On Perturbation of Strongly Stable Riesz-Spectral Operators

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Abstract Cauchy Problem

Consider an abstract differential equation

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0 \in X \tag{ACP}$$

on a Hilbert space X, where $A : \mathcal{D}(A) \subset X \to X$ is linear.

Abstract Cauchy Problem

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If A generates a *strongly continuous semigroup* $T(t)$:

$$x(t) = T(t)x_0.$$

Strongly Continuous Semigroup

Definition

A family $(T(t))_{t\geq 0} \subset \mathcal{L}(X)$ is a *strongly continuous semigroup* if

1. $T(0) = I$
2. $T(s+t) = T(s)T(t)$ for all $s, t \geq 0$
3. $T(\cdot)$ is strongly continuous at $t = 0$, i.e.

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Definition

Generator of a semigroup is a linear operator $A : \mathcal{D}(A) \subset X \to X$

$$Ax = \lim_{t\to 0+} \frac{T(t)x - x}{t}, \quad \mathcal{D}(A) = \left\{ x \;\middle|\; \lim_{t\to 0+} \frac{T(t)x - x}{t} \text{ exists} \right\}$$

Stability of Solutions

The semigroup is called *strongly stable* if for all $x_0 \in X$

$$\|T(t)x_0\| \to 0 \qquad \text{as} \quad t \to \infty,$$

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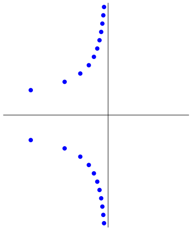
Lemma (Sufficient condition for strong stability)

1. *$\sigma(A) \subset \mathbb{C}^-$ (open left half-plane of $\mathbb{C}$);*
2. *For some $M > 0$ we have $\|T(t)\| \le M$ for all $t \ge 0$.*



Perturbed Abstract Cauchy Problem

Consider the abstract differential equation

$$\dot{\tilde{x}}(t) = (A + B)\tilde{x}(t), \qquad \tilde{x}(0) = \tilde{x}_0 \in X. \qquad \text{(pACP)}$$

where B is a bounded linear operator on X.

Perturbation B can represent modelling error, uncertainty etc.

Main Problem

Problem

Assume

- $A : \mathcal{D}(A) \subset X \to X$ *generates a strongly stable* C_0*-semigroup*
- $B \in \mathcal{L}(X)$.

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When is the C_0*-semigroup generated by* $A + B$ *strongly stable?*

Problem (Assumptions)

- $A = \sum_{k \in \mathbb{Z}} \lambda_k \langle \cdot, \phi_k \rangle \phi_k$, $\mathcal{D}(A) = \{ \, x \mid \sum_{k \in \mathbb{Z}} |\lambda_k|^2 |\langle x, \phi_k \rangle|^2 < \infty \, \}$
- $B = \langle \cdot, g \rangle b \in \mathcal{L}(X)$.

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- $B = \langle \cdot, g \rangle b \in \mathcal{L}(X)$.

Determine sets $M, N \subset X$ such that if $b \in M$ and $g \in N$, then

(a) $\sigma(A + \langle \cdot, g \rangle b) \subset \mathbb{C}^-$

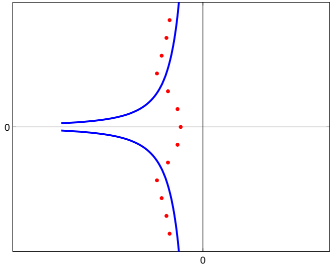
(b) $A + \langle \cdot, g \rangle b$ *generates a strongly stable C_0-semigroup.*

Asymptotic behaviour of $\sigma(A)$

Assumption (Geometric assumption on $\sigma(A)$)

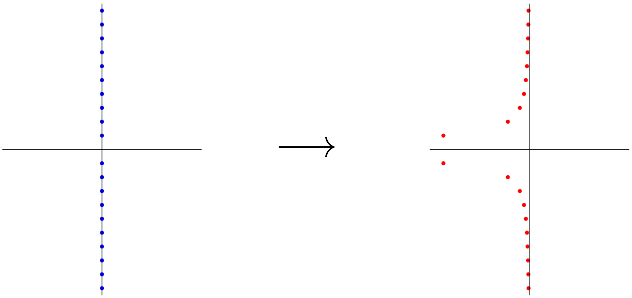
There exist constants $c, \alpha > 0$ *and* $y_0 > 0$ *such that*

$$\operatorname{Re}\lambda_k \leq -\frac{c}{|\operatorname{Im}\lambda_k|^\alpha} \qquad \textit{if} \quad |\operatorname{Im}\lambda_k| \geq y_0$$



Motivation

Robust Output Regulation: Strong stabilization leads to an operator with spectrum approaching $i\mathbb{R}$ asymptotically.



Robustness properties?

Main Results

Definition

For $\beta \geq 0$ define a Hilbert space $(M_\beta, \|\cdot\|_\beta)$ such that

$$M_\beta = \left\{ x \in X \mid \sum_{k \in \mathbb{Z}} |\lambda_k|^{2\beta} |\langle x, \phi_k \rangle|^2 < \infty \right\}$$

$$\|\cdot\|_\beta = \left(\sum_{k \in \mathbb{Z}} |\lambda_k|^{2\beta} |\langle \cdot, \phi_k \rangle|^2 \right)^{\frac{1}{2}}$$

Actually $M_\beta = \mathcal{D}((-A)^\beta)$ and $\|x\|_\beta = \|(-A)^\beta x\|$ since $-A$ is sectorial.

Perturbation of the Spectrum

Proposition

Let $\sigma(A)$ satisfy the geometric assumption for some $\alpha > 0$.

If $\beta + \gamma \geq \alpha$, there exists $\delta > 0$ such that

$$\sigma(A + \langle \cdot, g \rangle b) \subset \mathbb{C}^-$$

whenever $b \in M_\beta$ and $g \in M_\gamma$ with $\|b\|_\beta < \delta$ and $\|g\|_\gamma < \delta$.

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Proof.

Using the first Weinstein-Aronszajn formula. □

Corollary

Let $\sigma(A)$ satisfy the geometric assumption for some $\alpha > 0$.

If $n, m \in \mathbb{N}_0$ are such that $n + m \geq \alpha$, then there exists $\delta > 0$ such that

$$\sigma(A + \langle \cdot, g \rangle b) \subset \mathbb{C}^-$$

whenever $b \in \mathcal{D}(A^n)$ and $g \in \mathcal{D}(A^m)$ with $\|A^n b\| < \delta$ and $\|A^m g\| < \delta$.

Preservation of Uniform Boundedness

Proposition

Let $\sigma(A)$ satisfy the geometric assumption for some $\alpha > 0$.

Denote by $T_B(t)$ the C_0-semigroup generated by $A + \langle \cdot, g \rangle b$. There exists $\delta > 0$ such that

- *$T_B(t)$ is uniformly bounded whenever $b \in M_\alpha$ with $\|b\|_\alpha < \delta$;*
- *$T_B(t)$ is uniformly bounded whenever $g \in M_\alpha$ with $\|g\|_\alpha < \delta$.*

Preservation of Strong Stability

Proposition

Let $\sigma(A)$ satisfy the geometric assumption for some $\alpha > 0$.

If $\beta \geq \alpha$ or $\gamma \geq \alpha$, there exists $\delta > 0$ such that

$$A + \langle \cdot, g \rangle b$$

generates a strongly stable C_0-semigroup whenever $b \in M_\beta$ and $g \in M_\gamma$ with $\|b\|_\beta < \delta$ and $\|g\|_\gamma < \delta$.

Extensions

- $A = \sum \lambda_k \langle \cdot, \psi_k \rangle \phi_k$ is a Riesz spectral operator
  - Straightforward
  - Separate spaces $b \in M_\beta$ and $g \in N_\gamma$ corresponding to $\{\psi_k\}$ and $\{\phi_k\}$, respectively.

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- $A = \sum \lambda_k \langle \cdot, \psi_k \rangle \phi_k$ is a Riesz spectral operator
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  - Separate spaces $b \in M_\beta$ and $g \in N_\gamma$ corresponding to $\{\psi_k\}$ and $\{\phi_k\}$, respectively.
- A general finite-rank perturbation $B = \sum_{j=1}^{m} \langle \cdot, g_j \rangle b_j$
  - Using Weinstein-Aronszajn determinant

Strongly Stabilized Wave Equation

Consider a wave equation on $(0, 1)$:

$$\frac{\partial^2 w}{\partial t^2}(z,t) = \frac{\partial^2 w}{\partial z^2}(z,t) + b_0(z)u(t)$$
$$w(0,t) = w(1,t) = 0$$

where $b_0(z) = \sqrt{3}(1 - z)$.

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Define $x = \begin{bmatrix} w \\ \frac{\partial w}{\partial t} \end{bmatrix}$ and $A_0 = -\frac{\partial^2}{\partial z^2}$ with domain

$$\mathcal{D}(A_0) = \left\{ w \in L^2(0,1) \;\middle|\; w, w' \text{ abs. cont. } w'' \in L^2(0,1), \; w(0) = w(1) = 0 \right\}$$

Linear system

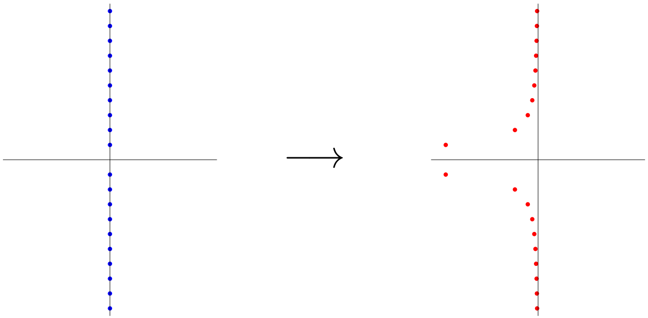
$$\dot{x} = Ax + Bu, \qquad x(0) = x_0$$

on a Hilbert space $X = \mathcal{D}(A_0^{\frac{1}{2}}) \times L^2(0,1)$ with

$$A = \begin{bmatrix} 0 & I \\ \frac{\partial^2}{\partial z^2} & 0 \end{bmatrix}, \qquad B = b = \begin{bmatrix} 0 \\ b_0 \end{bmatrix}.$$

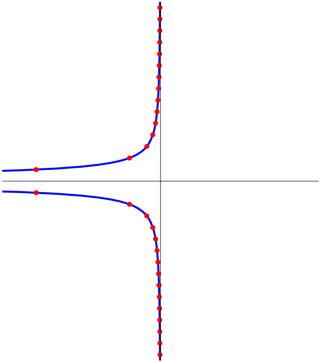
Strong Stabilization

Find feedback K such that $A + BK$ is a Riesz-spectral operator with eigenvalues $\sigma(A + BK) = \{-\frac{\pi}{|n|^2} + in\pi\}_{\substack{n\in\mathbb{Z}\\ n\neq 0}}$



Geometric Assumption

The spectrum of $A + BK$ satisfies the geometric assumption for $\alpha = 2$, $c = \pi^3$ and $y_0 = \pi$.



Perturbations

Consider perturbations

$$\frac{\partial^2 w}{\partial t^2}(z,t) = \frac{\partial^2 w}{\partial z^2}(z,t) + b_0(z)u(t) + d_0(z)\left(\langle w, g_1 \rangle_{L^2} + \langle \frac{\partial w}{\partial t}, g_2 \rangle_{L^2} \right)$$

where $d_0, g_2 \in \mathcal{D}(A_0)$ (abs. cont. etc.) and $g_1 \in L^2(0,1)$.

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where $d_0, g_2 \in \mathcal{D}(A_0)$ (abs. cont. etc.) and $g_1 \in L^2(0,1)$.

Then $\sigma(A + BK + \langle \cdot, g\rangle d) \subset \mathbb{C}^-$ whenever

$$13\|d_0\|_{L^2} + \|d_0'\|_{L^2} < 1$$

$$\|g_1\|_{L^2}^2 + \|g_2'\|_{L^2}^2 < \frac{1}{35^2}$$

$$\|g_1\|_{L^2}^2 + \pi^2\|g_2\|_{L^2}^2 < \frac{\pi^2}{35^2}$$

Conclusions

- Conditions for the preservation of the property $\sigma(A) \subset \mathbb{C}^-$ and the strong stability of a semigroup
- Application to a wave equation

Thank You!