On Polynomial Stability of Hyperbolic Partial Differential Equations

Lassi Paunonen

Tampere University, Finland

October 25th, 2019

joint work with R. Chill, D. Seifert, S. Stahn and Y. Tomilov.

A few words about Hans:

- The Curtain & Zwart book
- Research visit in 2011 and a joint paper in 2013
- Encouragement of young researchers
- CDC 2013 in Florence

Goal of the Talk

Introduce general conditions for non-uniform stability of ***damped*** *hyperbolic PDEs.*

Goal of the Talk

*Introduce general conditions for non-uniform stability of **damped** hyperbolic PDEs.*

Damped systems of the form

$$\dot{x}(t) = (A - BB^*)x(t) \quad \text{and} \quad \ddot{w}(t) + A_0 w(t) + B_0 B_0^* \dot{w}(t) = 0$$

Motivation:

- Polynomial and non-uniform stability often arise in damped wave/beam/plate equations with weak or partial dampings
- Most of the current literature based on case-by-case analysis

Main results:

- General **observability-type** sufficient conditions for stability

(B^*, A) **exactly** observable $\quad \Leftrightarrow \quad A - BB^*$ exponentially stable

(B^*, A) **approx.** observable $\quad \Leftrightarrow^* \quad A - BB^*$ strongly/weakly stable

[Slemrod, Levan, Russell, Benchimol, Guo–Luo, Lasiecka–Triggiani, Curtain–Weiss . . .]

(B^*, A) **exactly** observable $\Leftrightarrow$ $A - BB^*$ exponentially stable

(B^*, A) **non-uniformly** obs. $\Leftrightarrow$ $A - BB^*$ non-uniformly stable

(B^*, A) **approx.** observable $\Leftrightarrow^*$ $A - BB^*$ strongly/weakly stable

[Slemrod, Levan, Russell, Benchimol, Guo–Luo, Lasiecka–Triggiani, Curtain–Weiss . . .]

Earlier work: Ammari–Tucsnak 2001, Ammari et. al.

Main Assumptions (roughly, to keep things simple)

- A generates a contraction semigroup $T(t)$ on X Hilbert
- Either $B \in \mathcal{L}(U, X)$, or (A, B, B^*) is well-posed.
- $\Rightarrow A - BB^*$ generates a contraction semigroup $T_B(t)$

Main case:

$$\ddot{w}(t) + A_0 w(t) + B_0 B_0^* \dot{w}(t) = 0, \qquad \text{on} \quad X_0$$

where $A_0 > 0$, $B_0 \in \mathcal{L}(U, \mathcal{D}(A_0^{1/2})^*)$ leads to

$$A = \begin{bmatrix} 0 & I \\ A_0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ B_0 \end{bmatrix}, \qquad \text{on} \quad X = \mathcal{D}(A_0^{1/2}) \times X_0.$$

Well-posedness $\Leftrightarrow \lambda \mapsto \lambda B_0^* (\lambda^2 + A_0)^{-1} B_0$ is bounded on $\mathbb{C}_\varepsilon$.

Polynomial and Non-Uniform Stability

Definition

$T_B(t)$ generated by $A - BB^*$ is **non-uniformly stable** if there exist an increasing $M_T \colon [t_0, \infty) \to \mathbb{R}_+$ and $C > 0$ such that

$$\|T_B(t)x\| \leq \frac{C}{M_T(t)} \|(A - BB^*)x\| \qquad x \in \mathcal{D}(A - BB^*), t > t_0.$$

[. . . , Liu–Rao '05, Batty–Duyckaerts '08, Borichev–Tomilov '10, Rozendaal–Seifert–Stahn '19]

Application: $E(t) \sim \|T_B(t)x_0\|^2$ for many PDE systems.

Polynomial and Non-Uniform Stability

Definition

$T_B(t)$ generated by $A - BB^*$ is **non-uniformly stable** if there exist an increasing $M_T\colon [t_0,\infty) \to \mathbb{R}_+$ and $C > 0$ such that

$$\|T_B(t)x\| \le \frac{C}{M_T(t)}\|(A-BB^*)x\| \qquad x \in \mathcal{D}(A-BB^*), t > t_0.$$

[..., Liu–Rao '05, Batty–Duyckaerts '08, Borichev–Tomilov '10, Rozendaal–Seifert–Stahn '19]

Theorem

Assume $T_B(t)$ is bounded, $i\mathbb{R} \subset \rho(A-BB^*)$, and

$$\|(is - A + BB^*)^{-1}\| \le M(|s|), \qquad M \text{ non-decreasing.}$$

- *If $M(s) \lesssim 1 + s^\alpha$, then $M_T(t) = t^{1/\alpha}$*
- *If M has "positive increase", then $M_T(t) = M^{-1}(t)$.*

Main Problem

Damped systems of the form

$$\dot{x}(t) = (A - BB^*)x(t) \quad \text{and} \quad \ddot{w}(t) + A_0 w(t) + B_0 B_0^* \dot{w}(t) = 0$$

Problem

How do (A, B) or (A_0, B_0) determine the stability of the system?

Main results:

*Conditions based on **observability-type properties** of (B^*, A) and (B_0^*, iA_0).*

A "Non-uniform Hautus test"

Consider the Hautus-type condition [Miller 2012]

$$\|x\|^2 \le M_o(|s|)\|(is - A)x\|^2 + m_o(|s|)\|B^*x\|^2, \quad x \in \mathcal{D}(A), s \in \mathbb{R},$$

for some non-decreasing $M_o, m_o \colon [0, \infty) \to [r_0, \infty)$.

A "Non-uniform Hautus test"

Consider the Hautus-type condition [Miller 2012]

$$\|x\|^2 \leq M_o(|s|)\|(is-A)x\|^2 + m_o(|s|)\|B^*x\|^2, \quad x \in \mathcal{D}(A), s \in \mathbb{R},$$

for some non-decreasing $M_o, m_o \colon [0,\infty) \to [r_0,\infty)$.

Theorem

If the above condition holds, then $i\mathbb{R} \subset \rho(A-BB^*)$. *If* $M(s) := M_o(s) + m_o(s)$ *has positive increase, then*

$$\|T_B(t)x\| \leq \frac{C}{M^{-1}(t)}\|(A-BB^*)x\|, \quad x \in \mathcal{D}(A-BB^*), t \geq t_0.$$

Other Sufficient Conditions for Stability (An Overview)

For A skew-adjoint with spectral projection $P_{(a,b)}$ (for $i(a,b) \subset i\mathbb{R}$)

$$\|B^*x\| \geq \gamma(|s|)\|x\|, \qquad x \in \operatorname{Ran}(P_{(s-\delta(|s|),s+\delta(|s|))}), \ s \in \mathbb{R}$$

for some non-increasing $\delta, \gamma \colon [0,\infty) \to (0, r_0]$.

Such x are often called **"wavepackets"** of A.

(Used for exact observability, e.g., in Ramdani et. al. 2005, Miller 2012, Tucsnak–Weiss 2009.)

Other Sufficient Conditions for Stability (An Overview)

For

$$\ddot{w}(t) + A_0 w(t) + B_0 B_0^* \dot{w}(t) = 0, \qquad \text{on} \quad X_0$$

and $M_S, m_S \colon [0,\infty) \to [r_0,\infty)$ consider $(s \geq 0)$

$$\|w\|^2 \leq M_S(s)\|(s^2 - A_0)w\|^2 + m_S(s)\|B_0^* w\|^2, \quad w \in \mathcal{D}(A_0)$$

This is **observability of the "Schrödinger group"** (B_0^*, iA_0)
(generalises Anantharaman–Leataud 2014, Joly–Laurent 2019)

Other Sufficient Conditions for Stability (An Overview)

Time-domain observability conditions:

If $0 \in \rho(A)$, $\tau, c_\tau, \beta > 0$:

$$c_\tau \|(-A)^{-\beta} x\|^2 \leq \int_0^\tau \|B^* T(t) x\|^2 dt, \qquad x \in \mathcal{D}(A).$$

(cf. generalised observability conditions by Ammari–Tuscnak 2001, Ammari–Bchatnia–El Mufti 2017)

Examples: 2D Wave Equations

A wave equation with viscous damping on a convex $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary, $b \in L^\infty(\Omega)$

$$
\begin{aligned}
&w_{tt}(\xi,t) - \Delta w(\xi,t) + b(\xi)^2 w_t(\xi,t) = 0, && \xi \in \Omega,\ t>0,\\
&w(\xi,t) = 0, && \xi \in \partial\Omega,\ t>0,\\
&w(\cdot,0) = w_0(\cdot) \in H^2(\Omega)\cap H_0^1(\Omega), && w_t(\cdot,0) = w_1(\cdot) \in H_0^1(\Omega).
\end{aligned}
$$

- Several results exist for the exact observability of the Schrödinger group $(b, i\Delta)$ (Jaffard '90, Burq–Zworski '19) for rectangles/tori. Leads to polynomial decay $1/\sqrt{t}$.
- Precise lower bounds on b lead to generalised observability of the Schrödinger group via Burq–Zuily 2016.
- In general our results are sub-optimal, since conditions do not take into account the **smoothness** of b! (Burq–Hitrik '07)

1D Wave Equations

Consider a wave equation with weak damping (and Dirichlet BC)

$$w_{tt}(\xi,t) - w_{\xi\xi}(\xi,t) + b(\xi)\int_0^1 b(r)w_t(r,t)dr = 0, \quad \xi \in (0,1), \; t > 0,$$

- The wavepacket condition characterises (optimal) stability via lower bounds of the sine Fourier coefficients, e.g., $(c, \alpha > 0)$

$$\left|\int_0^1 b(\xi)\sin(n\pi\xi)d\xi\right| \geq \frac{c}{n^\alpha}$$

- Pointwise damping possible (formally $b(\xi) = \delta(\xi - \xi_0)$).
- Analogous results for Euler–Bernoulli / Timoshenko beams

A Fractional Klein–Gordon Equation

For $m > 0$, $0 < \beta \le 1$ and $b \in L^\infty(\mathbb{R})$, consider

$$w_{tt}(\xi,t) + (-\partial_{\xi\xi})^\beta w(\xi,t) + m w(\xi,t) + b(\xi)^2 w_t(\xi,t) = 0, \quad \xi \in \mathbb{R}.$$

- The wavepacket condition leads to optimal polynomial stability (with knowlege of exponential stability if $\beta = 1$)
- Interesting example, since the spectrum of A_0 is not discrete
- Considered in Malhi–Stanislavova '18, Green '19

Conclusions

In this presentation:

- General sufficient conditions for non-uniform stability of the semigroup generated by $A - BB^*$.
- Discussion of PDE examples and optimality of the results

R. Chill, LP, D. Seifert, R. Stahn, Y. Tomilov, "Non-Uniform Stability of Damped Unitary Groups," *in preparation*