Polynomial Stability of Coupled PDEs

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Joint work with Charles Batty, David Seifert and Filippo Dell'Oro

New Challenges in Operator Semigroups, Oxford, July 2022

Supported by Academy of Finland grants
298182 (2016-2019), 310489 (2017-2021), and 349002 (2022-2026).

Main Objectives

Problem

Consider different types of ***coupled PDE systems*** *from the point of view of* ***feedback structures*** *and control theory.*

Focus on **polynomial/non-uniform stability**, where

$$\|T(t)x\| \sim \frac{1}{t^\beta}, \qquad x \in \mathcal{D}(A),\ t \geq t_0.$$

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Motivation:

- Coupling of stable and unstable PDEs and ODEs often leads to rational decay of energy, i.e., polynomial stability.

Main results:

- New stability results for coupled PDEs.
- Disclaimer: Won't solve all problems (just the important ones)

Coupled PDE-PDE and PDE-ODE systems appear in models of

- Fluid-structure interactions
- Thermo-elasticity
- Mechanical systems, e.g., beams with tip masses
- Magnetohydrodynamics
- Acoustics
- Networks of 1D PDEs of mixed types (wave/heat/beam/ transport) with coupling BCs at vertices.

Couplings may either be

- Through the **boundary** (Fluid-structure, acoustics), or
- **inside a shared domain** (Thermo-elasticity, MHD)

Outline

(1) “Feedback structures” in coupled PDE systems

- Motivating examples

(2) Basic properties and conversion

- Examples on how to identify feedback structures

(3) Benefits

- General tools for proving non-uniform stability of coupled PDEs.

Part I

Motivating examples

Example class: Coupled Wave–Heat Systems

Models for fluid–structure and heat–structure interactions:

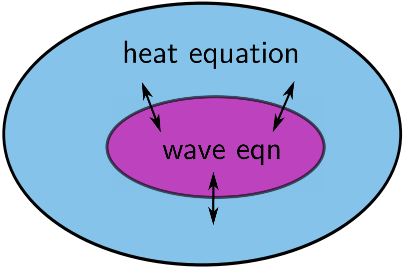
$$\frac{\partial^2 v}{\partial t^2}(x,t) = \Delta v(x,t)$$

$$\updownarrow$$

coupling BCs

$$\updownarrow$$

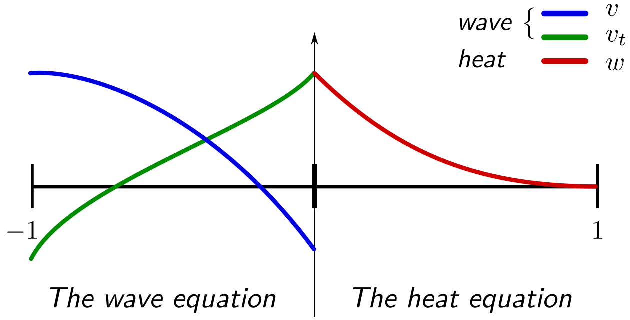
$$\frac{\partial w}{\partial t}(x,t) = \Delta w(x,t)$$



References: Avalos & Triggiani, Duyckaerts, Mercier, Nicaise, Ammari, Zhang & Zuazua, Guo, Ng & Seifert, and many others.

1D Wave–Heat Model

$$\begin{cases} v_{tt}(\xi,t) = v_{\xi\xi}(\xi,t), & -1<\xi<0,\ t>0, \\ w_t(\xi,t) = w_{\xi\xi}(\xi,t), & 0<\xi<1,\ t>0, \\ v_\xi(0,t) = w_\xi(0,t),\ v_t(0,t) = w(0,t),\ v_\xi(-1,t) = w(1,t) = 0 \end{cases}$$



1D Wave–Heat Model

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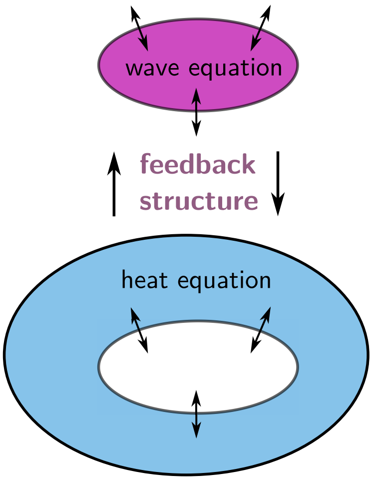
Total energy:

$$E(t) = \frac{1}{2}\int_{-1}^{0} |u_\xi(\xi,t)|^2 + |u_t(\xi,t)|^2 \, d\xi + \frac{1}{2}\int_0^1 |w(\xi,t)|^2 \, d\xi$$

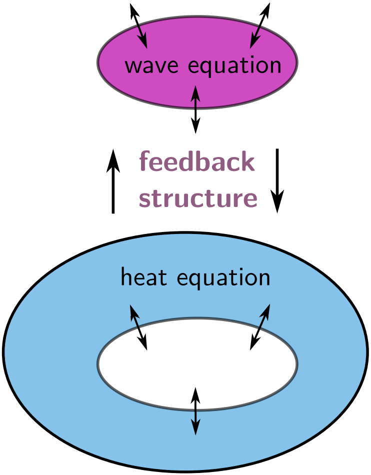
Theorem (Zhang–Zuazua '04, Batty–P–Seifert '16)

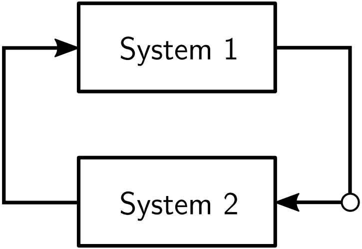
Total energy $E(t)$ of every classical solution decays at the rate t^{-4}.

Coupled Wave–Heat Systems



Coupled Wave–Heat Systems





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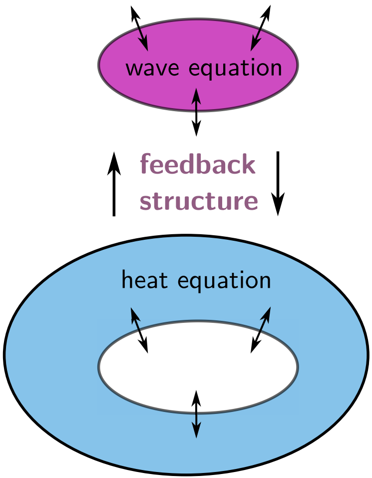
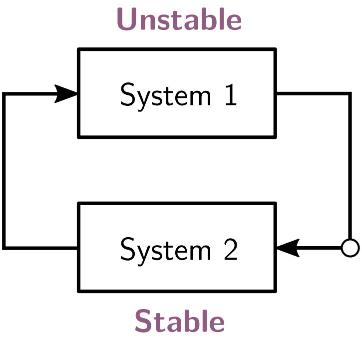
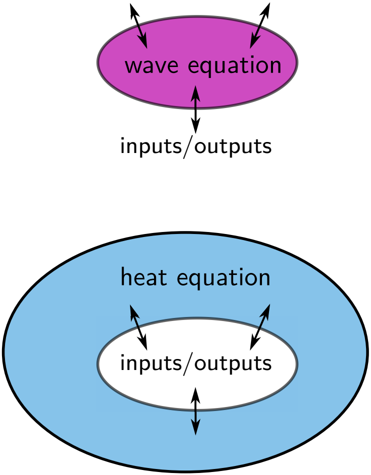
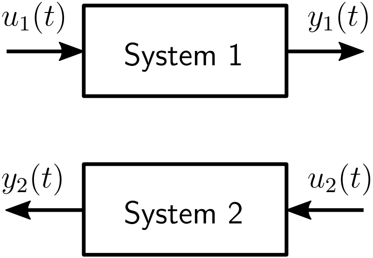




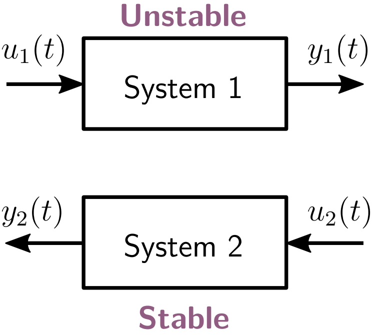
Inputs and Outputs





Problem

Use the properties of the two systems to deduce stability of the coupled PDE.



Benefits:

- "*Divide and conquer*"
- Reduce to well-known parts
- Modularity

“Impedance Passive” Systems

We focus on systems of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad x(0) = x_0 \in X \\ y(t) &= B^*x(t) \end{aligned}$$

where X is Hilbert, A generates a **contraction semigroup**, and $B \in \mathcal{L}(U, V^*)$ for some suitable spaces U and $V^* \supseteq X$.

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where X is Hilbert, A generates a **contraction semigroup**, and $B \in \mathcal{L}(U, V^*)$ for some suitable spaces U and $V^* \supseteq X$.

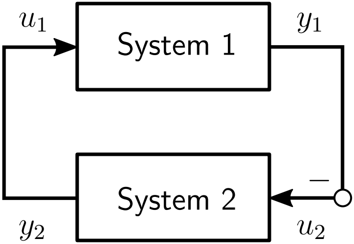
Such systems are “**impedance passive**”, which in particular means they have “**no internal sources of energy**”,

$$\frac{d}{dt}\|x(t)\|^2 \leq 2\,\mathrm{Re}\langle u(t), y(t)\rangle_Y$$

- This class contains models with “collocated” input and output
- Examples: Many mechanical systems, RLC circuits, ...

Feedback Theory of Passive Systems

Property: "Power-preserving interconnection" preserves passivity!



$\Rightarrow$ Closed-loop semigroup contractive on Hilbert $X_1 \times X_2$.

Coupled Passive Systems

If for $k = 1, 2$ we let

$$\dot{x}_k(t) = A_k x_k(t) + B_k u_k(t), \qquad x_k(0) \in X_k$$
$$y_k(t) = B_k^* x_k(t),$$

then the "**power-preserving interconnection**" leads to

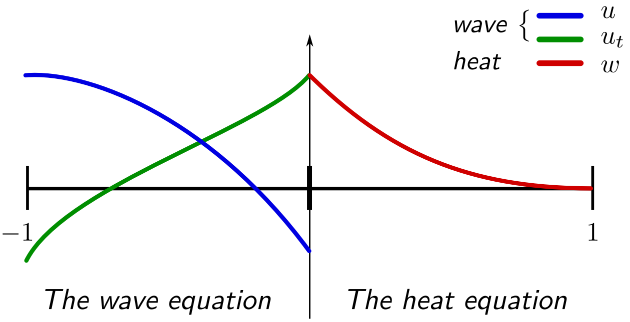
$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A_1 & B_1 B_2^* \\ -B_2 B_1^* & A_2 \end{bmatrix}}_{=: A} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Question

How to choose U, B_1, and B_2?

Example: 1D Wave–Heat Model

$$\begin{cases} v_{tt}(\xi,t) = v_{\xi\xi}(\xi,t), & \xi \in (-1,0),\ t>0, \\ w_t(\xi,t) = w_{\xi\xi}(\xi,t), & \xi \in (0,1),\ t>0, \\ v_\xi(0,t) = w_\xi(0,t), \quad v_t(0,t) = w(0,t), & t>0, \end{cases}$$



The Feedback Structure Based on the Coupling

Coupling BCs:

$$\begin{cases} v_t(0,t) = w(0,t) \\ v_\xi(0,t) = w_\xi(0,t) \end{cases}$$

vs.

Power-preserving Feedback:

$$\left. \begin{aligned} u_1(t) &= y_2(t) \\ u_2(t) &= -y_1(t) \end{aligned} \right\}$$

The Feedback Structure Based on the Coupling

Coupling BCs: Power-preserving Feedback:

$$\begin{cases} \alpha v_t(0,t) = \alpha w(0,t) \\ \beta v_\xi(0,t) = \beta w_\xi(0,t) \end{cases} \qquad \begin{matrix} \longleftrightarrow \\ \longleftrightarrow \end{matrix} \qquad \begin{aligned} u_1(t) &= y_2(t) \\ u_2(t) &= -y_1(t) \end{aligned}$$

or

$$\begin{cases} \alpha v_t(0,t) = \alpha w(0,t) \\ \beta v_\xi(0,t) = \beta w_\xi(0,t) \end{cases} \qquad \qquad \begin{aligned} u_1(t) &= y_2(t) \\ u_2(t) &= -y_1(t) \end{aligned}$$

The Feedback Structure Based on the Coupling

Coupling BCs: Power-preserving Feedback:

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or

$$\begin{cases} \alpha v_t(0,t) = \alpha w(0,t) \\ \beta v_\xi(0,t) = \beta w_\xi(0,t) \end{cases} \qquad \begin{array}{l} u_1(t) = y_2(t) \\ u_2(t) = -y_1(t) \end{array}$$

Example: 1D Wave-Heat — Open-Loop Splitting

Wave system on $(-1,0)$:

$$\begin{aligned} v_{tt}(\xi,t) &= v_{\xi\xi}(\xi,t) \\ u_1(t) &= v_t(0,t) \\ y_1(t) &= v_\xi(0,t) \end{aligned}$$

Unstable

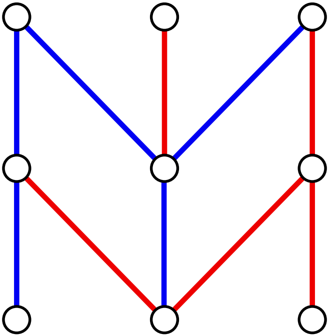
Heat system on $(0,1)$:

$$\begin{aligned} w_t(\xi,t) &= w_{\xi\xi}(\xi,t) \\ u_2(t) &= -w_\xi(0,t) \\ y_2(t) &= w(0,t) \end{aligned}$$

Stable

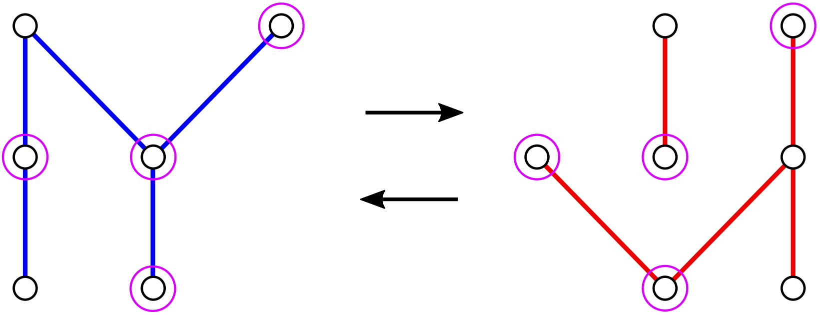
The systems **are** impedance passive. We have $U=\mathbb{C}$ and B_1 and B_2 are unbounded.

Networks of Wave and Heat Equations



Wave equations and heat equations

Networks of Wave and Heat Equations



Power-preserving interconnection $\longleftrightarrow$ Kirchoff-type couplings

Part II

Stability Analysis

Polynomial and Non-Uniform Stability

Theorem (Borichev & Tomilov '10)

Let $T(t)$ be a uniformly bounded C_0-semigroup on a Hilbert space X. Let A be the generator of $T(t)$ and $\sigma(A) \cap i\mathbb{R} = \emptyset$.

For any constant $\alpha > 0$, the following are equivalent:

$$\|T(t)x_0\| \leq \frac{M}{t^{1/\alpha}}\|Ax_0\| \qquad \textit{for some } M > 0$$

$$\|(is - A)^{-1}\| \leq M_R(1 + |s|^\alpha), \qquad \textit{for some } M_R > 0$$

General: Batty & Duyckaerts '08, Rozendaal, Seifert & Stahn '17.

Application: $E(t) \sim \|T(t)x_0\|^2$ for many PDE systems.

Polynomial and Non-Uniform Stability

Since our coupled systems are contractive by default,

*"Polynomial stability **only** requires a resolvent estimate"*

$$\|(is - A)^{-1}\| \leq M(s), \qquad s \in \mathbb{R}$$

Problem

Derive a resolvent estimate of the form $\|(is - A)^{-1}\| \le M(s)$ *for*

$$A := \begin{bmatrix} A_1 & B_1 B_2^* \\ -B_2 B_1^* & A_2 \end{bmatrix}$$

in terms of the properties of

- (A_1, B_1, B_1^*) [**Unstable**]
- (A_2, B_2, B_2^*) [**Exp. Stable**]

Assumption

- A_1 *is skew-adjoint and has compact resolvent.*
- A_2 *generates an exponentially stable semigroup.*

Problem

Derive a resolvent estimate for

$$A := \begin{bmatrix} A_1 & B_1B_2^* \\ -B_2B_1^* & A_2 \end{bmatrix}$$

- (A_1, B_1, B_1^*) [**Unstable**] *and* (A_2, B_2, B_2^*) [**Exp. Stable**]

Overview of results:

$$\|(is - A)^{-1}\| \lesssim M_1(|s|)M_2(|s|)$$

Problem

Derive a resolvent estimate for

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- $M_1(\cdot)$ from "observability properties" of (B_1^*, A_1) (next slide)

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Overview of results:

$$\|(is - A)^{-1}\| \lesssim M_1(|s|)M_2(|s|)$$

- $M_1(\cdot)$ from "observability properties" of (B_1^*, A_1) (next slide)
- When $P_2(is) = B_2^*(is - A_2)^{-1}B_2$ is the "transfer function",

$$M_2(s) \sim \|\left[\operatorname{Re} P_2(is)\right]^{-1}\| \qquad \left(M_2(s) \sim \frac{1}{\operatorname{Re} P_2(is)}\right)$$

Observability-Type Conditions on (B_1^*, A_1)

Goal

$$\|(is - A)^{-1}\| \lesssim M_1(|s|)M_2(|s|)$$

Assumed: $\sigma_p(A_1) = \{is_k\}$ has no finite accumulation points and

$$A_1 = \sum_{k \in \mathbb{Z}} is_k \langle \cdot, \phi_k \rangle \phi_k.$$

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Proposition (Simple version)

If $d_{gap} := \inf_{k \neq l} |s_k - s_l| > 0$ *and if there exists* $\gamma : \mathbb{R} \to (0,1)$ *s.t.*

$$\|B_1^* \phi_k\| \geq \gamma(s_k) \|\phi_k\|, \qquad \forall k \in \mathbb{Z},$$

then $M_1(s) \lesssim \gamma(s)^{-2}$.

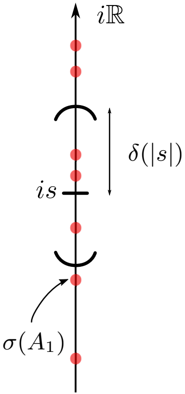
Classical: (B_1^*, A_1) is **exactly observable** iff $\sup_k \|B_1^* \phi_k\| > 0$.

Observability-Type Conditions on (B_1^*, A_1)

A_1 has spectral projections $P_{(a,b)}$ (for $i(a,b) \subset i\mathbb{R}$)

$$(*) \quad \|B_1^* x\| \geq \gamma(|s|)\|x\|, \qquad x \in \mathcal{R}(P_{(s-\delta(|s|),s+\delta(|s|))}), \ s \in \mathbb{R}$$

for some non-increasing $\delta, \gamma \colon \mathbb{R}_+ \to (0,1)$.



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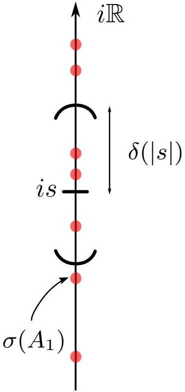


Proposition (Full version)

If $(*)$ *holds, then* $M_1(s) \lesssim \delta(s)^{-2}\gamma(s)^{-2}$.

Main Result

Consider $T(t)$ generated by A, where

$$A := \begin{bmatrix} A_1 & B_1 B_2^* \\ -B_2 B_1^* & A_2 \end{bmatrix}, \qquad P_2(\lambda) = B_2^*(\lambda - A_2)^{-1} B_2$$

Theorem (P. 2019)

Let $\delta, \gamma, \eta : \mathbb{R}_+ \to (0,1)$ *be decreasing so that*

$$\|B_1^* x\| \geq \gamma(|s|)\|x\|, \qquad x \in \mathcal{R}(P_{(s-\delta(|s|), s+\delta(|s|))})$$

$$\operatorname{Re} P_2(is) \geq \eta(|s|) I \qquad \textit{for } s \approx s_k.$$

Then

$$\|(is - A)^{-1}\| \lesssim \frac{1}{\delta(|s|)^2 \gamma(|s|)^2 \eta(|s|)}, \qquad s \in \mathbb{R}.$$

Important Special Case

Consider $T(t)$ generated by

$$A := \begin{bmatrix} A_1 & B_1 B_2^* \\ -B_2 B_1^* & A_2 \end{bmatrix}, \qquad P_2(\lambda) = B_2^*(\lambda - A_2)^{-1} B_2$$

Proposition

Assume

- (B_1^*, A_1) *is exactly observable* $\quad$ ($\Leftrightarrow A_1 - B_1 B_1^*$ *exp. stable*)
- *there exists* $\alpha \geq 0$ *such that*

$$\operatorname{Re} P_2(is) \gtrsim \frac{1}{1+|s|^\alpha} I$$

Then $T(t)$ *is polynomially stable,* $\|(is - A)^{-1}\| \lesssim 1 + |s|^\alpha$*,*

$$\|T(t)x_0\| \leq \frac{M}{t^{1/\alpha}} \|Ax_0\|, \qquad x_0 \in \mathcal{D}(A).$$

Comments:

- Philosophy in line with "Dissipativity Theory" of Willems: *Deducing closed-loop properties from properties of component systems.*
- Theorem requires some admissibility and well-posedness assumptions (swept under the carpet here). Limits 2D-nD BC.

Optimality

- Obtained rate is not always optimal, especially if A_1 has no spectral gap (2D, nD waves)
- A nice way of getting (possibly) suboptimal rates easily.

Example: 1D Wave-Heat System

Wave system on $(-1,0)$:

$$\rho(\xi)v_{tt}(\xi,t) = (T(\xi)v_\xi)_\xi(\xi,t)$$
$$u_1(t) = v_t(0,t)$$
$$y_1(t) = T(0)v_\xi(0,t)$$

Heat system on $(0,1)$:

$$w_t(\xi,t) = w_{\xi\xi}(\xi,t)$$
$$u_2(t) = -w_\xi(0,t)$$
$$y_2(t) = w(0,t)$$

- A_1 skew-adjoint, (B_1^*, A_1) exactly observable.
- $P_2(is) = B_2^*(is-A_2)^{-1}B_2$ satisfies $\operatorname{Re} P_2(is) \sim |s|^{-1/2}$.

Thus the closed-loop system is polynomially stable,

$$\|(is-A)^{-1}\| \lesssim 1+|s|^{1/2} \qquad \text{and} \qquad \|T(t)x_0\| \le \frac{M}{t^2}\|Ax_0\|.$$

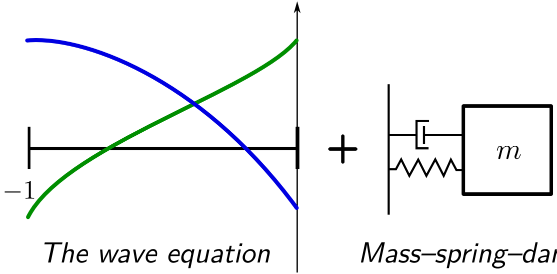
A mild generalisation of [Zhang–Zuazua, Batty–P–Seifert].

Example: Wave equation with an Acoustic BC

$$\begin{aligned}
\rho(\xi)v_{tt}(\xi,t) &= (T(\xi)v_\xi(\xi,t))_\xi, \qquad -1 < \xi < 0,\\
m\delta_{tt}(t) &= -d\delta_t(t) - k\delta(t) - \beta v_t(0,t)\\
v_\xi(0,t) &= \delta_t(t), \qquad v_t(-1,t) = 0.
\end{aligned}$$

1D Wave with an Acoustic Boundary Condition

$$\begin{cases} \rho(\xi)v_{tt}(\xi,t) = (T(\xi)v_\xi(\xi,t))_\xi, \qquad -1 < \xi < 0, \\ m\delta_{tt}(t) = -k\delta(t) - d\delta_t(t) - v_t(0,t) \\ v_\xi(0,t) = \delta_t(t), \qquad v_t(-1,t) = 0. \end{cases}$$



The wave equation | *Mass–spring–damper*

Example: Wave equation with an Acoustic BC

Wave system on $(-1,0)$:

$$\rho(\xi)v_{tt}(\xi,t) = (T(\xi)v_\xi)_\xi(\xi,t)$$
$$u_1(t) = T(0)v_\xi(0,t)$$
$$y_1(t) = v_t(0,t)$$

The ODE part

$$m\ddot{\delta}(t) + k\delta(t) + d\dot{\delta}(t) = \beta u_2(t)$$
$$y_2(t) = T(0)\dot{\delta}(t).$$

- A_1 skew-adjoint, (B_1^*, A_1) exactly observable.
- $P_2(is) = B_2^*(is - A_2)^{-1}B_2$ satisfies $\operatorname{Re} P_2(is) \sim s^{-2}$.

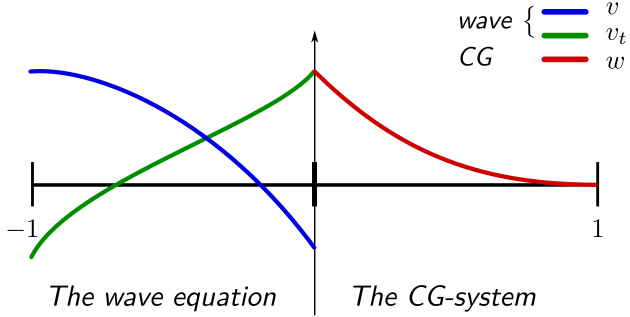
Thus the closed-loop system is polynomially stable,

$$\|(is - A)^{-1}\| \lesssim 1 + s^2 \qquad \text{and} \qquad \|T(t)x_0\| \le \frac{M}{\sqrt{t}}\|Ax_0\|.$$

Reproduces results of [Muños Rivera–Qin '03, Abbas–Nicaise '13]

Result: Wave-Coleman-Gurtin System

$$\begin{cases} v_{tt}(\xi,t) = v_{\xi\xi}(\xi,t), & -1<\xi<0, \\ w_t(\xi,t) = w_{\xi\xi}(\xi,t) + \int_0^\infty g(s) w_{\xi\xi}(\xi,t-s)ds, & 0<\xi<1, \\ v_\xi(0,t) = w_\xi(0,t) + \int_0^\infty g(s) w_\xi(0,t-s)ds, \ \ v_t(0,t) = w(0,t) \end{cases}$$



Result: Wave-Coleman-Gurtin System

$$\begin{cases} v_{tt}(\xi,t) = v_{\xi\xi}(\xi,t), & -1<\xi<0,\\ w_t(\xi,t) = w_{\xi\xi}(\xi,t) + \int_0^\infty g(s) w_{\xi\xi}(\xi,t-s)ds, & 0<\xi<1,\\ v_\xi(0,t) = w_\xi(0,t) + \int_0^\infty g(s) w_\xi(0,t-s)ds, \; v_t(0,t) = w(0,t) \end{cases}$$

- Decomposition into two passive systems: Wave + CG
- Analyse CG-part using the Dafermos history space formulation

Theorem (Dell'Oro–P–Seifert '21)

Under a mild condition on the kernel g, we have $\operatorname{Re} P_2(is) \sim s^{-1/2}$ and the coupled system is polynomially stable

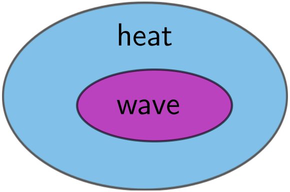
$$\|(is-A)^{-1}\| \lesssim 1 + |s|^{1/2} \qquad \textit{and} \qquad \|T(t)x_0\| \le \frac{M}{t^2}\|Ax_0\|.$$

Coupled PDEs on 2-Dimensional Domains

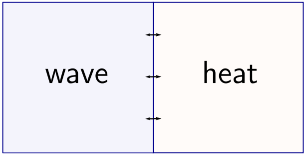
Comments on 2D wave-heat systems:

- Studied in detail by Duyckaerts, Zhang–Zuazua, Avalos–Lasiecka–Triggiani.
- In the general results, the well-posedness conditions limit applicability to boundary coupled systems in 2D (and nD), but same principles hold.
- The "observability" in the wave equation is related to the Geometric Control Condition.

2D Wave–Heat Systems



$\longleftrightarrow$



Optimal: $\alpha = 1$

Optimal: $\alpha = 3$

Polynomial stability:

$\|(is - A)^{-1}\| \lesssim 1 + |s|$

[Avalos–Lasiecka–Triggiani 2016, ...]

$\|(is - A)^{-1}\| \lesssim 1 + |s|^3$

[Batty–P–Seifert 2019]

Conclusions

In this presentation:

- Discussion of coupled PDE and PDE-ODE systems from the viewpoint of systems theory
- General conditions for polynomial stability of coupled systems.

LP, "Stability and Robust Regulation of Passive Linear Systems" SIAM J. Control Optim. 2019, `arXiv:1706.03224`

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R. Chill, LP, D. Seifert, R. Stahn, Y. Tomilov "Non-uniform stability of damped contraction semigroups" Analysis & PDE, accepted, `arXiv:1911.04804`

F. Dell'Oro, LP, D. Seifert. "Optimal decay for a wave-heat system with Coleman–Gurtin thermal law", in review, `arXiv:2101.11397`