Energy Decay in Systems of Partial Differential Equations

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Joint work with D. Seifert and S. Nicaise.

Main Objectives

Problem

Study dynamics of coupled systems consisting of Partial Differential Equations (PDEs), with focus on ***long-time behaviour of the energy****.*

Motivation from PDE networks:

- Dynamics of networked systems are interesting!
- Both network structure and component dynamics contribute
- We focus on what happens when $t \to \infty$

What are PDE networks?

Throughout the talk a “**PDE network**” refers to

A collection of Partial Differential Equation models whose the connectivity can be described by a graph.

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Perhaps some examples will illustrate the concept. . .

Beam structures

- A truss bridge — beams which are joined at their ends.
- Deflection of each beam described by a PDE — **beam equation**
- Can study either dynamic behaviour or the steady state.

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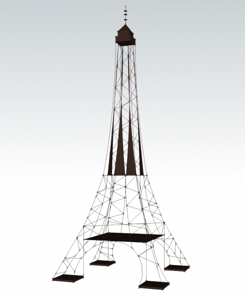
Truss structures

Structure determines a graph where **the beams are the edges and the meeting points are the nodes**.

$\rightsquigarrow$



We focus on such PDE networks, i.e., where

PDE $\leftrightarrow$ edge connection $\leftrightarrow$ node/vertex

Pipe Networks

Pipe delivery networks for gas, water, and petrol can be modelled as PDE networks. PDEs are **transport equations**.

- Waterflow in channels: **wave equations**
- Heat or material diffusion in structures: **heat or convection-diffusion equations**

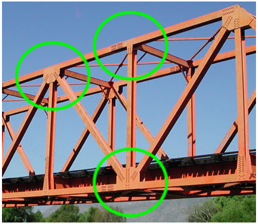
Couplings

Question

How do the PDEs in the network communicate?

Each PDE in the network involves:

- The actual differential equation (deflection along the beam)
- **Boundary conditions** (deflection, slope, strain etc **at the endpoints**)



$\rightsquigarrow$ The beam models interact with each other at the network nodes via their **boundary conditions**.

More Examples of Boundary Couplings

Physical system: Gas pipe network

- PDE type: Transport equation
- Coupling at nodes: Incoming material = outgoing mater.

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Physical system: Network of strings or water waves

- PDE type: Wave equation
- Coupling: Deflections are equal **and** forces sum to 0

More Examples of Boundary Couplings

Physical system: Gas pipe network

- PDE type: Transport equation
- Coupling at nodes: Incoming material = outgoing mater.

Physical system: Network of strings or water waves

- PDE type: Wave equation
- Coupling: Deflections are equal **and** forces sum to 0

Physical system: Truss bridge

- PDE type: Beam equation
- Coupling: Deflections are equal **and** forces sum to 0 **and**

All expressed in terms of boundary conditions, and the number depends on the PDE type.

Dynamics and Stability

Definition (Rough, but close enough)

We say the coupled PDE system is **stable** if all its solutions converge to a steady state as $t \to \infty$.

- **Linear case** $\rightsquigarrow$ zero is the unique steady state
- Stability is often equivalent to the property that the **energy** of the solutions satisfies $E(t) \to 0$ as $t \to \infty$.

Goal

*We are especially interested in the **rate** of the convergence.*

Mixed-Type PDE Networks

Goal

*We focus on studying **stability** of systems which involve **two different types of PDEs**.*

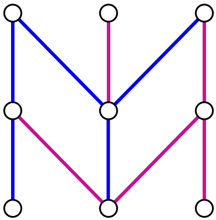
- Beams and strings (suspension bridge).
- Wave and heat equations (fluid-structure interactions)
- Damped and undamped strings or beams.
- (PDE networks with dynamic couplings)

Stability of Mixed-Type PDE Networks

Goal

We focus on studying ***stability*** *of coupled systems which involve* ***two different types of PDEs****.*

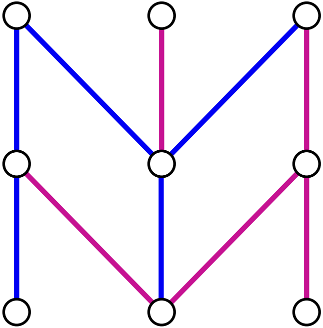
- Stability of mixed-type systems is particularly interesting!
- Convergence of solutions often *slower than exponential* $\rightsquigarrow$ "**polynomial stability**"
- Motivates deriving accurate (rational) convergence rates for $E(t) \to 0$



Part II

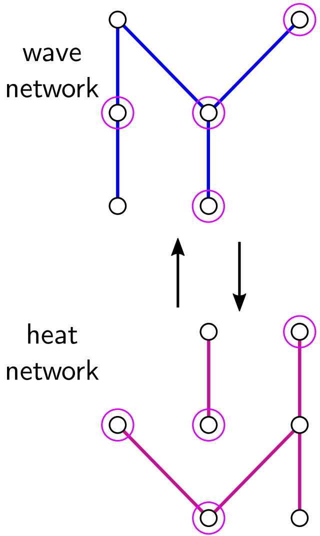
Stability Analysis

Dynamics of Mixed-Type Networks



Wave equations and heat equations

Wave–Heat Networks – Decoupled



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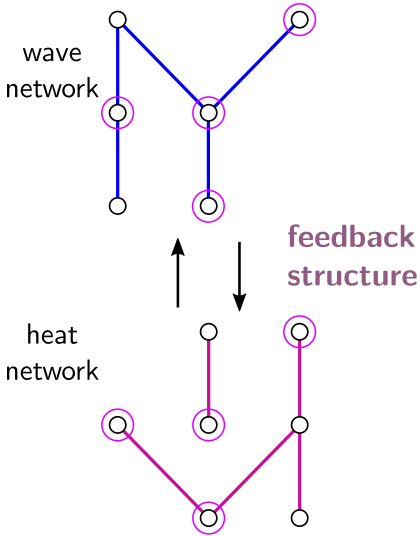

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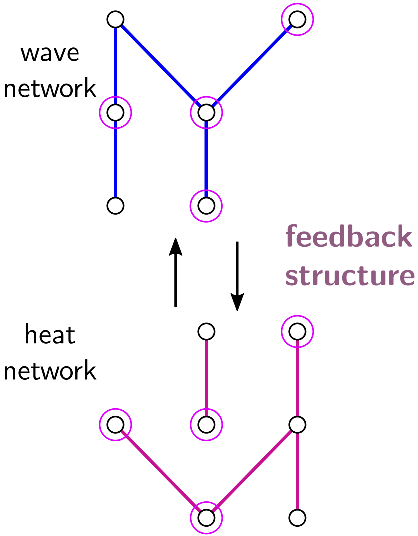

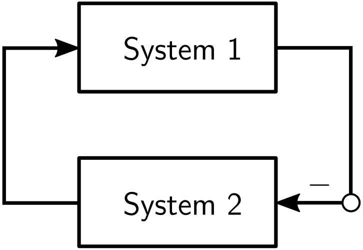

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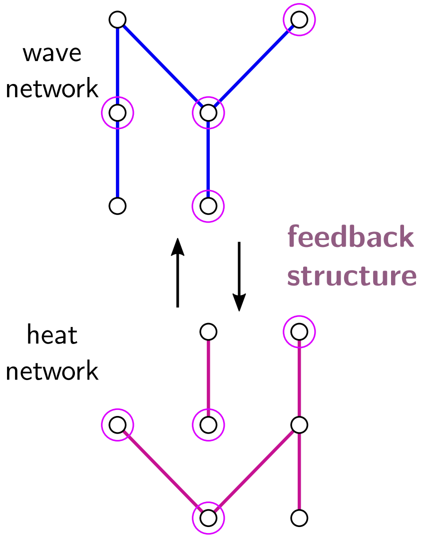
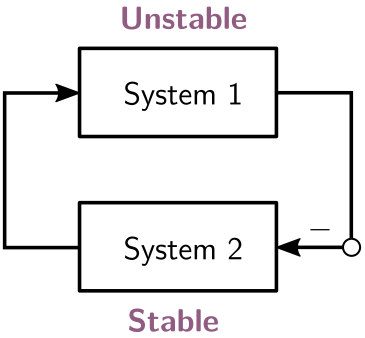
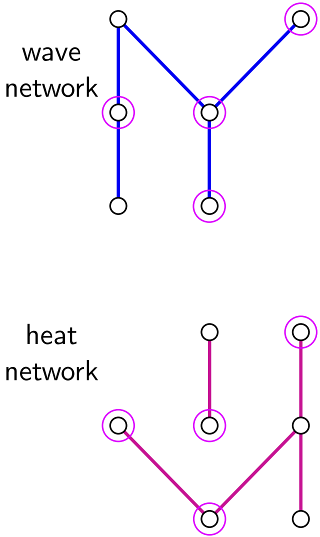
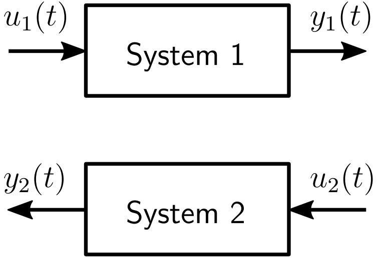




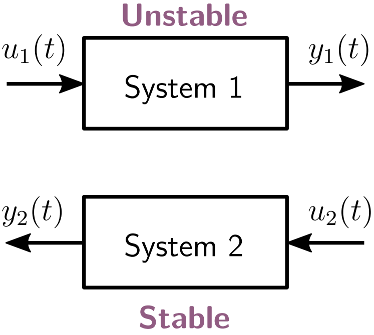
Inputs and Outputs





Problem

Use the properties of the two systems to deduce stability of the coupled PDE.

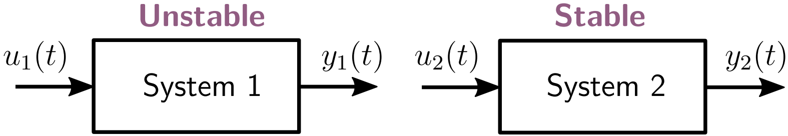


Benefits:

- "*Divide and conquer*"
- Reduce to well-known parts
- Modularity

Assumption

- *The two systems are **impedance passive***
  - *They have no "internal sources of energy"*



Assumption

- *The two systems are **impedance passive***
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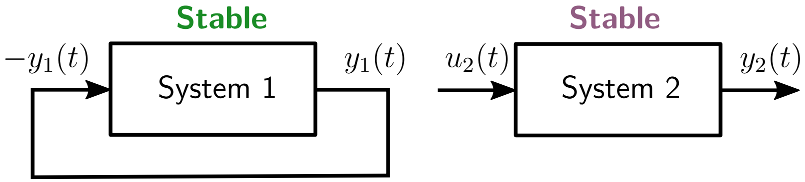
- ***System 1** is stable under **(neg) output feedback***
  - *This often represents addition of **damping** to System 1.*



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- ***System 1** is stable under **(neg) output feedback***
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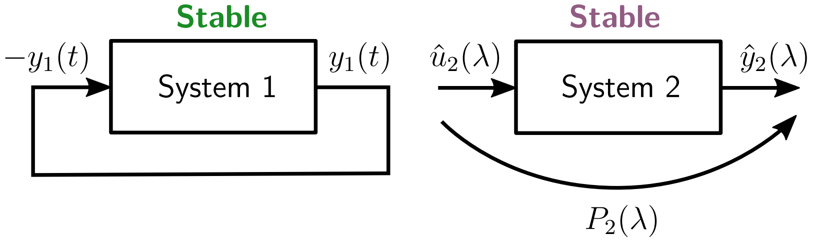
- *Knowledge of the **transfer function** $P_2(\lambda)$ of **System 2***
  - *$P_2(\lambda)$ describes how "Laplace transform of u_2 is mapped to Laplace transform of y_2", i.e., $\hat{y}_2(\lambda) = P_2(\lambda)\hat{u}_2(\lambda)$.*



Main Stability Result

Theorem (Nicaise, P., Seifert '25)

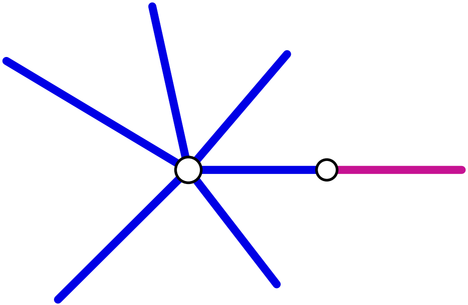
Assume the following:

- *The two systems are* ***impedance passive***
- *System 1 becomes stable under* ***(neg) output feedback*** *and we know the energy decay rate* $E_1(t) \to 0$
- *We have an estimate for the* ***transfer function*** $P_2(\lambda)$ *of System 2 on the imaginary axis*

Then:

The network is stable and we have
a very good estimate *for the decay rate of* $E(t) \to 0$.

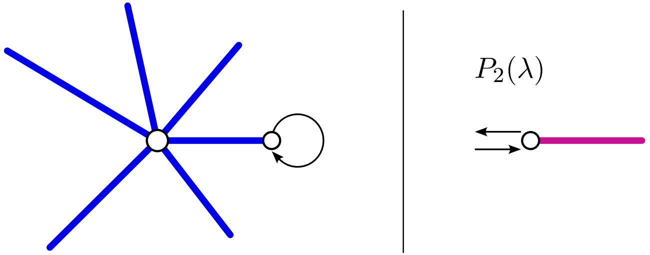
Application to Wave-Heat Networks



Question

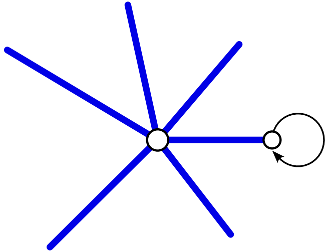
Decay rate for energy $E(t)$ in a network of N wave equations and 1 heat equation?

Wave/Heat Decoupling



- In the wave network (System 1) the "output feedback" is equivalent to damping at a single exterior node.
  $\rightsquigarrow$ Stability results in the literature
- Transfer function $P_2(\lambda)$ of the heat equation (System 2) can be computed explicitly.

Wave/Heat Decoupling

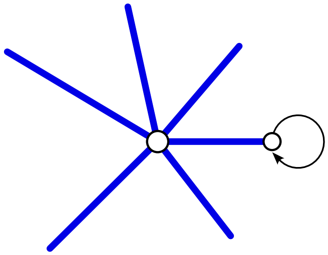


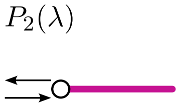
Theorem

*If decay rate for classical solutions in the **decoupled** wave-network is $E_1(t) \sim t^{-\alpha}$, $\alpha > 0$, then the energy $E(t)$ of classical solutions of the **wave-heat network** has decay*

$$E(t) \sim t^{-\frac{4\alpha}{4+\alpha}}, \qquad t \to \infty. \qquad \textit{(slower)}$$

Wave/Heat Decoupling



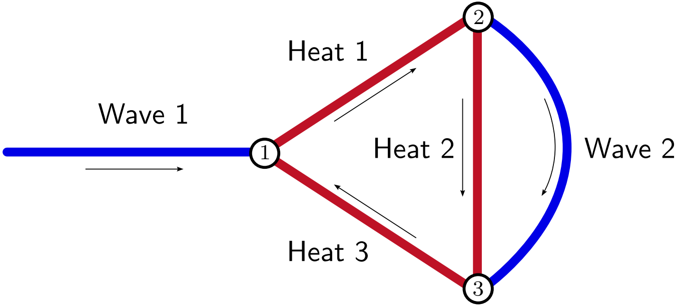


Theorem (Using Valein–Zuazua '09)

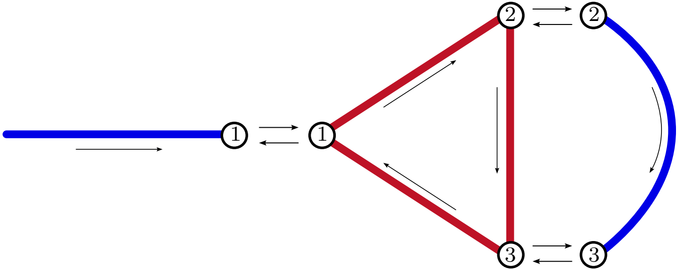
Assume that ***decoupled*** *wave-network is* ***star-shaped*** *with same wave speeds, and ratios of lengths of the waves are irrational numbers "of constant type". Then energy of classical solutions has decay*

$$E(t) \sim t^{-\frac{4}{4N-3}}, \qquad t \to \infty.$$

A Network with Multiple Heat Equations



A Network with Multiple Heat Equations



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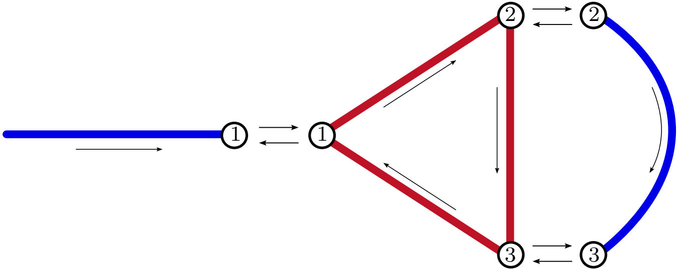


Theorem

The energy of classical solutions has decay

$$E(t) \sim \frac{1}{t^4}, \qquad t \to \infty.$$

Future Prospects: nD Wave–Heat Systems

Multidimensional models for fluid–structure interactions:

$$\frac{\partial^2 v}{\partial t^2}(x,t) = \Delta v(x,t)$$

$$\updownarrow$$

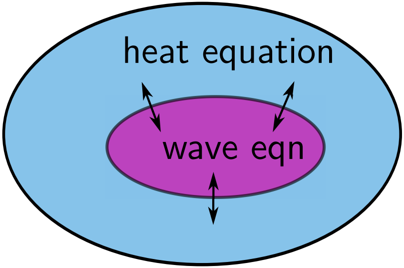
coupling BCs:

$$v_t(x,t) = w(x,t)$$

$$n \cdot \nabla v(x,t) = n \cdot \nabla w(x,t)$$

$$\updownarrow$$

$$\frac{\partial w}{\partial t}(x,t) = \Delta w(x,t)$$



References: Avalos & Triggiani, Duyckaerts, Mercier, Nicaise, Ammari, Zhang & Zuazua, and others.

A Quick Literature Survey

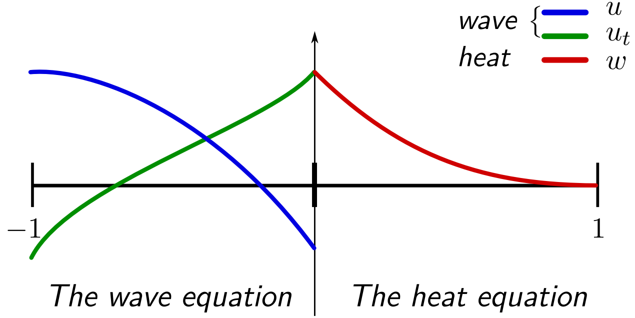
- PDE networks — single-type
  - Valein–Zuazua, Ammari–Jellouli, Ammari–Shel, Nicaise–Valein, Kramar-Fijavž et. al., Augner, ...
- Coupled PDEs — two equations or simple network structures (3–4 PDEs)
  - Zhang–Zuazua, Duyackerts, Li-Wang, Augner, Avalos–Lasiecka–Triggiani, Rao–Zhang, Avalos–Geredeli, ...
- Coupled abstract systems
  - Ben Ait Hassi et. al., Ammari et. al., Feng–Guo, LP, Boulouz–Bounit–Hadd, Dell'Oro–LP–Seifert, ...

Part III

- Details on how PDE models are analysed as coupled "systems"
- Description of the underlying abstract result
- Sketch of the proof

Example: 1D Wave–Heat Model

$$
\begin{cases}
v_{tt}(\xi,t) = v_{\xi\xi}(\xi,t), & \xi \in (-1,0),\ t>0, \\
w_t(\xi,t) = w_{\xi\xi}(\xi,t), & \xi \in (0,1),\ t>0, \\
v_\xi(0,t) = w_\xi(0,t), \quad v_t(0,t) = w(0,t), & t>0, \\
v_\xi(-1,t) = 0, \qquad w(1,t) = 0, & t>0,
\end{cases}
$$



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Energy of a solution:

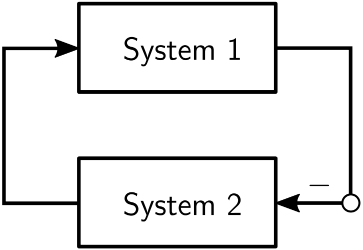
$$E(t) = \frac{1}{2}\int_{-1}^{0}\Big[|v_t(\xi,t)|^2 + |v_\xi(\xi,t)|^2\Big]d\xi + \frac{1}{2}\int_0^1 |w(\xi,t)|^2 d\xi$$

Aim: Find a rate for the decay of $E(t)$ as $t \to \infty$.

Example: 1D Wave–Heat Model

$$\begin{cases} v_{tt}(\xi,t) = v_{\xi\xi}(\xi,t), & \xi \in (-1,0),\ t>0,\\ w_t(\xi,t) = w_{\xi\xi}(\xi,t), & \xi \in (0,1),\ t>0,\\ v_\xi(0,t) = w_\xi(0,t), \quad v_t(0,t) = w(0,t), & t>0, \end{cases}$$

- System 1: Wave equation
- System 2: Heat equation
- Signals: Boundary at $\xi = 0$



Example: 1D Wave-Heat — Open-Loop Splitting

Wave system on $(-1,0)$:

$$\begin{aligned} v_{tt}(\xi,t) &= v_{\xi\xi}(\xi,t) \\ v_t(0,t) &= u_1(t) \\ y_1(t) &= v_\xi(0,t) \end{aligned}$$

Unstable

Heat system on $(0,1)$:

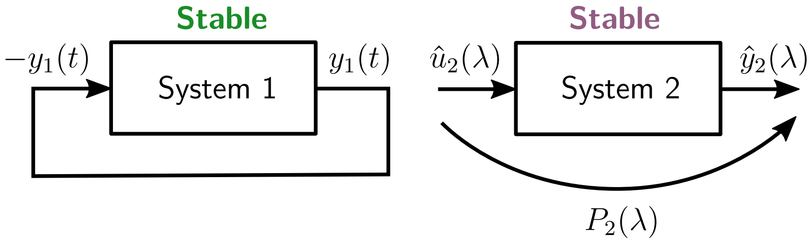
$$\begin{aligned} w_t(\xi,t) &= w_{\xi\xi}(\xi,t) \\ -w_\xi(0,t) &= u_2(t) \\ y_2(t) &= w(0,t) \end{aligned}$$

Stable

$$\begin{cases} u_1(t) = y_2(t) \\ u_2(t) = -y_1(t) \end{cases} \quad \text{is exactly} \quad \begin{cases} v_t(0,t) = w(0,t) \\ v_\xi(0,t) = w_\xi(0,t) \end{cases}$$

Assumption

- *The two systems are **impedance passive***
- ***System 1** is stable under **(neg) output feedback**, and have an estimate for the rate $E_1(t) \to 0$*
- *Knowledge of the **transfer function** $P_2(\lambda)$ of **System 2***
  - *$P_2(is)$ describes how "Laplace transform of u_2 is mapped to Laplace transform of y_2", i.e., $\hat{y}_2(\lambda) = P_2(\lambda)\hat{u}_2(\lambda)$.*



Example: 1D Wave-Heat — Assumptions

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Heat system on $(0,1)$:

$$\begin{aligned} w_t(\xi,t) &= w_{\xi\xi}(\xi,t) \\ -w_\xi(0,t) &= u_2(t) \\ y_2(t) &= w(0,t) \end{aligned}$$

Negative feedback in System 1: If we connect $u_1(t) = -y_1(t)$ in the wave equation, we obtain the PDE

$$\begin{aligned} v_{tt}(\xi,t) &= v_{\xi\xi}(\xi,t) \\ v_t(0,t) + v_\xi(0,t) &= 0 \\ v_\xi(-1,t) &= 0 \end{aligned}$$

Boundary damping at $\xi = 0$ leads to stability, $E_1(t) = O(e^{-\alpha t})$

Example: 1D Wave-Heat — Assumptions

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Transfer function of System 2: Taking Laplace transforms shows that $\hat{y}_2(\lambda) = P_2(\lambda)\hat{u}_2(\lambda)$ if

$$\begin{cases} \lambda\hat{w}(\xi,\lambda) = \hat{w}_{\xi\xi}(\xi,\lambda) \\ -\hat{w}_\xi(0,\lambda) = \hat{u}_2(\lambda) \\ \hat{w}(1,\lambda) = 0 \\ \hat{y}_2(\lambda) = \hat{w}(0,\lambda) \end{cases}$$

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We have $\operatorname{Re} P_2(is) \geq c/(1+s^2)$ for some $c>0$.

Result

The one-dimensional wave-heat system:

$$\begin{cases} v_{tt}(\xi,t) = v_{\xi\xi}(\xi,t), & \xi \in (-1,0),\ t>0, \\ w_t(\xi,t) = w_{\xi\xi}(\xi,t), & \xi \in (0,1),\ t>0, \\ v_\xi(0,t) = w_\xi(0,t), \quad v_t(0,t) = w(0,t) & t>0, \end{cases}$$

Theorem

The energy of classical solutions has decay

$$E(t) \sim \frac{1}{t^4}, \qquad t \to \infty.$$

Abstract Framework

System 1 ("Wave")

$$\frac{d}{dt}x_1(t) = A_1 x_1(t) + B_1 u_1(t)$$
$$y_1(t) = B_1^* x_1(t)$$

System 2 ("Heat")

$$\frac{d}{dt}x_2(t) = A_2 x_2(t) + B_2 u_2(t)$$
$$y_2(t) = B_2^* x_2(t)$$

Abstract Framework

System 2 ("Heat")

$$
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\end{aligned}
$$

Heat system on $(0,1)$:

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$$x_2(t) = w(\cdot,t) \in L^2(0,1)$$

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$$x_1(t) = (v(\cdot,t), v_t(\cdot,t))^\top$$

Heat system on $(0,1)$:

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$$

System 2 (“Heat”)

$$
\begin{aligned}
\frac{d}{dt}x_2(t) &= A_2x_2(t) + B_2u_2(t) \\
y_2(t) &= B_2^*x_2(t)
\end{aligned}
$$

Abstract Framework

System 1 ("Wave")

$$\frac{d}{dt}x_1(t) = A_1 x_1(t) + B_1 u_1(t)$$
$$y_1(t) = B_1^* x_1(t)$$

System 2 ("Heat")

$$\frac{d}{dt}x_2(t) = A_2 x_2(t) + B_2 u_2(t)$$
$$y_2(t) = B_2^* x_2(t)$$

Interconnecting $u_1(t) = y_2(t)$ and $u_2(t) = -y_1(t)$ we get

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} A_1 & B_1 B_2^* \\ -B_2 B_1^* & A_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Energy of the solution: $E(t) = \frac{1}{2}\left(\|x_1(t)\|^2 + \|x_2(t)\|^2\right)$.

Abstract Result

Goal: Analyse the asymptotic (as $t \to \infty$) behaviour of

$$\frac{d}{dt}x(t) = \begin{bmatrix} A_1 & B_1 B_2^* \\ -B_2 B_1^* & A_2 \end{bmatrix} x(t), \qquad x(0) \in X$$

with $x(t) = (x_1(t), x_2(t))$ based on information on

1. Knowledge of dissipativity: $\mathrm{Re}\langle A_k x, x\rangle \leq 0$ for $k = 1, 2$
2. Behaviour of System 1 with feedback $u_1(t) = -y_1(t)$
3. Transfer function of System 2

Key Background Result

Result (Hilbert spaces, our case)

Classical sol. convergence rates of $\|x(t)\| \to 0$ *as* $t \to \infty$ *in*

$$\frac{d}{dt}x(t) = Ax(t), \qquad x(0) = x_0$$

are characterised by growth bounds of $\|(is - A)^{-1}\|$ *for* $s \in \mathbb{R}$.

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are characterised by growth bounds of $\|(is - A)^{-1}\|$ *for* $s \in \mathbb{R}$.

- $\sup_{s\in\mathbb{R}} \|(is - A)^{-1}\| < \infty$ *equivalent to* $\|x(t)\| = O(e^{-\alpha t})$
- $\|(is - A)^{-1}\| \le M(1 + |s|^\alpha) \quad \Leftrightarrow \quad \|x(t)\| = O(t^{-1/\alpha})$.

Result via Resolvent Growths

Consequence

It suffices to find an estimate for $\|(is-A)^{-1}\|$ for the operator

$$A = \begin{bmatrix} A_1 & B_1 B_2^* \\ -B_2 B_1^* & A_2 \end{bmatrix}.$$

We will do this in terms of

- Growth of $\|(is - A_1 + B_1 B_1^*)^{-1}\|$, which describes energy decay in System 1 after feedback $u_1(t) = -y_1(t)$
- The transfer function $P_2(\lambda) = B_2^*(\lambda - A_2)^{-1} B_2$ of System 2.

Main Result

Denote $P_2(\lambda) = B_2^*(is - A_2)^{-1}B_2$ and $A = \begin{bmatrix} A_1 & B_1 B_2^* \\ -B_2 B_1^* & A_2 \end{bmatrix}$

Theorem (Nicaise, P., Seifert '25)

Assume that $\sup_{s\in\mathbb{R}} \|(is - A_2)^{-1}\| < \infty$ *and* $\eta : \mathbb{R} \to (0, 1]$ *is such that*

$$\mathrm{Re}\langle P_2(is)u, u\rangle \geq \eta(s)\|u\|^2, \qquad u \in U, s \in \mathbb{R}$$

Then

$$\|(is - A)^{-1}\| \lesssim \frac{\|(is - A_1 + B_1 B_1^*)^{-1}\|}{\eta(s)}, \qquad s \in \mathbb{R}.$$

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Stability of A using stability of System 1 under neg. feedback and the transfer function of System 2.

Sketch of the Proof (Main Steps)

$$A = \begin{bmatrix} A_1 & B_1 B_2^* \\ -B_2 B_1^* & A_2 \end{bmatrix}, \quad \|(is - A)^{-1}\| \lesssim \frac{\|(is - A_1 + B_1 B_1^*)^{-1}\|}{\eta(s)}$$

The proof consists of 3 main steps:

1. Employing Schur complements (for the block matrix A)
2. Adding 0 (always nice!!)
3. A dissipativity argument

The proof works for operators arising from PDEs, but you can alternatively think of **matrices** A_1, A_2, B_1, and B_2.

Sketch of the Proof

To estimate the inverse of

$$is - A = \begin{bmatrix} is - A_1 & -B_1 B_2^* \\ B_2 B_1^* & is - A_2 \end{bmatrix},$$

we can use Schur complements. Denoting $R_2 = (is - A_2)^{-1}$

$$is - A = \begin{bmatrix} I & -B_1 B_2^* R_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} S(is) & 0 \\ 0 & is - A_2 \end{bmatrix} \begin{bmatrix} I & 0 \\ R_2 B_2 B_1^* & I \end{bmatrix},$$

where

$$S(is) = is - A_1 + B_1 \underbrace{B_2^* (is - A_2)^{-1} B_2}_{=P_2(is)} B_1^*.$$

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where

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Result

$$\|(is - A)^{-1}\| \lesssim \|S(is)^{-1}\|, \qquad s \in \mathbb{R}.$$

Proof (adding 0)

Let $S(is) = is - A_1 + B_1 P_2(is) B_1^*$, and $y = S(is)x$. Our aim is then to show

$$\|x\| \lesssim \eta(s)^{-1} \|(is - A_1 + B_1 B_1^*)^{-1}\| \|y\|.$$

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We can write

$$\begin{aligned} y = S(is)x &= (is - A_1 + B_1 P_2(is) B_1^*)x \\ &= (is - A_1 + B_1 B_1^*)x + B_1(P_2(is) - I)B_1^* x \end{aligned}$$

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and “solve” for x:

$$x = (is - A_1 + B_1 B_1^*)^{-1}(y + B_1(I - P_2(is))B_1^* x)$$

Proof

If we denote $M_1(s) = \|(is - A_1 + B_1 B_1^*)^{-1}\|$, then

$$x = (is - A_1 + B_1 B_1^*)^{-1}(y + B_1(I - P_2(is))B_1^* x)$$

implies

$$\|x\| \lesssim M_1(s)\|y\| + \|(is - A_1 + B_1 B_1^*)^{-1} B_1\| \|B_1^* x\|$$

Proof (dissipativity argument)

Progress

Claim: $\|x\| \lesssim M_1(s)\eta(s)^{-1}\|y\|$

Progress: $\|x\| \lesssim M_1(s)\|y\| + \|(is - A_1 + B_1 B_1^*)^{-1} B_1\| \|B_1^* x\|$

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Since $y = S(is)x = (is - A_1 + B_1P_2(is)B_1^*)x$, we have

$$\begin{aligned}\mathrm{Re}\langle y, x\rangle &= \mathrm{Re}(is\|x\|^2) - \mathrm{Re}\langle A_1x, x\rangle + \mathrm{Re}\langle P_2(is)B_1^*x, B_1^*x\rangle \\ &\geq \mathrm{Re}\langle P_2(is)B_1^*x, B_1^*x\rangle \geq \eta(s)\|B_1^*x\|^2\end{aligned}$$

Since $\mathrm{Re}\langle A_1x, x\rangle \leq 0$ and $\mathrm{Re}\langle P_2(is)u, u\rangle \geq \eta(s)\|u\|^2$ by ass.

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Since $\mathrm{Re}\langle A_1x, x\rangle \leq 0$ and $\mathrm{Re}\langle P_2(is)u, u\rangle \geq \eta(s)\|u\|^2$ by ass.

$$\|B_1^*x\|^2 \leq \eta(s)^{-1}\,\mathrm{Re}\langle y, x\rangle \leq \eta(s)^{-1}\|x\|\|y\|.$$

Proof

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Claim: $\|x\| \lesssim M_1(s)\eta(s)^{-1}\|y\|$

Progress: $\|x\| \lesssim M_1(s)\|y\| + \|(is - A_1 + B_1B_1^*)^{-1}B_1\|\|B_1^*x\|$

Using $\|B_1^*x\|^2 \le \eta(s)^{-1}\|x\|\|y\|$ and Young's $ab \le \frac{\varepsilon a^2}{2} + \frac{b^2}{2\varepsilon}$ leads to

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Using $\|B_1^*x\|^2 \leq \eta(s)^{-1}\|x\|\|y\|$ and Young's $ab \leq \frac{\varepsilon a^2}{2} + \frac{b^2}{2\varepsilon}$ leads to

$$\|x\| \lesssim M_1(s)\|y\| + \|(is - A_1 + B_1B_1^*)^{-1}B_1\|^2\eta(s)^{-1}\|y\|.$$

The magic of dissipativity implies

$$\|(is - A_1 + B_1B_1^*)^{-1}B_1\|^2 \leq M_1(s),$$

and thus $\|x\| \lesssim M_1(s)\eta(s)^{-1}\|y\|$. $\square$

Sketch of the Proof

Theorem (Nicaise, P., Seifert '25)

With $\mathrm{Re}\langle P_2(is)u, u\rangle \geq \eta(s)\|u\|^2$, *we have*

$$\|(is-A)^{-1}\| \lesssim \frac{\|(is-A_1+B_1B_1^*)^{-1}\|}{\eta(s)}.$$

The proof consisted of 3 main steps:

1. Employing Schur complements (for the block matrix A)
2. Adding 0
3. A dissipativity argument (plus another similar one)

Conclusions

In this talk:

- Study of dynamics and stability of coupled partial differential equations of mixed type.
- Main result on energy decay rate for the full system in terms of the properties of simpler subsystems.

S. Nicaise, LP, D. Seifert "Stability of Abstract Coupled Systems", *J. Functional Analysis*, 2025.

M. Fkirine, LP "Polynomial stability of wind turbine tower models", *SIAM J. Control and Optimization*, to appear.

Thank You!