Stability of Waves with Boundary Damping: Sharpening the Observability Knife

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joint work with R. Chill, D. Seifert, R. Stahn, Y. Tomilov and N. Vanspranghe.

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Goal of the Talk

Objective: *Investigate energy decay in 2-dimensional wave equations with boundary damping.*

Steps:

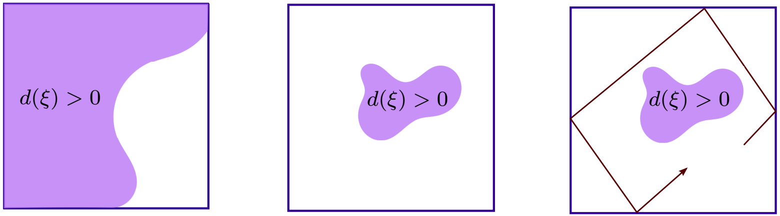
- Discuss the PDE problem
- Introduce abstract tools based on observability estimates
- New results for sharpening the observability results for waves
- Discussion on sub-optimality due to overdamping

Damped Wave Equations

Consider the wave equation on a "nice" domain $\Omega \subset \mathbb{R}^2$,

$$\begin{aligned} \ddot{w}(\xi,t) - \Delta w(\xi,t) + d(\xi)\dot{w}(\xi,t) &= 0, \qquad && \xi \in \Omega, \quad t > 0 \\ w(\xi,t) &= 0 && \xi \in \partial\Omega \end{aligned}$$

Then the stability of the wave equation depends on geometry of Ω and $\omega := \{\, \xi \in \Omega \mid d(\xi) > 0 \,\}$:



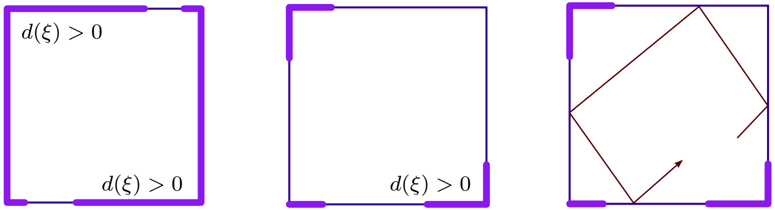
Exponential stability | Non-uniform stability | Geometric Control Condition

Wave Equations with Boundary Damping

Consider the wave equation on a domain $\Omega \subset \mathbb{R}^2$, with boundary Γ,

$$\begin{aligned} \ddot{w}(\xi,t) - \Delta w(\xi,t) &= 0, && \xi \in \Omega, \quad t > 0 \\ \nu \cdot \nabla w(\xi,t) + d(\xi)\dot{w}(\xi,t) &= 0 && \xi \in \Gamma \end{aligned}$$

Then the stability of the wave equation again depends on Ω and now on $\omega := \{\, \xi \in \Gamma \mid d(\xi) > 0 \,\}$ [Bardos, Lebeau, Rauch '92]:

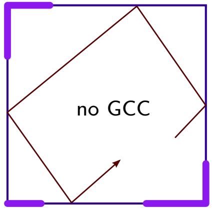


Exponential stability Non-uniform stability GCC

Wave Equations with Boundary Damping

Wave equation with boundary damping:

$$\ddot{w}(\xi,t) - \Delta w(\xi,t) = 0, \qquad \xi \in \Omega$$
$$\nu \cdot \nabla w(\xi,t) + d(\xi)\dot{w}(\xi,t) = 0, \quad \xi \in \Gamma$$



- Lack of exponential stability understood in the non-GCC case
- Concrete decay rates in the non-GCC have not been investigated much (unlike for in-domain dampings!)
- Recent abstract results yield rates based on observability estimates

Abstract Formulation

The wave equation on Ω can be formulated as an "abstract damped wave equation"

$$\begin{cases} \ddot{w}(t) + Lw(t) + DD^*\dot{w}(t) = 0 \\ w(0) = w_0, \quad \dot{w}(0) = w_1 \end{cases}$$

on the Hilbert space $H = L^2(\Omega)$ with $L = -\Delta_{\mathsf{Neu}} \geq 0$. In boundary damping, D is "unbounded" in the sense that $\mathrm{Ran}(D) \not\subset L^2(\Omega)$, but instead $D \in \mathcal{L}(U, \mathrm{Dom}(L^{1/2})^*)$.

Problem

Formulate conditions on (L, D) such that for all initial conditions

$$\|L^{1/2}w(t)\| + \|\dot{w}(t)\| \to 0 \qquad \textit{as} \quad t \to \infty$$

*and especially study the **rate** of the convergence. ($\sim \sqrt{E(t)}$)*

Polynomial and Non-Uniform Stability

$$\ddot{w}(t) + Lw(t) + DD^*\dot{w}(t) = 0, \qquad w(0) = w_0, \quad \dot{w}(0) = w_1$$

Definition (Non-Uniform Stability)

There exists an increasing unbounded $M(\cdot) : [t_0, \infty) \to (0, \infty)$ s.t.

$$\|L^{1/2}w(t)\| + \|\dot{w}(t)\| \leq \frac{1}{M(t)} \left(\|Lw_0 + DD^*w_1\| + \|L^{1/2}w_1\| \right),$$

for $t \geq t_0$ and for all initial conditions $w_0 \in H$, $w_1 \in \mathrm{Dom}(L^{1/2})$ satisfying $Lw_0 + DD^*w_1 \in H$.

- w_0, w_1 correspond to **classical solutions** of the PDE.
- In **Uniform Exponential Stability** all (mild) solutions decay at an exponential rate for all $w_0 \in \mathrm{Dom}(L^{1/2})$ and $w_1 \in H$.
- In **Polynomial Stability** $M(t) = ct^{\beta}$ for some $\beta, c > 0$.

Polynomial and Non-Uniform Stability

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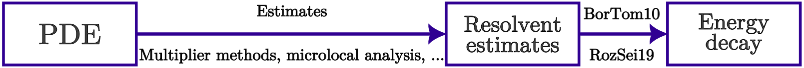
where $w_0 \in H$, $w_1 \in \mathrm{Dom}(L^{1/2})$ satisfy $Lw_0 + DD^*w_1 \in H$.

Borichev–Tomilov 2010, Rozendaal–Seifert–Stahn '19:
Non-uniform stability and M are characterised by resolvent bounds $\|(is - A)^{-1}\| \lesssim N(s)$ for the semigroup generator A

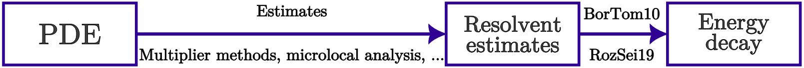
$$A = \begin{bmatrix} 0 & I \\ -L & -DD^* \end{bmatrix}$$

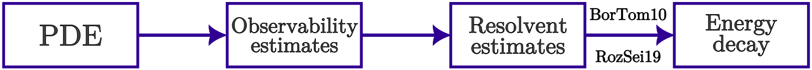
and $M \approx N^{-1}$. E.g., if $N(s) = 1 + s^\alpha$, then $M(t) = ct^{1/\alpha}$.

Deriving Non-Uniform Decay Rates



Deriving Non-Uniform Decay Rates





- "Observability estimates" aim to reduce the derivation of resolvent estimates to a simpler problem.
- Instead of the damped problem, these involve the undamped equation "with an output"
- Earlier for NU stability: Ammari–Tucsnak 2001, Ammari et. al., Anantharaman–Léautaud 2014, Joly–Laurent 2019

Main Results

Main Results: The Plan

- I'll present two observability-type conditions, both lead to energy decay rates.
- Since we are interested in **boundary damping**, we highlight features that arise from D being unbounded.
- First step: we define a non-decreasing $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$1 + s\|D^*((1+is)^2 - L)^{-1}D\| \leq \mu(s), \qquad s \geq 0.$$

  For the boundary damped wave equation, μ is **unbounded**.
- The resolvent identity implies that $\mu(s) \lesssim 1 + s^2$ **always**, but in reality growth can be much slower.

A Non-Uniform (Schrödinger) Hautus Test

Consider the Hautus-type condition [Miller 2012]

$$\|w\|^2 \leq M_0(s)\|(s^2 - L)w\|^2 + m_0(s)\|D^* w\|^2, \quad w \in \mathrm{Dom}(L), s \geq 0$$

for some non-decreasing $M_0, m_0 \colon [0, \infty) \to [r_0, \infty)$.

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for some non-decreasing $M_0, m_0 \colon [0,\infty) \to [r_0,\infty)$.

Theorem (Chill–P–Seifert–Stahn–Tomilov '23)

If the condition holds and $N(s) := M_0(s)m_0(s)\mu(s)^2(1+s^2)$*, then*

$$\|L^{1/2}w(t)\| + \|\dot{w}(t)\| \le \frac{C}{N^{-1}(t)} \left(\|Lw_0 + DD^* w_1\| + \|L^{1/2}w_1\|\right),$$

for some $C, t_0 > 0$.

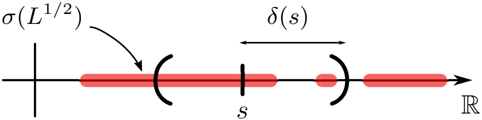
- Recall that $1 + s\|D^*((1+is)^2 - L)^{-1}D\| \le \mu(s)$
- Generalises Anantharaman–Léautaud 2014, Joly–Laurent 2019

A "Wavepacket Condition"

Operator $L^{1/2} \geq 0$ has spectral projections $P_{(a,b)}$ (for $(a,b) \subset \mathbb{R}_+$). Assume

$$\|D^* w\| \geq \gamma(s)\|w\|, \qquad w \in \operatorname{Ran}(P_{(s-\delta(s),s+\delta(s))}), \ s > 0$$

for some non-increasing $\delta, \gamma \colon [0,\infty) \to (0, r_0]$.



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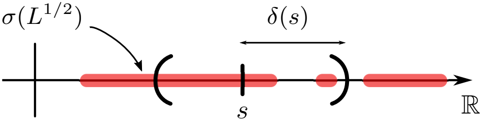

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Theorem (Chill–P–Seifert–Stahn–Tomilov '23)

If $N(s) := \mu(s)^2/(\gamma(s)^2\delta(s)^2)$ *has "positive increase", then*

$$\|L^{1/2}w(t)\| + \|\dot{w}(t)\| \leq \frac{C}{N^{-1}(t)} \left(\|Lw_0 + DD^*w_1\| + \|L^{1/2}w_1\|\right),$$

Results: Summary

We get decay rates $\sim 1/N^{-1}(t)$ from both conditions, where

$$\begin{cases} N(s) := M_0(s)m_0(s)\mu(s)^2(1+s^2) & \text{Schrödinger Hautus test} \\ N(s) := \dfrac{\mu(s)^2}{\delta(s)^2\gamma(s)^2} & \text{Wavepacket condition} \end{cases}$$

That is, we get energy decay rates based on

- The observability condition parameters: (M_0, m_0) or (δ, γ)
- The function μ s.t. $1 + s\|D^*((1+is)^2 - L)^{-1}D\| \leq \mu(s)$
- If we don't have a good estimate for μ, we can use $\mu(s)^2 \lesssim 1 + s^4$. But this may result in a **significant loss of optimality**, and the "**observability knife is very blunt**".
- The presence of μ in the conditions is not a technicality, but examples illustrate that **it cannot be omitted**.

Discussion on Observability Conditions

In the observability condition approach we analyse

$$\ddot{w}(t) + Lw(t) + DD^*\dot{w}(t) = 0,$$

by studying its **undamped** version, but equipped with an **input** u and **output** y:

$$\begin{aligned} \ddot{w}(t) + Lw(t) &= Du(t), \\ y(t) &= D^*\dot{w}(t). \end{aligned}$$

- We get the damped equation with “feedback” $u(t) = -y(t)$.
- Motivation: The equation without damping is simpler.

Discussion on Observability Conditions

"Observability" measures (roughly) how well the state $(w(t), \dot{w}(t))$ can be reconstructed based on the output signal y in

$$\begin{aligned} \ddot{w}(t) + Lw(t) &= 0, \\ y(t) &= D^* \dot{w}(t). \end{aligned}$$

- (M_0, m_0) and (δ, γ) measure the observability properties.
- The growth of $\mu(s)$ (i.e. $s\|D^*((1+is)^2 - L)^{-1}D\|$) measures the **level of unboundedness of** D.
- The role of μ in the results can be interpreted roughly as: "stronger unboundedness of D may cause worse overdamping"

Sharpening the Knife: Accurate Estimation of μ

Summary of main results:

- The "baseline estimate" $\mu(s)^2 \lesssim 1 + s^4$ can be dramatically improved for actual PDE models $\rightsquigarrow$ improved decay rates.
- New tools for deriving these (frequency-domain) estimates based on time-domain estimates

The Wave Equation with Boundary Damping

The wave equation on $\Omega \subset \mathbb{R}^2$ with boundary Γ, $d \in L^\infty$, $d \geq 0$

$$\begin{aligned} \ddot{w}(\xi,t) - \Delta w(\xi,t) &= 0, & \xi &\in \Omega, \quad t > 0 \\ \nu \cdot \nabla w(\xi,t) + d(\xi)\dot{w}(\xi,t) &= 0 & \xi &\in \Gamma. \end{aligned}$$

Estimate $1 + s\|D^*((1+is)^2 - L)^{-1}D\| \leq \mu(s)$.

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Estimate $1 + s\|D^*((1+is)^2 - L)^{-1} D\| \leq \mu(s)$.

Proposition (LP, D. Seifert, N. Vanspranghe, '24)

We have $\mu(s) = 1 + s^\eta$ in the following cases:

$\eta = 1/2 + \varepsilon$	*for Ω rectangle*
$\eta = 1/2$	*when Γ is smooth and flat*
$\eta = 1/3$	*when Γ is smooth and concave*
$\eta = 2/3$	*when Γ is smooth*

The Wave Equation with Boundary Damping

The wave equation on $\Omega \subset \mathbb{R}^2$ with boundary Γ, $d \in L^\infty$, $d \geq 0$

$$\begin{aligned} \ddot{w}(\xi,t) - \Delta w(\xi,t) &= 0, & \xi \in \Omega, \quad t > 0 \\ \nu \cdot \nabla w(\xi,t) + d(\xi)\dot{w}(\xi,t) &= 0 & \xi \in \Gamma. \end{aligned}$$

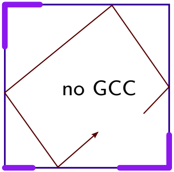
Corollary (LP, D. Seifert, N. Vanspranghe, '24)

With the observability estimates, decay rate $\sim 1/N^{-1}(t)$ *where* $N(s) = M_0(s)m_0(s)(1+s^{2+2\eta})$ *or* $N(s) = (1+s^{2\eta})/(\delta(s)\gamma(s))^2$

$\eta = 1/2 + \varepsilon$	*for* Ω *rectangle*
$\eta = 1/2$	*when* Γ *is smooth and flat*
$\eta = 1/3$	*when* Γ *is smooth and concave*
$\eta = 2/3$	*when* Γ *is smooth*

Wave Equation on a Rectangle

$$\begin{aligned} \ddot{w}(\xi,t) - \Delta w(\xi,t) &= 0, & \xi &\in \Omega \\ \nu \cdot \nabla w(\xi,t) + d(\xi)\dot{w}(\xi,t) &= 0, & \xi &\in \Gamma \end{aligned}$$



Proposition

Assume that Ω is a rectangle and there exists $\omega \subset \Gamma$ such that $\operatorname{ess\,sup}_{\xi\in\omega} d(\xi) > 0$. Then for any $\varepsilon > 0$ we have

$$\|\nabla w(\cdot,t)\|_{L^2} + \|\dot{w}(\cdot,t)\|_{L^2} = o\left(\frac{1}{t^{1/\alpha}}\right)$$

with $\alpha = 3 + \varepsilon$ for all classical solutions.

Previous [Abbas–Nicaise '15]: $\alpha = 2$ if damping on single edge

Wave Equation on a Rectangle

$$\ddot{w}(\xi,t) - \Delta w(\xi,t) = 0, \qquad \xi \in \Omega$$
$$\nu \cdot \nabla w(\xi,t) + d(\xi)\dot{w}(\xi,t) = 0, \qquad \xi \in \Gamma$$

Proposition

Assume that Ω is a rectangle and $\exists \omega \subset \Gamma$ s.t. $\operatorname{ess\,sup}_{\xi\in\omega} d(\xi) > 0$. Then for any $\varepsilon > 0$ we have $\|\nabla w(\cdot,t)\|_{L^2} + \|\dot{w}(\cdot,t)\|_{L^2} = o(t^{-1/\alpha})$ with $\alpha = 3+\varepsilon$ for all classical solutions.

Proof.

The “Schrödinger group (D^*, iL)” is “exactly observable”, and thus M_0 and m_0 can be chosen as constants. We have $\mu(s) = 1 + s^{1/2+\varepsilon}$ and thus $N(s) \lesssim 1 + s^{2+1+2\varepsilon}$. $\square$

Where Do The Values of η Come From?

An abstract result connecting μ to time-domain input-output estimates for

$$\begin{cases} \ddot{w}(t) + Lw(t) = Du(t), & w_0 = 0,\ w_1 = 0 \\ \qquad\quad y(t) = D^*\dot{w}(t). \end{cases} \qquad (*)$$

Theorem (LP, D. Seifert, N. Vanspranghe, '24)

Let $\eta \in [0,1]$. Then $1 + s\|D^*((1+is)^2 - L)^{-1}D\| \lesssim 1 + s^{2\eta}$ if and only if there exist $M, T > 0$ s.t. the solutions of $(*)$ satisfy

$$\|y\|_{H^{-2\eta}(0,T;U)} \le M\|u\|_{L^2(0,T;U)}, \qquad u \in L^2(0,T;U).$$

Such estimates the wave equation have been established by Lasiecka–Triggiani '91 and Tataru '98 $\rightsquigarrow$ The concrete values of η.

Comments on (Sub-)Optimality

- The optimal rate in the rectangle case is most likely $\alpha = 2$.
- In our results, these observability estimates (M_0, m_0) and (δ, γ) and the measure μ of unboundedness are **decoupled**.
- This **explains** suboptimality in several cases:
  - The observability conditions **need to prepare for the worst**.
  - In reality, the "observability" and "unboundedness" aspects interact, and they may compensate for each other beneficially.

Despite these comments, the take-home message could be:

Observability estimates combined with accurate analysis of μ can lead to reasonably sharp energy decay rates in the 2D boundary damped wave equations.

Conclusion

In this presentation:

- General sufficient conditions for stability of boundary damped wave equations
- New and improved decay rates based on estimation of the level of unboundedness of D in the results

R. Chill, LP, D. Seifert, R. Stahn, Y. Tomilov, "Non-Uniform Stability of Damped Contraction Semigroups," *Analysis & PDE*, 2023, `https://arxiv.org/abs/1911.04804`

LP, D. Seifert, N. Vanspranghe, "Admissibility theory in abstract Sobolev scales and transfer function growth at high frequencies," arXiv, `https://arxiv.org/abs/2412.14786`

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