analysis

Main results:

- General **observability-type** sufficient conditions for stability

(B^*, A) **exactly** observable $\Leftrightarrow$ $A - BB^*$ exponentially stable

$$\|x(t)\| \leq Me^{-\omega t}\|x_0\| \ \forall x_0$$

(B^*, A) **approx.** observable $\Leftrightarrow^*$ $A - BB^*$ strongly/weakly stable

$$x(t) \to 0 \ \forall x_0$$

[Slemrod, Levan, Russell, Benchimol, Guo–Luo, Lasiecka–Triggiani, Curtain–Weiss . . .]

(B^*, A) **exactly** observable $\Leftrightarrow$ $A - BB^*$ exponentially stable
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(B^*, A) **non-uniformly** obs. $\Leftrightarrow$ $A - BB^*$ non-uniformly stable

(B^*, A) **approx.** observable $\Leftrightarrow^*$ $A - BB^*$ strongly/weakly stable
$x(t) \to 0 \ \forall x_0$

[Slemrod, Levan, Russell, Benchimol, Guo–Luo, Lasiecka–Triggiani, Curtain–Weiss ...]

Earlier work: Ammari–Tucsnak 2001, Ammari et. al.
Anantharaman–Léautaud 2014, Joly–Laurent 2019

Main Results

A Non-Uniform Hautus Test

Consider the Hautus-type condition [Miller 2012]

$$\|x\|^2 \le M(|s|)\|(is-A)x\|^2 + m(|s|)\|B^*x\|^2, \quad x \in D(A), s \in \mathbb{R},$$

for some non-decreasing $M, m\colon [0,\infty) \to [r_0,\infty)$.

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for some non-decreasing $M, m \colon [0, \infty) \to [r_0, \infty)$.

Theorem

If the above condition holds, then $i\mathbb{R} \subset \rho(A - BB^*)$. If $N(s) := M(s) + m(s)$ has positive increase, then

$$\|e^{(A-BB^*)t}x_0\| \leq \frac{C}{N^{-1}(t)}\|(A - BB^*)x_0\|, \quad x_0 \in D(A - BB^*)$$

Observability of the Schrödinger Group

For

$$\ddot{w}(t) + Lw(t) + DD^*\dot{w}(t) = 0, \qquad \text{on} \quad H$$

and $M_S, m_S \colon [0,\infty) \to [r_0,\infty)$ consider ($s \geq 0$)

$$\|w\|^2 \leq M_S(s)\|(s^2 - L)w\|^2 + m_S(s)\|D^*w\|^2, \quad w \in D(L)$$

This is **observability of the "Schrödinger group"** (D^*, iL) (generalises Anantharaman–Léautaud 2014, Joly–Laurent 2019)

Theorem

A similar result, decay rate determined by $N^{-1}(t)$, where

$$N(s) := M_S(s)m_S(s)(1+s^2).$$

The Wavepacket Condition

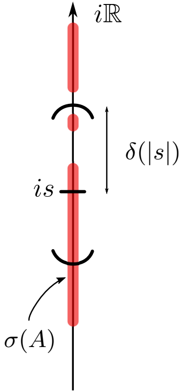
For A skew-adjoint with spectral projection $P_{(a,b)}$ (for $i(a,b) \subset i\mathbb{R}$)

$$\|B^*x\| \geq \gamma(|s|)\|x\|, \qquad x \in \operatorname{Ran}(P_{(s-\delta(|s|),s+\delta(|s|))}), \; s \in \mathbb{R}$$

for some non-increasing $\delta, \gamma \colon [0,\infty) \to (0, r_0]$.

Such x are often called **"wavepackets"** of A.

(Used for exact observability, e.g., in Ramdani et. al. 2005, Miller 2012, Tucsnak–Weiss 2009.)



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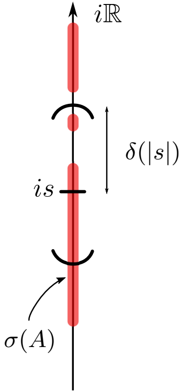


Theorem

If $A^* = -A$ and if $N(s) := \delta(s)^{-2}\gamma(s)^{-2}$ has positive increase, then

$$\|e^{(A-BB^*)t}x_0\| \leq \frac{C}{N^{-1}(t)}\|(A-BB^*)x_0\|.$$

Time-Domain Non-Uniform Observability

Time-domain observability conditions:

For $\tau, c_\tau, \beta > 0$:

$$c_\tau \|(1-A)^{-\beta} x_0\|^2 \le \int_0^\tau \|B^* e^{At} x_0\|^2 dt, \qquad x_0 \in D(A).$$

(cf. generalised observability conditions by Ammari–Tuscnak 2001, Ammari–Bchatnia–El Mufti 2017)

Theorem

Assume $D(A^*) = D(A)$, $B \in \mathcal{L}(U, X)$ and $0 < \beta \le 1$. If the above condition holds, then $i\mathbb{R} \subset \rho(A - BB^*)$, and

$$\|e^{(A-BB^*)t} x_0\| \le \frac{C}{t^{1/(2\beta)}} \|Ax_0\|, \quad x_0 \in D(A)$$

Examples: 2D Wave Equations

A wave equation with viscous damping on a convex $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary, $d \in L^\infty(\Omega)$, $d \geq 0$

$$
\begin{aligned}
&w_{tt}(\xi,t) - \Delta w(\xi,t) + d(\xi) w_t(\xi,t) = 0, && \xi \in \Omega,\ t > 0,\\
&w(\xi,t) = 0, && \xi \in \partial\Omega,\ t > 0,\\
&w(\cdot,0) = w_0(\cdot) \in H^2(\Omega) \cap H_0^1(\Omega), && w_t(\cdot,0) = w_1(\cdot) \in H_0^1(\Omega).
\end{aligned}
$$

- Several results exist for the exact observability of the Schrödinger group $(D^*, i\Delta)$ (Jaffard '90, Burq–Zworski '19) for rectangles/tori. Leads to polynomial decay $1/\sqrt{t}$.
- Precise lower bounds on d lead to generalised observability of the Schrödinger group via Burq–Zuily 2016.
- In general our results are sub-optimal, since conditions do not take into account the **smoothness** of d! (Burq–Hitrik '07)

1D Wave Equations

Consider a wave equation with weak damping (and Dirichlet BC)

$$w_{tt}(\xi,t) - w_{\xi\xi}(\xi,t) + d(\xi)\int_0^1 d(r)w_t(r,t)dr = 0, \quad \xi \in (0,1), \; t > 0,$$

- The wavepacket condition characterises (optimal) stability via lower bounds of the sine Fourier coefficients, e.g., $(c, \alpha > 0)$

$$\left|\int_0^1 d(\xi)\sin(n\pi\xi)d\xi\right| \geq \frac{c}{n^\alpha}$$

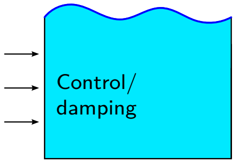
- Pointwise damping possible (formally $d(\xi) = \delta(\xi - \xi_0)$).
- Analogous results for Euler–Bernoulli / Timoshenko beams

Application: Water Waves System

In the reference

Su–Tucsnak–Weiss "Stabilizability properties of a linearized water waves system," *Systems & Control Letters*, 2020.

the results were applied to prove non-uniform stabilizability of a "**water waves system**" in a 2D domain.



- The PDE system models small amplitude gravity water waves
- Stability and convergence rate proved using the "Wavepacket condition", A has eigenvalues $\lambda_k \approx i\sqrt{k}$
- $\delta(|s|) \to 0$ so that $(is - \delta(|s|), is + \delta(|s|))$ reduce to 1D spectral subspaces.
- The stability result is likely to be optimal.

Conclusions

In this presentation:

- General sufficient conditions for non-uniform stability of the semigroup generated by $A - BB^*$.
- Discussion of PDE examples and optimality of the results

R. Chill, LP, D. Seifert, R. Stahn, Y. Tomilov, "Non-Uniform Stability of Damped Contraction Semiroups," *Analysis & PDE*, 2023 (https://arxiv.org/abs/1911.04804)

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Thank You!

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