Operator methods in the control of infinite-dimensional systems

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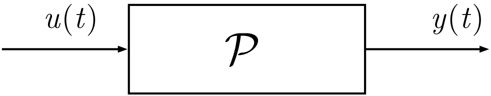
1. Introduction and the Output Regulation Problem
   - The classes of abstract linear systems
   - Introduction to main mathematical tools
2. Selected results on output regulation
   - Sylvester-type regulator equations
3. Conclusions and further research directions

The System To Be Controlled



- The System $\mathcal{P}$: Abstract linear differential equation
  - Linear ordinary and partial differential equations
  - Delay equations, etc.

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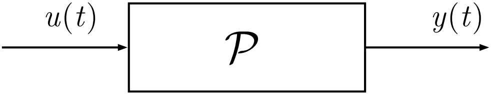
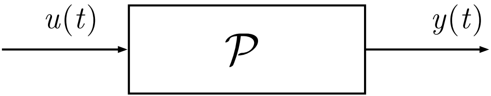


- The System $\mathcal{P}$: Abstract linear differential equation
  - Linear ordinary and partial differential equations
  - Delay equations, etc.
- Goal: For a given *reference signal* $y_{ref}(t)$,

  choose *input* $u(t)$ in such a way that *output* $y(t)$ satisfies

$$\|y(t) - y_{ref}(t)\| \to 0 \qquad \text{as} \quad t \to \infty.$$

The System To Be Controlled



Main tools:

- Linear functional analysis
  - Theory of semigroups, stability
  - Spectral theory of linear operators
  - Sylvester operator equations
- Complex analysis, function theory, PDE's

The System To Be Controlled

The controlled system on the Banach space X is of the form

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t), \qquad x(0) = x_0 \in X \\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

Here

- $x(t)$ is the *state*, $u(t)$ the *input*, and $y(t)$ the *output*
- A generates a *strongly continuous semigroup* e^{At} on X
- operators A, B, C, and D are linear and bounded

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State $x(t)$ can be expressed using the semigroup e^{At}:

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}Bu(s)ds$$

An Example of a Controlled System: A Heat Equation

Consider a one-dimensional heat equation on $(0,1)$. Choose $X = L^2(0,1)$ and (with appropriate b.c.'s)

$$\frac{d}{dt}x(t,z) = \frac{d^2}{dz^2}x(t,z) + b(z)u(t)$$

$$y(t) = \int_0^1 c(z)x(t,z)\,dz$$

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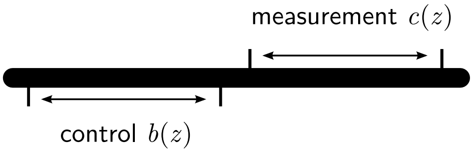

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Now $x(t) = x(t,\cdot) \in L^2(0,1)$, $u(t), y(t) \in \mathbb{C}$, and the operators

$$A = \frac{d^2}{dz^2}, \qquad Bu = b(\cdot)u, \qquad Cf = \int_0^1 c(z)f(z)\,dz$$

A Finite-Dimensional System

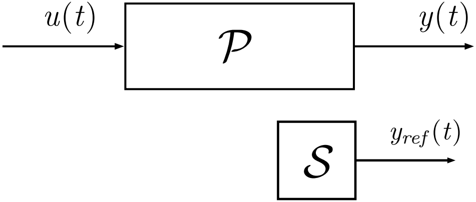
In a simpler case, A, B, C, and D are matrices

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t), \qquad x(0) = x_0 \in X\\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

Then

- The system is a linear ordinary differential equation
- The eigenvalues of A determine asymptotic behavior of e^{At}
- In particular, if $\operatorname{Re}\lambda < 0$ for all $\lambda \in \sigma(A)$, then $\|e^{At}\| \to 0$ exponentially fast as $t \to \infty$.

The Exosystem



Goal:

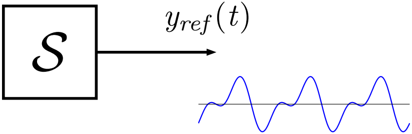
Choose u *such that* y *satisfies* $\|y(t) - y_{ref}(t)\| \to 0$ *as* $t \to \infty$.

The Exosystem (Signal Generator)

The reference signals are generated by the *exosystem*

$$\dot{v}(t) = Sv(t), \qquad v(0) = v_0 \in W$$
$$y_{ref}(t) = Fv(t)$$

on the space W. Operator S generates a *group* e^{St}.



Example

For $W = \mathbb{C}^q$, signals are combinations of trigonometric functions.

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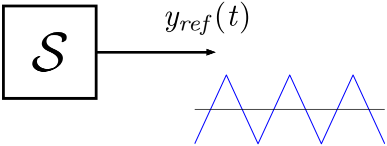
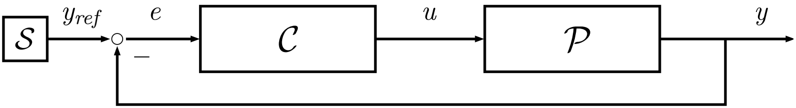


Example

With $\dim W = \infty$ we can consider continuous periodic functions.

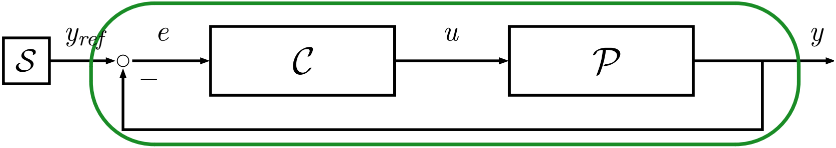
Feedback Controller



The error feedback controller on a Banach space Z is of the form

$$\dot{z}(t) = \mathcal{G}_1 z(t) + \mathcal{G}_2 e(t), \qquad z(0) = z_0 \in Z$$
$$u(t) = Kz(t),$$

where $\mathcal{G}_1$ generates a semigroup and $\mathcal{G}_2$ and K are bounded.


$\mathcal{S}$
y_{ref}
e
$-$
$\mathcal{C}$
u
$\mathcal{P}$
y

The Closed-Loop System

The closed-loop system with state $(x(t), z(t))^T \in X \times Z$

$$\begin{aligned} \dot{x}_e(t) &= A_e x_e(t) + B_e v(t), \qquad x_e(0) = (x_0, z_0)^T \\ e(t) &= C_e x_e(t) + D_e v(t). \end{aligned}$$

- $e(t) = y(t) - y_{ref}(t)$ is the *regulation error*
- $v(t)$ is the state of the exosystem $\dot{v} = Sv$.

Output Regulation Problem

Problem (Output Regulation Problem)

Choose controller parameters $(\mathcal{G}_1, \mathcal{G}_2, K)$ *such that*

(i) *The closed-loop system operator* A_e *generates a strongly stable* C_0*-semigroup on* $X \times Z$;

(i.e., $\|e^{A_e t}x\| \to 0$ *for all* $x \in X_e$*)*

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(ii) *For all initial states* x_0, z_0 *and* v_0 *the regulation error* $e(t) = y(t) - y_{ref}(t)$ *decays to zero as* $t \to \infty$*;*

Theorem (Characterization of solvability of the ORP)

Assume the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ *stabilizes the closed-loop system strongly and that the Sylvester equation*

$$\Sigma S = A_e \Sigma + B_e$$

has a bounded solution Σ*.*

Then the controller solves the ORP if and only if Σ *satisfies*

$$C_e \Sigma + D_e = 0.$$

The Idea of The Proof

- If the Sylvester equation

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- If $e^{A_e t}$ is stable, then it turns out

$$e(t) \longrightarrow 0 \quad \forall v_0 \in W \qquad \Leftrightarrow \qquad C_e \Sigma + D_e = 0.$$

The Sylvester Equation

Comments on the Sylvester equation

$$\Sigma S = A_e \Sigma + B_e \tag{1}$$

for unbounded operator S (of the exosystem).

1. (FACT) the operator $\mathcal{G}_1$ (controller) must contain the eigenvalues of S.
2. Operator S may have an infinite number of imaginary eigenvalues.
3. In this case the operator A_e can not be exponentially stable (i.e., $\|e^{A_e t}\| \to 0$ exponentially fast as $t \to \infty$)
4. $\Rightarrow$ no general results on solvability of (1).

The Sylvester Equation

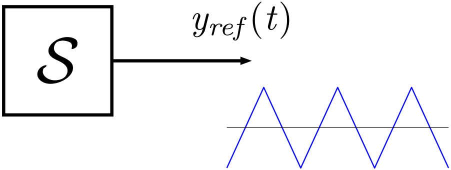
Results on the Sylvester equation

$$\Sigma S = A_e \Sigma + B_e \tag{1}$$

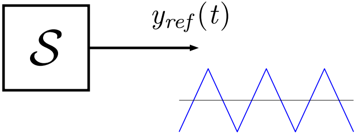
for unbounded operator S (of the exosystem).

- Solvability for particular types of ∞-dimensional exosystems
  - S diagonal, block diagonal
  - $S = iS_0$, where S_0 self-adjoint operator.
- Uniqueness for exosystems generating periodic and *almost periodic* reference signals

Other Types of Exosystems



Further/Other Research Directions



A *periodic exosystem*

$$\dot{v}(t) = S(t)v(t), \qquad v(0) = v_0$$
$$y_{ref}(t) = F(t)v(t)$$

where $S(\cdot)$ and $F(\cdot)$ are periodic functions.

Leads to the use of theory of nonautonomous infinite-dimensional systems and *evolution families*.

Further/Other Research Directions

The closed-loop system becomes time-dependent

$$\dot{x}_e(t) = A_e(t)x_e(t) + B_e(t)u(t), \qquad x_e(0) = x_{e0}.$$

The state can be expressed using the *strongly continuous evolution family* $U_e(t,s)$ as

$$x_e(t) = U_e(t,0)x_{e0} + \int_0^t U_e(t,s)B_e(s)u(s)ds.$$

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$$\dot{x}_e(t) = A_e(t)x_e(t) + B_e(t)u(t), \qquad x_e(0) = x_{e0}.$$

The Sylvester equation $\Sigma S = A_e\Sigma + B_e$ in the theory is replaced by an infinite-dimensional *Sylvester differential equation*

$$\dot{\Sigma}(t) + \Sigma(t)S(t) = A_e(t)\Sigma(t) + B_e(t).$$

In This Presentation

- Output regulation theory for infinite-dimensional systems
- Comments on the main mathematical tools
- Solvability of the associated Sylvester equations