The Infinite-Dimensional Sylvester Differential Equation and Periodic Output Regulation

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The Main Problem

Problem

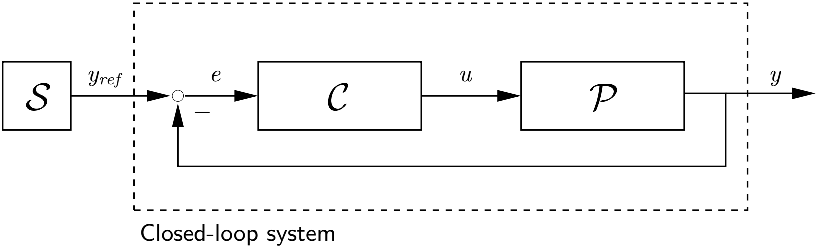
Study

$$\dot{\Sigma}(t) + \Sigma(t)S = A_e(t)\Sigma(t) + B_e(t). \qquad \text{(SDe)}$$

Motivation:

- The Periodic Output Regulation Problem
- Characterization of controllers solving the PORP.

Introduction



Motivation

Definition (The periodic signal generator $\mathcal{S}$)

The exosystem is of form

$$\dot{v} = Sv, \qquad v(0) \in W = \mathbb{C}^q,$$
$$y_{ref}(t) = -F(t)v(t)$$

$S \in \mathcal{L}(W)$ and $F \in C_T^1(\mathbb{R}, \mathcal{L}(W, Y))$.

Here

$$C_T^1(\mathbb{R}, X) = \{\, f \in C^1(\mathbb{R}, X) \mid f(t+T) = f(t) \text{ for all } t \in \mathbb{R} \,\}.$$

The Closed-Loop System

The closed-loop system is of form

$$\begin{aligned} \dot{x}_e &= A_e(t)x_e + B_e(t)v, \qquad x_e(0) \in X_e \\ e &= C_e(t)x_e + D_e(t)v. \end{aligned}$$

Assume there exists a *strongly continuous evolution family* $U_e(t,s)$ associated to the family $(A_e(t), \mathcal{D}(A_e(t)))$.

The Evolution Family $U_e(t, s)$

Definition (A Strongly Continuous Evolution Family)

1. $U_e(t, t) = I$
2. $U_e(t, r)U_e(r, s) = U_e(t, s)$ for $s \leq r \leq t$
3. $\{ (t, s) \mid t \geq s \} \ni (t, s) \mapsto U_e(t, s)$ is strongly continuous.

In the finite-dimensional case: $U_e(t, s) = e^{\int_s^t A_e(r) dr}$.

If $A_e(t) \equiv A_e$, generates a semigroup: $U_e(t, s) = T_e(t - s)$.

The Evolution Family $U_e(t,s)$

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The state of the closed-loop system

$$x_e(t) = U_e(t,0)x_e(0) + \int_0^t U_e(t,s)B_e(s)v(s)ds.$$

The Periodic Output Regulation Problem

Problem (Periodic Output Regulation Problem)

Choose the controller such that

- *The CL system is exponentially stable, i.e. there exist* $M_e, \omega_e > 0$ *s.t.*

$$\|U_e(t,s)\| \leq M_e e^{-\omega_e(t-s)}, \qquad t \geq s$$

- *For all initial states* $x_e(0) \in X_e$ *and* $v(0) \in W$ *the regulation error satisfies*

$$e(t) \longrightarrow 0$$

as $t \to \infty$.

Theorem (Theorem 1)

Assume the controller stabilizes the closed-loop system exponentially.

1. *The periodic Sylvester differential equation*

$$\dot{\Sigma}(t) + \Sigma(t)S = A_e(t)\Sigma(t) + B_e(t)$$

   has a unique periodic solution $\Sigma_\infty(\cdot)$.

2. *The controller solves the PORP if and only if this solution satisfies*

$$C_e(t)\Sigma_\infty(t) + D_e(t) = 0$$

   for all $t \in [0, T]$.

The Main Problem

Problem

Study

$$\dot{\Sigma}(t) + \Sigma(t)S = A_e(t)\Sigma(t) + B_e(t). \tag{SDe}$$

What are the required assumptions for Theorem 1.

- The unique periodic solution
- The type of SDe (strong/weak)?
- Corresponding additional conditions for (2) in Theorem 1?

The Main Problem

Problem

Study

$$\dot{\Sigma}(t) + \Sigma(t)S = A_e(t)\Sigma(t) + B_e(t). \qquad \text{(SDe)}$$

Main results:

1. Solvability of the strong SDe + additional conds for (2)
2. Solvability of the weak SDe + additional conds for (2).

Standing Assumptions

Assumption

- $A_e(t) = A_{sg} + A_b(t)$ *on a Hilbert space* X_e, *where*
  - A_{sg} *gen. a* C_0*-semigroup*
  - $A_b(\cdot)x \in C_T^1$ *for all* $x \in X_e$
  - $\mathcal{D}(A_e(t)) \equiv \mathcal{D}(A_{sg}) =: \mathcal{D}(A_e)$
  - $\mathcal{D}(A_e(t)^*) \equiv \mathcal{D}(A_{sg}^*) =: \mathcal{D}(A_e^*)$
- $B_e(\cdot) \in C_T^1(\mathbb{R}, X_e)$
- $U_e(t,s)$ *is exp. stable, i.e.* $\|U_e(t,s)\| \leq M_e e^{-\omega_e(t-s)}$, $\omega_e > 0$
- $\|e^{St}\| \geq M_S > 0$ *for* $t \geq 0$.

Strong Solution

Theorem (The Strong SDe)

If for every $v \in W$ we have

$$\int_{-\infty}^{0} U_e(0,s)B_e(s)e^{Ss}v\,ds \;\in\; \mathcal{D}(A_e),$$

then the equation

$$\dot{\Sigma}(t) + \Sigma(t)S = A_e(t)\Sigma(t) + B_e(t)$$

has a unique periodic solution $\Sigma_\infty(\cdot) \in C_T^1$ given by

$$\Sigma_\infty(t) = \int_{-\infty}^{t} U_e(t,s)B_e(s)e^{S(s-t)}\,ds$$

such that $\mathcal{R}(\Sigma_\infty(t)) \subset \mathcal{D}(A_e)$ for all $t \in [0, T]$.

The Parabolic Case

Corollary

If $A_e(t) = A_{sg} + A_b(t)$ *where* A_{sg} *generates an analytic semigroup, the equation*

$$\dot{\Sigma}(t) + \Sigma(t)S = A_e(t)\Sigma(t) + B_e(t)$$

has a unique periodic solution $\Sigma_\infty(\cdot) \in C_T^1$ *given by*

$$\Sigma_\infty(t) = \int_{-\infty}^{t} U_e(t,s)B_e(s)e^{S(s-t)}\,ds$$

such that $\mathcal{R}(\Sigma_\infty(t)) \subset \mathcal{D}(A_e)$ *for all* $t \in [0, T]$.

Assumptions for Theorem 1

Theorem (Additional assumptions for (2) in Theorem 1)

If (SDe) *has a strong solution, no additional assumptions required:*

The controller solves the PORP if and only if

$$C_e(t)\Sigma_\infty(t) + D_e(t) \equiv 0.$$

The Weak Sylvester Differential Equation

Problem

Study

$$\frac{d}{dt}\langle \Sigma(t)v, y\rangle + \langle \Sigma(t)Sv, y\rangle = \langle \Sigma(t)v, A_e(t)^* y\rangle + \langle B_e(t)v, y\rangle,$$

for all $v \in W$, $y \in \mathcal{D}(A_e^*)$.

Weak Solution

Theorem (The Weak SDe)

The equation

$$\frac{d}{dt}\langle \Sigma(t)v, y\rangle + \langle \Sigma(t)Sv, y\rangle = \langle \Sigma(t)v, A_e(t)^* y\rangle + \langle B_e(t)v, y\rangle,$$

for all $v \in W$, $y \in \mathcal{D}(A_e^*)$, *has a unique periodic solution*

$$\Sigma_\infty(t) = \int_{-\infty}^{t} U_e(t,s)B_e(s)e^{S(s-t)}\,ds.$$

Assumptions for Theorem 1

Theorem (Additional assumptions for (2) in Theorem 1)

Assume

- $U_e(t,s)^*(\mathcal{D}(A_e^*)) \subset \mathcal{D}(A_e^*)$
- $\frac{d}{ds} U_e(t,s)^* y = -A_e(s)^* U_e(t,s)^* y$ *for all* $y \in \mathcal{D}(A_e^*)$.

The controller solves the PORP iff $C_e(t)\Sigma_\infty(t) + D_e(t) \equiv 0$.

For $A_e(t) = A_{sg} + A_b(t)$ these are satisfied if $\mathcal{R}(A_b(t)^*) \subset \mathcal{D}(A_{sg}^*)$ and if

$$t \mapsto A_{sg}^* A_b(t)^*$$

is strongly continuous.

Assumptions for Theorem 1

Theorem (Additional assumptions for (2) in Theorem 1)

Assume

- $U_e(t,s)^*(\mathcal{D}(A_e^*)) \subset \mathcal{D}(A_e^*)$
- $\frac{d}{ds} U_e(t,s)^* y = -A_e(s)^* U_e(t,s)^* y$ *for all* $y \in \mathcal{D}(A_e^*)$.

The controller solves the PORP iff $C_e(t)\Sigma_\infty(t) + D_e(t) \equiv 0$.

For an observer-based controller of a DPS (A, B, C):

- $\mathcal{R}(C^*) \subset \mathcal{D}(A^*)$
- There exists $K \in \mathcal{L}(X, U)$ with $\mathcal{R}(K^*) \subset \mathcal{D}(A^*)$ such that $A + BK$ is exponentially stable.

More General Sylvester Differential Equations

Problem

Consider

$$\dot{\Sigma}(t)v + \Sigma(t)S(t)v = A(t)\Sigma(t)v + B(t)v, \quad \Sigma(0) = \Sigma_0$$

for all $v \in \mathcal{D}(S)$ *where*

- *The families* $(A(t), \mathcal{D}(A))$ *and* $(S(t), \mathcal{D}(S))$ *have associated evolution families* $U_A(t,s)$ *and* $U_S(t,s)$
- $B(\cdot)v \in C_{ub}$.

The solution is of form

$$\Sigma(t) = U_A(t,0)\Sigma_0 U_S(0,t) + \int_0^t U_A(t,s)B(s)U_S(s,t)\,ds.$$

Conclusion

In this presentation:

- Periodic Sylvester differential equation
- Conditions for
  - Solvability
  - Theorem 1 (Characterization of the solvability of the PORP).

Further research topics:

- Further conditions
- Possible to get rid of exponential stability?
- Banach space.