Polynomial Stability of Abstract Linear Differential Equations

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Abstract Cauchy Problems

An *abstract Cauchy problem* (ACP) is a linear initial value problem

$$\frac{d}{dt}x(t) = Ax(t), \qquad x(0) = x_0 \in X$$

on a Banach space X.

- A is a possibly unbounded operator on X with domain $\mathcal{D}(A)$, i.e., $A : \mathcal{D}(A) \subset X \to X$.
- Solution $x(\cdot) : [0, \infty) \to X$

Abstract Cauchy Problems

An *abstract Cauchy problem* (ACP) is a linear initial value problem

$$\frac{d}{dt}x(t) = Ax(t), \qquad x(0) = x_0 \in X$$

on a Banach space X.

The (ACP) may have different types of solutions, most notably

- Classical solution: $x(\cdot) \in C^1(0, \infty; X)$
- Mild/(weak) solution: $x(\cdot) \in C(0, \infty; X)$ satisfies

$$x(t) = A \int_0^t x(s)ds + x_0$$

Strongly Continuous Semigroups

The abstract Caucy problem

$$\frac{d}{dt}x(t) = Ax(t), \qquad x(0) = x_0 \in X$$

is well-posed if and only if the operator A generates a *strongly continuous semigroup* e^{At} on X.

This is a mapping $t \mapsto e^{At} : [0, \infty) \to \mathcal{L}(X)$ with properties

- $e^{A0} = I$
- $e^{A(t+s)} = e^{At}e^{As}$
- $e^{At}x \to e^{At_0}x$ as $t \to t_0$ for every $x \in X$ (strong continuity)
- $\frac{d}{dt}e^{At}x = Ae^{At}x = e^{At}Ax$ for every $x \in \mathcal{D}(A)$.

NOTE: e^{At} generalizes the matrix exponential function.

Solution of the (ACP)

If the operator A generates a strongly stable semigroup e^{At}, then the (ACP)

$$\frac{d}{dt}x(t) = Ax(t), \qquad x(0) = x_0 \in X$$

has a unique mild solution given by

$$x(t) = e^{At}x_0, \qquad \forall t \geq 0.$$

Moreover, if $x_0 \in \mathcal{D}(A)$, then $x(\cdot)$ is a classical solution (i.e., $x(\cdot) \in C^1$).

A Linear Heat Equation

Consider a 1D heat equation (on the interval $[0,1]$)

$$\begin{aligned} \frac{dw}{dt}(z,t) &= \frac{d^2w}{dz^2}(z,t) \\ w(0,t) &= w(1,t) = 0 \\ w(z,0) &= w_0(z) \in L^2(0,1). \end{aligned}$$

This PDE can be written as an (ACP) on $X = L^2(0,1)$ by choosing

$$(Ax)(z) = \frac{d^2x}{dz^2}(z),$$

$$\mathcal{D}(A) = \left\{ x \in L^2(0,1) \;\middle|\; x,\, \frac{dx}{dz} \text{ abs. cont.},\; x(0) = x(1) = 0 \right\}.$$

A Damped Wave Equation

Consider a 2D wave equation (on a square $\Omega = [0,1] \times [0,1]$)

$$v_{tt}(z,t) = \Delta v(z,t) - a(z) v_t(z,t)$$

$$\begin{aligned} v(z,t) &= 0 \qquad z \in \partial\Omega \\ v(z,0) &= v_0(z), \quad v_t(z,0) = v_1(z) \end{aligned}$$

where the function $a(\cdot)$ in the damping term is essentially bounded.

To form the (ACP), denote

$$H_0^1(\Omega) = \left\{ v \in L^2(\Omega) \;\middle|\; \frac{dv}{dz} \in L^2(\Omega), \quad v(z) = 0 \text{ for } z \in \partial\Omega \right\}.$$

A Damped Wave Equation

The wave equation can be written as an (ACP) on a Hilbert space

$$X = H_0^1(\Omega) \times L^2(\Omega)$$

by choosing

$$A = \begin{pmatrix} 0 & I \\ \Delta & -a(z) \end{pmatrix}, \quad \mathcal{D}(A) = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \;\middle|\; x_2 \in H_0^1(\Omega),\ \Delta x_1 \in L^2(\Omega) \right\}.$$

Then the PDE becomes an (ACP)

$$\frac{d}{dt}\begin{pmatrix} v(z) \\ v_t(z) \end{pmatrix} = \begin{pmatrix} 0 & I \\ \Delta & -a(z) \end{pmatrix}\begin{pmatrix} v(z) \\ v_t(z) \end{pmatrix}, \qquad \begin{pmatrix} v(z) \\ v_t(z) \end{pmatrix} = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}.$$

Stability of Abstract Cauchy Problems

Problem

Study the behaviour of the solutions of the (ACP)

$$\frac{d}{dt}x(t) = Ax(t), \qquad x(0) = x_0 \in X$$

as $t \to \infty$.

Definition

The (ACP) (and the semigroup e^{At}) is called *stable* if

$$x(t) = e^{At}x_0 \to 0 \quad \text{as} \quad t \to \infty$$

for every $x_0 \in X$.

Stability of Finite-Dimensional Equations

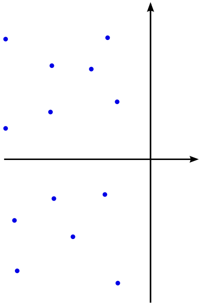
On a finite-dimensional space $X = \mathbb{C}^n$: An abstract Cauchy problem where A is an $n \times n$ matrix

$$\frac{d}{dt}x(t) = Ax(t), \qquad x(0) = x_0 \in X$$

The semigroup e^{At} is stable if and only if

$$\operatorname{Re}\lambda < 0$$

for all eigenvalues λ of A.



Stability of Finite-Dimensional Equations

On a finite-dimensional space $X = \mathbb{C}^n$: An abstract Cauchy problem where A is an $n \times n$ matrix

$$\frac{d}{dt}x(t) = Ax(t), \qquad x(0) = x_0 \in X$$

If e^{At} is stable, then solutions decay at a uniform exponential rate:

There exists $\omega, M > 0$ such that

$$\|x(t)\| = \|e^{At}x_0\| \leq Me^{-\omega t}\|x_0\|$$

for all $x_0 \in X$.

The Banach Space Case

For (ACP)

$$\frac{d}{dt}x(t) = Ax(t), \qquad x(0) = x_0 \in X$$

on a Banach space X, there are different types of stability:

The strongest type (same as in finite-dimensional case):

Definition (Exponential Stability)

The semigroup e^{At} is *exponentially stable* if there exist $\omega, M > 0$ such that

$$\|x(t)\| = \|e^{At}x_0\| \leq Me^{-\omega t}\|x_0\| \qquad \forall t \geq 0$$

for every $x_0 \in X$.

The Banach Space Case

For (ACP)

$$\frac{d}{dt}x(t) = Ax(t), \qquad x(0) = x_0 \in X$$

on a Banach space X, there are different types of stability:

Weaker type, no uniform rate required.

Definition (Strong Stability)

The semigroup e^{At} is *strongly stable* if

$$\|x(t)\| = \|e^{At}x_0\| \longrightarrow 0 \quad \text{as} \quad t \to \infty$$

for every $x_0 \in X$.

The Banach Space Case

For (ACP)

$$\frac{d}{dt}x(t) = Ax(t), \qquad x(0) = x_0 \in X$$

on a Banach space X, there are different types of stability:

Intermediate, decay rate is not for every $x_0 \in X$.

Definition (Polynomial Stability)

The semigroup e^{At} is *polynomially stable* if there exist $\alpha, M > 0$ such that

$$\|x(t)\| = \|e^{At}x_0\| \le \frac{M}{t^{1/\alpha}}\|Ax_0\| \qquad \forall t \ge 0$$

for every $x_0 \in \mathcal{D}(A)$.

Stability of the Heat Equation

The 1D heat equation

$$\begin{aligned} \frac{dw}{dt}(z,t) &= \frac{d^2 w}{dz^2}(z,t) \\ w(0,t) &= w(1,t) = 0 \\ w(z,0) &= w_0(z) \in L^2(0,1). \end{aligned}$$

is exponentially stable, the solutions satisfy

$$\|w(\cdot,t)\|_{L^2} \leq e^{-\pi t} \|w_0(\cdot)\|_{L^2}.$$

for every initial state $w_0 \in L^2(0,1)$.

Stability of the 2D Wave Equation on $\Omega = [0,1] \times [0,1]$

$$v_{tt}(z,t) = \Delta v(z,t) - a(z) v_t(z,t)$$

$$v(z,t) = 0 \qquad z \in \partial\Omega$$

$$v(z,0) = v_0(z), \quad v_t(z,0) = v_1(z)$$

depends on the function $a(\cdot)$ in the damping term:

Stability of the 2D Wave Equation on $\Omega = [0,1] \times [0,1]$

$$v_{tt}(z,t) = \Delta v(z,t) - a(z) v_t(z,t)$$

$$v(z,t) = 0 \qquad z \in \partial\Omega$$

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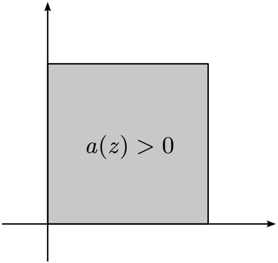


If

$$a(z) > 0$$

uniformly in Ω, then the equation is exponentially stable, i.e.,

$$\|v(\cdot,t)\|_{L^2} \xrightarrow{t\to\infty} 0$$

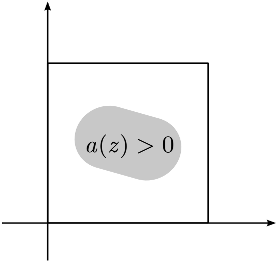
exponentially fast.

Stability of the 2D Wave Equation on $\Omega = [0,1] \times [0,1]$

$$v_{tt}(z,t) = \Delta v(z,t) - a(z) v_t(z,t)$$

$$v(z,t) = 0 \qquad z \in \partial\Omega$$

$$v(z,0) = v_0(z), \quad v_t(z,0) = v_1(z)$$



If

$$a(z) > 0$$

uniformly in a region of Ω, then the equation is strongly stable,

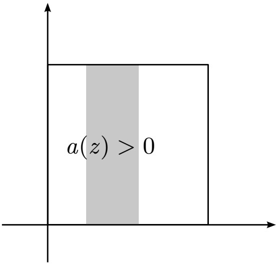
$$\|v(\cdot,t)\|_{L^2} \xrightarrow{t\to\infty} 0$$

Stability of the 2D Wave Equation on $\Omega = [0,1] \times [0,1]$

$$v_{tt}(z,t) = \Delta v(z,t) - a(z) v_t(z,t)$$

$$v(z,t) = 0 \qquad z \in \partial\Omega$$

$$v(z,0) = v_0(z), \quad v_t(z,0) = v_1(z)$$



If
$$a(z) > 0$$

uniformly in a strip, then the equation is polynomially stable

$$\|v(\cdot,t)\|_{L^2} \le \frac{\tilde{M}}{t^{1/2}} \xrightarrow{t\to\infty} 0$$

for IC's $v_0 \in H_0^2$, $v_1 \in H_0^1$.

Comparison of Stability Types

Exponential	Polynomial	Strong
+ Very good properties		− Very few properties
− Often unachievable		+ Very often achievable

Comparison of Stability Types

Exponential	Polynomial	Strong
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Conclusions

In this presentation:

- Uniform presentation for linear PDE's in abstract form.
- Introduction and comparison of stability types.

Thank You!