Non-Uniform Stability of Damped Contraction Semigroups

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Goal of the Talk

Introduce general conditions for non-uniform stability of ***damped*** *hyperbolic Cauchy problems (and PDEs).*

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*Introduce general conditions for non-uniform stability of **damped** hyperbolic Cauchy problems (and PDEs).*

$$\begin{cases} \dot{x}(t) = (A - BB^*)x(t) \\ x(0) = x_0 \end{cases}$$

and

$$\begin{cases} \ddot{w}(t) + A_0 w(t) + B_0 B_0^* \dot{w}(t) = 0 \\ w(0) = w_0, \quad \dot{w}(0) = w_1 \end{cases}$$

Problem

Formulate conditions on (A, B) *and* (A_0, B_0) *such that*

$$\|x(t)\| \to 0, \qquad \textit{or} \qquad \|w(t)\| \to 0 \qquad \textit{as} \quad t \to \infty$$

*and especially study the **rate** of the convergence.*

Goal of the Talk

Introduce general conditions for non-uniform stability of ***damped*** *hyperbolic Cauchy problems and PDEs.*

Damped systems of the form

$$\dot{x}(t) = (A - BB^*)x(t) \quad \text{and} \quad \ddot{w}(t) + A_0 w(t) + B_0 B_0^* \dot{w}(t) = 0$$

Motivation:

- So-called "polynomial" and "non-uniform" stability often arise in wave/beam/plate equations with weak or partial dampings
- Most of the current literature based on case-by-case analysis

Main results:

- General **observability-type** sufficient conditions for stability

(B^*, A) **exactly** observable $\Leftrightarrow$ $A - BB^*$ exponentially stable

$\|x(t)\| \le Me^{-\omega t}\|x_0\| \ \forall x_0$

(B^*, A) **approx.** observable $\Leftrightarrow^*$ $A - BB^*$ strongly/weakly stable

$x(t) \to 0 \ \forall x_0$

[Slemrod, Levan, Russell, Benchimol, Guo–Luo, Lasiecka–Triggiani, Curtain–Weiss . . .]

(B^*, A) **exactly** observable $\quad\Leftrightarrow\quad$ $A - BB^*$ exponentially stable

$$\|x(t)\| \le Me^{-\omega t}\|x_0\| \ \forall x_0$$

(B^*, A) **non-uniformly** obs. $\quad\Leftrightarrow\quad$ $A - BB^*$ non-uniformly stable

(B^*, A) **approx.** observable $\quad\Leftrightarrow^*\quad$ $A - BB^*$ strongly/weakly stable

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Earlier work: Ammari–Tucsnak 2001, Ammari et. al.

Main Assumptions (roughly, to keep things simple)

- A generates a contraction semigroup e^{At} on X Hilbert, i.e., $t \mapsto e^{At}$ is strongly continuous and $\|e^{At}\| \le 1$.
- Either $B \in \mathcal{L}(U,X)$, or (A,B,B^*) is a "well-posed system".
- $\Rightarrow A - BB^*$ generates a contraction semigroup $e^{(A-BB^*)t}$

Main case:

$$\ddot{w}(t) + A_0 w(t) + B_0 B_0^* \dot{w}(t) = 0, \qquad \text{on} \quad X_0$$

where $A_0 > 0$, $B_0 \in \mathcal{L}(U, D(A_0^{1/2})^*)$ leads to

$$A = \begin{bmatrix} 0 & I \\ A_0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ B_0 \end{bmatrix}, \qquad \text{on} \quad X = D(A_0^{1/2}) \times X_0.$$

"Well-posedness" $\Leftrightarrow \lambda \mapsto \lambda B_0^* (\lambda^2 + A_0)^{-1} B_0$ bounded for $\lambda = 1 + is$

Polynomial and Non-Uniform Stability

Definition

$e^{(A-BB^*)t}$ generated by $A - BB^*$ is **non-uniformly stable** if there exist an increasing $M_T \colon [t_0,\infty) \to \mathbb{R}_+$ and $C > 0$ such that

$$\|e^{(A-BB^*)t}x_0\| \leq \frac{C}{M_T(t)}\|(A-BB^*)x_0\| \qquad x_0 \in D(A-BB^*)$$

[. . . , Liu–Rao '05, Batty–Duyckaerts '08, Borichev–Tomilov '10, Rozendaal–Seifert–Stahn '19]

Application: $E(t) \sim \|e^{(A-BB^*)t}x_0\|^2$ for many PDE systems.

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Theorem (BT'10, RSS'19)

Assume $e^{(A-BB^*)t}$ *is contractive,* $i\mathbb{R} \subset \rho(A-BB^*)$*, and*

$$\|(is - A + BB^*)^{-1}\| \leq M(|s|), \qquad M \text{ non-decreasing.}$$

- *If* $M(s) \lesssim 1 + s^\alpha$*, then* $M_T(t) = t^{1/\alpha}$
- *If* M *has "positive increase", then* $M_T(t) = M^{-1}(t)$.

Main Problem

Damped systems of the form

$$\dot{x}(t) = (A - BB^*)x(t) \quad \text{and} \quad \ddot{w}(t) + A_0 w(t) + B_0 B_0^* \dot{w}(t) = 0$$

Problem

How do (A, B) or (A_0, B_0) determine the stability of the system?

Main results:

Conditions based on ***observability-type properties*** *of (B^*, A) and (B_0^*, iA_0).*

A "Non-uniform Hautus test"

Consider the Hautus-type condition [Miller 2012]

$$\|x\|^2 \le M_o(|s|)\|(is-A)x\|^2 + m_o(|s|)\|B^*x\|^2, \quad x \in D(A), s \in \mathbb{R},$$

for some non-decreasing $M_o, m_o \colon [0,\infty) \to [r_0,\infty)$.

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Theorem

If the above condition holds, then $i\mathbb{R} \subset \rho(A-BB^*)$*. If* $M(s) := M_o(s) + m_o(s)$ *has positive increase, then*

$$\|e^{(A-BB^*)t}x_0\| \leq \frac{C}{M^{-1}(t)}\|(A-BB^*)x_0\|, \quad x_0 \in D(A-BB^*)$$

Observability of the Schrödinger Group

For

$$\ddot{w}(t) + A_0 w(t) + B_0 B_0^* \dot{w}(t) = 0, \qquad \text{on} \quad X_0$$

and $M_S, m_S \colon [0,\infty) \to [r_0, \infty)$ consider ($s \geq 0$)

$$\|w\|^2 \leq M_S(s) \|(s^2 - A_0)w\|^2 + m_S(s)\|B_0^* w\|^2, \quad w \in D(A_0)$$

This is **observability of the "Schrödinger group"** (B_0^*, iA_0) (generalises Anantharaman–Leataud 2014, Joly–Laurent 2019)

Theorem

A similar result, decay rate determined by $M^{-1}(t)$, where

$$M(s) := M_S(s) m_S(s)(1+s^2).$$

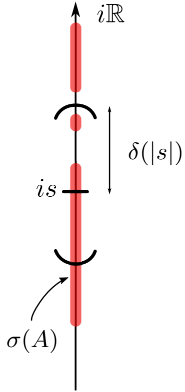
A "Wavepacket Condition"

For A skew-adjoint with spectral projection $P_{(a,b)}$ (for $i(a,b) \subset i\mathbb{R}$)

$$\|B^*x\| \geq \gamma(|s|)\|x\|, \qquad x \in \mathrm{Ran}(P_{(s-\delta(|s|),s+\delta(|s|))}),\ s \in \mathbb{R}$$

for some non-increasing $\delta, \gamma\colon [0,\infty) \to (0, r_0]$.

Such x are often called **"wavepackets"** of A.
(Used for exact observability, e.g., in Ramdani et. al. 2005, Miller 2012, Tucsnak–Weiss 2009.)



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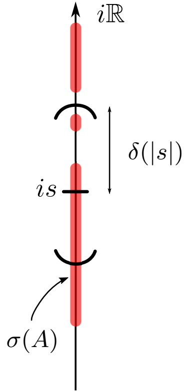


Theorem

If $A^* = -A$ and if $M(s) := \delta(s)^{-2}\gamma(s)^{-2}$ has positive increase, then

$$\|e^{(A-BB^*)t}x_0\| \leq \frac{C}{M^{-1}(t)}\|(A-BB^*)x_0\|.$$

Time-Domain Non-Uniform Observability

Time-domain observability conditions:

For $\tau, c_\tau, \beta > 0$:

$$c_\tau \|(1-A)^{-\beta} x_0\|^2 \le \int_0^\tau \|B^* e^{At} x_0\|^2 dt, \qquad x_0 \in D(A).$$

(cf. generalised observability conditions by Ammari–Tuscnak 2001, Ammari–Bchatnia–El Mufti 2017)

Theorem

Assume $D(A^*) = D(A)$, $B \in \mathcal{L}(U, X)$ and $0 < \beta \le 1$. If the above condition holds, then $i\mathbb{R} \subset \rho(A - BB^*)$, and

$$\|e^{(A-BB^*)t} x_0\| \le \frac{C}{t^{1/(2\beta)}} \|Ax_0\|, \quad x_0 \in D(A)$$

Examples: 2D Wave Equations

A wave equation with viscous damping on a convex $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary, $b \in L^\infty(\Omega)$

$$\begin{aligned} &w_{tt}(\xi,t) - \Delta w(\xi,t) + b(\xi)^2 w_t(\xi,t) = 0, && \xi \in \Omega,\ t>0,\\ &w(\xi,t) = 0, && \xi \in \partial\Omega,\ t>0,\\ &w(\cdot,0) = w_0(\cdot) \in H^2(\Omega)\cap H_0^1(\Omega), && w_t(\cdot,0) = w_1(\cdot) \in H_0^1(\Omega). \end{aligned}$$

- Several results exist for the exact observability of the Schrödinger group $(b, i\Delta)$ (Jaffard '90, Burq–Zworski '19) for rectangles/tori. Leads to polynomial decay $1/\sqrt{t}$.
- Precise lower bounds on b lead to generalised observability of the Schrödinger group via Burq–Zuily 2016.
- In general our results are sub-optimal, since conditions do not take into account the **smoothness** of b! (Burq–Hitrik '07)

1D Wave Equations

Consider a wave equation with weak damping (and Dirichlet BC)

$$w_{tt}(\xi,t) - w_{\xi\xi}(\xi,t) + b(\xi)\int_0^1 b(r)w_t(r,t)dr = 0, \quad \xi \in (0,1), \; t > 0,$$

- The wavepacket condition characterises (optimal) stability via lower bounds of the sine Fourier coefficients, e.g., $(c, \alpha > 0)$

$$\left|\int_0^1 b(\xi)\sin(n\pi\xi)d\xi\right| \geq \frac{c}{n^\alpha}$$

- Pointwise damping possible (formally $b(\xi) = \delta(\xi - \xi_0)$).
- Analogous results for Euler–Bernoulli / Timoshenko beams

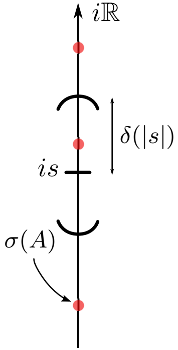
Application: Water Waves System

In the reference

Su–Tucsnak–Weiss "Stabilizability properties of a linearized water waves system," *Systems & Control Letters*, 2020.

the results were applied to prove polynomial stabilizability of a "water waves system" in a 2D rectangular domain.

- Models small amplitude gravity water waves
- A has discrete spectrum $\subset i\mathbb{R}$, but no uniform gap, $\lambda_k \approx i\sqrt{k}$
- Decay rate derived using the "Wavepacket condition"
- $\delta(s) \to 0$ so that $(s - \delta(|s|), s + \delta(|s|)) \cap \sigma(A)$ are singleton sets for all $s \in \mathbb{R}$.
- Optimality possible (..?)



Conclusions

In this presentation:

- General sufficient conditions for non-uniform stability of the semigroup generated by $A - BB^*$.
- Discussion of PDE examples and optimality of the results

R. Chill, LP, D. Seifert, R. Stahn, Y. Tomilov, "Non-Uniform Stability of Damped Contraction Semiroups," *in review* (https://arxiv.org/abs/1911.04804)

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