Boundary Control Systems and Controller Design for PDEs

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Introduction

Goal of the Talk

1. *Discuss the challenges in using abstract systems in controller design for PDEs*
2. *Propose abstract Boundary Control Systems as a remedy to some of the issues.*

Motivating Example

Wave equation on $(0,1)$ with boundary input and outputs:

$$\begin{aligned} w_{tt}(\xi,t) &= w_{\xi\xi}(\xi,t) \\ -w_\xi(0,t) &= u(t), \qquad w_\xi(1,t) = 0 \\ y(t) &= (w(1,t), w_t(1,t))^\top \end{aligned}$$

Problem

Construct a stabilising output feedback control law such that $(w(\cdot,t), w_t(\cdot,t)) \to 0$ *as* $t \to \infty$.

- Input at $\xi = 0$ and output at $\xi = 1$: Stabilisation not possible using static output feedback $\rightsquigarrow$ dynamic controller design.
- Decide: Work with the PDE, or represent it as an ∞-dimensional system to apply general results?

Alternatives: PDE Techniques vs. Abstract Approach

Abstract approach workflow:

1. Represent the wave equation as an infinite-dimensional system (within a suitable class)
2. Find a controller design result for the class from the literature
3. Tune the parameters of the controller to fit our wave equation. Closed-loop stability is then guaranteed by the abstract result
4. Interpret the abstract controller as a PDE model

A Very Formal Sketch of the Abstract Approach

1. **Representing:** If $x(t) = (w(\cdot,t), w_t(\cdot,t))$, our wave system can be represented as a system on $X = H^1(0,1) \times L^2(0,1)$ of the form

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

Theorem (2. Abstract controller result)

Choose K and L in such a way that the dynamics described by $A+BK$ and $A+LC$ are exponentially stable. Then the controller

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(C\hat{x}(t) - y(t))$$
$$u(t) = K\hat{x}(t)$$

stabilises the system.

A Very Formal Sketch of the Abstract Approach

3. **Tuning:** The parameters K and L need to be chosen so that

$$\dot{x}(t) = (A + BK)x(t) \qquad \text{and} \qquad \dot{x}(t) = (A + LC)x(t)$$

are exponentially stable (apply PDE techniques or observability etc)

4. **Interpreting:** Since (A, B, C) describes a wave equation, also

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + L(C\hat{x}(t) - y(t)) \\ u(t) &= K\hat{x}(t) \end{aligned}$$

should be an abstract representation of a wave-type PDE.

Pros and Cons of the Abstract Approach

Benefits:

- A clear structure for the controller
- Stability guaranteed by the associated abstract result
- In parameter tuning, choosing K and L to stabilise

  $$\dot{x}(t) = (A + BK)x(t) \qquad \text{and} \qquad \dot{x}(t) = (A + LC)x(t)$$

  is in general easier than proving the closed-loop stability in the PDE-approach
- Approach is systematic, different types of PDEs can be handled in a uniform way

Pros and Cons of the Abstract Approach

Challenges in the case of PDEs with boundary inputs and outputs:

- Representing the PDE as an abstract system is often tricky! Especially B is unbounded in the sense that $\mathrm{Ran}(B) \not\subset X$.
  - The appropriate class can be very theoretically demanding, requiring heavy operator theory
  - Pronounced when the PDE is multi-dimensional
  - Leads to situation that advanced methods from both PDEs and FA are required for the solution of the problem
- Is the controller design result available for the abstract class?
- The designed controller is an abstract system, and interpretation step may be highly nontrivial as well

Solution: An Abstract Class Closer to the PDE

Alternatively, the wave equation

$$\begin{aligned} w_{tt}(\xi,t) &= w_{\xi\xi}(\xi,t) \\ \begin{pmatrix} -w_\xi(0,t) \\ w_\xi(1,t) \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t) \\ y(t) &= (w(1,t), w_t(1,t))^\top \end{aligned}$$

can be expressed as an **abstract Boundary Control System**

$$\begin{aligned} \dot{x}(t) &= \mathfrak{A}x(t) \\ \mathfrak{B}x(t) &= Qu(t) \\ y(t) &= \mathfrak{C}x(t) \end{aligned}$$

with $x(t) = (w(\cdot,t), w_t(\cdot,t))$ on $X = H^1(0,1) \times L^2(0,1)$.

Boundary Control System Representation

$$\frac{d}{dt}\begin{pmatrix} w(\xi,t) \\ w_t(\xi,t) \end{pmatrix} = \begin{pmatrix} w_t(\xi,t) \\ w_{\xi\xi}(\xi,t) \end{pmatrix}$$

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$$y(t) = \begin{pmatrix} w(1,t) \\ w_t(1,t) \end{pmatrix}$$

vs.

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$$y(t) = \mathfrak{C}x(t)$$

We choose $x(t) = (w(\cdot,t), w_t(\cdot,t))$ on $X = H^1(0,1) \times L^2(0,1)$,

$$\mathfrak{A}x = \begin{pmatrix} g \\ f'' \end{pmatrix}, \qquad ,$$

for $x = (f,g) \in D(\mathfrak{A}) := H^2(0,1) \times H^1(0,1)$.

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Boundary Control Systems

In our wave equation's representation

$$\begin{aligned} \dot{x}(t) &= \mathfrak{A}x(t) \\ \mathfrak{B}x(t) &= Qu(t) \\ y(t) &= \mathfrak{C}x(t) \end{aligned}$$

the operators have the characteristic properties.

(i) $\mathfrak{A} : D(\mathfrak{A}) \subset X \to X$, $\mathfrak{B}, \mathfrak{C} \in \mathcal{L}(D(\mathfrak{A}), \mathbb{C}^2)$, and $Q \in \mathcal{L}(\mathbb{C}, \mathbb{C}^2)$

(ii) $A := \mathfrak{A}|_{\mathcal{N}(\mathfrak{B})}$ with $D(A) = \mathcal{N}(\mathfrak{B})$ generates semigroup on X

(iii) $\mathfrak{B}$ has a bounded right inverse $\mathfrak{B}^r \in \mathcal{L}(\mathbb{C}^2, D(\mathfrak{A}))$

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Definition

We call $(\mathfrak{B}, \mathfrak{A}, \mathfrak{C}, Q)$ a **boundary node** on the Hilbert space X.

Benefits of Boundary Control Systems

The construction is relatively simple for the representation

$$\begin{aligned} \dot{x}(t) &= \mathfrak{A}x(t) \\ \mathfrak{B}x(t) &= Qu(t) \\ y(t) &= \mathfrak{C}x(t) \end{aligned}$$

- $\mathfrak{B}$ is a boundary trace operator which arises naturally from the boundary conditions of the PDE.
- It does not need to be "inverted" to obtain the unbounded "B", and extrapolation spaces are not needed.
- Similarly, the BCS can easily be converted back to a PDE.
- Main caveat: BCS do not have as many results as other abstract classes. **But this can change!**

Basic Properties: Solutions

$$\begin{aligned} \dot{x}(t) &= \mathfrak{A}x(t), \qquad x(0) = x_0 \\ \mathfrak{B}x(t) &= Qu(t) \\ y(t) &= \mathfrak{C}x(t) \end{aligned}$$

Theorem (Malinen–Staffans 2007)

If $x_0 \in D(\mathfrak{A})$ and $u \in C^2([0,\infty);\mathbb{C})$ satisfy $\mathfrak{B}x_0 = Qu(0)$, then the BCS has a unique ***classical solution*** *such that $x \in C^1([0,\infty);X)$, $x(t) \in D(\mathfrak{A})$ and the equations hold for $t \geq 0$.*

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$(\mathfrak{B},\mathfrak{A},\mathfrak{C},Q)$ is called **well-posed** if for one/all $\tau > 0$ there exists $M_\tau > 0$ such that

$$\|x(t)\|_X^2 + \int_0^\tau \|y(s)\|^2 ds \leq M_\tau \left(\|x_0\|_X^2 + \int_0^\tau \|u(s)\|^2 ds \right)$$

Basic Properties: Other Results

- If $(\mathfrak{B}, \mathfrak{A}, \mathfrak{C}, Q)$ is well-posed, then BCS has well-defined and unique **generalised solutions** (as limits of classical solutions on intervals $[0, \tau]$) and **mild solutions**
- If $(\mathfrak{B}, \mathfrak{A}, \mathfrak{C}, Q)$ is well-posed, the maps

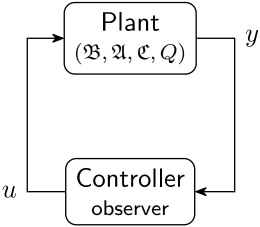
$$x_0 \mapsto x(t), \quad u \mapsto x(t), \quad x_0 \mapsto y, \quad u \mapsto y$$

define a **well-posed linear system**.

Controller Design Result

Observer-based controller is a copy of the plant with **output injections**

$$\begin{aligned}\dot{\hat{x}} &= \mathfrak{A}\hat{x} + L_i(\hat{y} - y)\\ \mathfrak{B}\hat{x} &= Qu + L(\hat{y} - y)\\ \hat{y} &= \mathfrak{C}\hat{x}, \qquad u = \mathfrak{K}\hat{x}\end{aligned}$$



Parameters:

- $\mathfrak{K}$ — state feedback gain
- L, L_i — output injection gains

BCS Stabilisation Result

Theorem (Fkirine–LP 2026)

Let $(\mathfrak{B}, \mathfrak{A}, \left[\begin{smallmatrix}\mathfrak{C}\\ \mathfrak{K}\end{smallmatrix}\right], [Q, L])$ *be a well-posed boundary node and that a minimal transfer function condition holds. Then the closed-loop system is well-posed and*

$$A_K := \mathfrak{A}|_{\mathcal{N}(\mathfrak{B}+Q\mathfrak{K})} \qquad \text{and} \qquad A_L := (\mathfrak{A} + L_i\mathfrak{C})|_{\mathcal{N}(\mathfrak{B}+L\mathfrak{C})}$$

generate semigroups T_K *and* T_L*. If* T_K *and* T_L *are exponentially stable, then the closed-loop system is exponentially stable.*

- A_K and A_L correspond to the stabilisation problem for the PDE using state feedback and output injections, respectively.

Stabilisation of the Wave Equation

1. We formulated the wave equation as a BCS on $X = H^1 \times L^2$,

$$\begin{aligned} w_{tt}(\xi,t) &= w_{\xi\xi}(\xi,t) \\ -w_\xi(0,t) &= u(t), \qquad w_\xi(1,t) = 0 \\ y(t) &= (w(1,t), w_t(1,t))^\top \end{aligned}$$

2. $(\mathfrak{B}, \mathfrak{A}, \mathfrak{C}, Q)$ is well-posed (existing PDE results).
$\rightsquigarrow$ We can use our BCS stabilisation result.

3. Tuning. Stability of A_K corresponds to the PDE

$$\begin{aligned} w_{tt}(\xi,t) &= w_{\xi\xi}(\xi,t) \\ -w_\xi(0,t) &= \mathfrak{K}(w(\cdot,t), w_t(\cdot,t)), \qquad w_\xi(1,t) = 0 \end{aligned}$$

Find $\mathfrak{K}$, e.g., via backstepping.

The Controller

Our result guarantees that closed-loop stability is achieved with

$$
\begin{aligned}
\dot{\hat{x}} &= \mathfrak{A}\hat{x} + L_i(\hat{y} - y) \\
\mathfrak{B}\hat{x} &= Qu + L(\hat{y} - y) \\
\hat{y} &= \mathfrak{C}\hat{x}, \qquad u = \mathfrak{K}\hat{x}
\end{aligned}
$$

4. Interpret. The controller is the abstract representation of

$$
\begin{aligned}
\hat{w}_{tt}(\xi,t) &= \hat{w}_{\xi\xi}(\xi,t) - \ell_i\big(\hat{w}(1,t) - w(1,t)\big) \\
-\hat{w}_{\xi}(0,t) &= u(t), \quad \hat{w}_{\xi}(1,t) = -\ell_b\big(\hat{w}_t(1,t) - w_t(1,t)\big) \\
u(t) &= -\kappa_0\hat{w}(0,t) - \kappa_1\hat{w}_t(0,t) - \kappa_0\kappa_1\int_0^1 \hat{w}_t(\xi,t)\,d\xi
\end{aligned}
$$

with $\kappa_0, \kappa_1, \ell_b > 0$, and with $\ell_i > 0$ sufficiently small.

Conclusion

Take-home messages:

- Controller design using abstract methods has the workflow: **Represent** $\rightarrow$ **Construct** $\rightarrow$ **Tune** $\rightarrow$ **Interpret**
- Often challenging for PDEs with boundary I/O
- Choosing a suitable framework can help, and Boundary Control Systems have a lot of potential in this role

M. Fkirine, LP, "Dynamic Stabilisation of Boundary Control Systems," preprint, `https://arxiv.org/abs/2605.28189`

S. Nicaise, LP, D. Seifert, "Stability of Abstract Coupled Systems," *J. Functional Analysis*, 2025.

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