Robust Output Regulation for PDE Systems

Lassi Paunonen

including joint work with
Konsta Huhtala, Weiwei Hu, Jukka-Pekka Humaloja, and Duy Phan

Tampere University, Finland

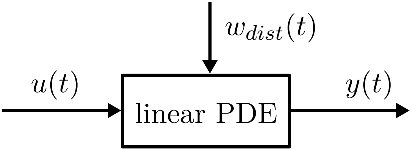
September 2022

Funded by Academy of Finland grants
298182 (2016–2019), 310489 (2017–2021), 349002 (2022–2026)

Introduction

Problem

*Study **robust output regulation** of linear PDE models.*



Output Regulation = Tracking + Disturbance Rejection:
Design a controller such that the output $y(t)$ of the system converges to a reference signal despite the disturbance $w_{dist}(t)$, i.e.,

$$\|y(t) - y_{ref}(t)\| \to 0, \qquad \text{as} \quad t \to \infty$$

Robustness: The controller is required to tolerate uncertainty in the parameters of the system.

Applications

Applications of regulation for PDEs:

- Temperature tracking control, e.g., in manufacturing processes
- Tracking control of flexible robotic manipulators
- Rejection of unwanted periodic noises or vibrations

Robustness:

- Tolerance to the unavoidable **uncertainty** in models.
- Allows reliable use of **approximate** controller parameters.

Outline

Part I:

- Introduction and background
- General controller design methods

Part II:

- Reduced-order controllers for parabolic PDEs
- Regulation of thermal fluid flows

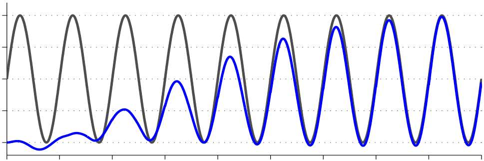
The Reference and Disturbance Signals

The reference and disturbance signals are of the form

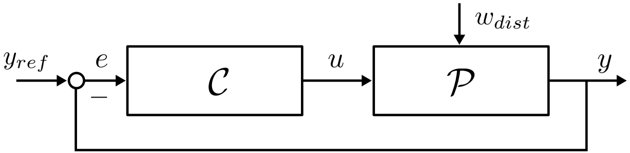
$$y_{\mathit{ref}}(t) = \sum_{k=0}^{q} a_k \cos(\omega_k t + \theta_k)$$

$$w_{\mathit{dist}}(t) = \sum_{k=0}^{q} b_k \cos(\omega_k t + \varphi_k)$$

with **known frequencies** $0 = \omega_0 < \omega_1 < \cdots < \omega_q$ and unknown amplitudes and phases.



The Dynamic Error Feedback Controller



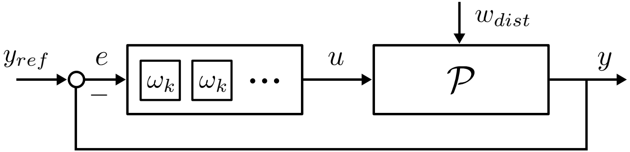
We consider a dynamic error feedback controller which is a (ideally finite-dimensional) linear system.

Result

The Robust Output Regulation Problem is solvable if the system

- *is stabilizable and detectable*
- *does not have transmission zeros at the frequencies* $\pm i\omega_k$ *of* $y_{\textit{ref}}(t)$ *and* $w_{\textit{dist}}(t)$.

The Internal Model Principle



Theorem (Francis–Wonham, Davison 1970's, ...)

The following are equivalent:

- *The controller solves the robust output regulation problem.*
- *Closed-loop system is stable and the controller has **an internal model** of the frequencies $\{\omega_k\}_k$ of $w_{\text{dist}}(t)$ and $y_{\text{ref}}(t)$.*

*"**Internal Model**": For every k, the complex frequencies $\pm i\omega_k$ must be eigenmodes of the controller dynamics with at least $p = \dim Y$ independent eigenvectors.*

Internal Model Based Controller Design

The robust output regulation problem can be solved in two parts:

Step 1° Include a suitable internal model into the controller

Step 2° Use the rest of the controller's parameters to stabilize the closed-loop system.

Internal model has fixed structure (easy), the closed-loop stability can be achieved in several ways (often the main challenge).

Historical Highlights Related to PDEs

Internal Model Principle (*characterization* of controllers) for PDEs:

- **Starting point:** IMP for linear finite-dimensional systems
  - Francis–Wonham '75, Davison '76:

Historical Highlights Related to PDEs

Internal Model Principle (*characterization* of controllers) for PDEs:

- **Starting point:** IMP for linear finite-dimensional systems
  - Francis–Wonham '75, Davison '76:
- **Extension to PDEs** with distributed control and observation
  - Bhat '76: Geometric approach, PhD with Koivo and Wonham
  - Immonen '05–'07: Approach using Sylvester equations
  - P.–Pohjolainen '10: The "classical" form of the IMP

Historical Highlights Related to PDEs

Internal Model Principle (*characterization* of controllers) for PDEs:

- **Starting point:** IMP for linear finite-dimensional systems
  - Francis–Wonham '75, Davison '76:
- **Extension to PDEs** with distributed control and observation
  - Bhat '76: Geometric approach, PhD with Koivo and Wonham
  - Immonen '05–'07: Approach using Sylvester equations
  - P.–Pohjolainen '10: The "classical" form of the IMP
- **Extension to PDEs** with boundary control and observation
  - P. '14, '16: The class of Regular Linear Systems
  - Humaloja–Kurula–P. '19: Boundary Control Systems

Historical Highlights Related to PDEs

Internal Model Principle (*characterization* of controllers) for PDEs:

- **Starting point:** IMP for linear finite-dimensional systems
  - Francis–Wonham '75, Davison '76:
- **Extension to PDEs** with distributed control and observation
  - Bhat '76: Geometric approach, PhD with Koivo and Wonham
  - Immonen '05–'07: Approach using Sylvester equations
  - P.–Pohjolainen '10: The "classical" form of the IMP
- **Extension to PDEs** with boundary control and observation
  - P. '14, '16: The class of Regular Linear Systems
  - Humaloja–Kurula–P. '19: Boundary Control Systems
- **Frequency domain extensions** of the IMP
  - Nett '84, Yamamoto–Hara '88, Vidyasagar '88, Laakkonen '13

Developments: Internal Model based control design for PDEs

- **Classes of linear PDEs** with distributed/boundary control
  - Pohjolainen ’83, Logemann–Townley ’97, Hämäläinen–Pohjolainen ’00, ’06, ’10, Rebarber–Weiss ’03, Boulite et. al. ’09, Harkort–Deutscher ’11, ’17, P. ’16, ’17
- **Parabolic PDEs**
  - Chentouf et. al. ’08, ’10, Deutscher ’13, ’15, ’16, Guo-Meng ’20, Huhtala–P.–Hu ’22, ...
- **Hyperbolic PDEs**
  - Guo–Guo ’13, ’16, Guo–Krstic ’17, ’18, Deutscher–Gabriel ’18–21, Wang et. al. ’18, ’21, Guo-Meng ’21, ’22, ...
- **PIDEs, Coupled systems, Networks, ...**

In addition: Controllers for regulation without robustness

- Schumacher ’83, Byrnes et. al. ’00, Boulite–Saij et. al. ’13, ’18, Natarajan et. al. ’14, Xu–Dubljevic ’16, ’17, ...

Internal Model Based Control for PDEs

The robust output regulation problem in two parts:

Step 1° Include a suitable internal model into the controller

Step 2° Use the rest of the controller's parameters to stabilize the closed-loop system.

For controlling a PDE model, there are (at least) **two approaches:**

- "PDE-based": Construct a controller directly based on the individual PDE model.
  - Controller often has a "PDE part" which acts as an observer.
- "Abstract": Represent the PDE as an infinite-dimensional linear system and use existing results for controller design.
  - Controller is an abstract system with PDE-type dynamics

The Abstract Approach: Preliminaries

In the “abstract approach”, the PDE is reformulated as

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + B_d w_{dist}(t), \qquad x(0) = x_0 \in X\\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

on a Banach or Hilbert space X.

- A generates a strongly continuous semigroup on X.
- $u(t) \in U$ input, $y(t) \in Y$ output, $w_{dist}(t) \in U_d$ disturbance
- input operators B and B_d and output operator C are either **bounded** (distributed I/O) or **unbounded** (boundary I/O).

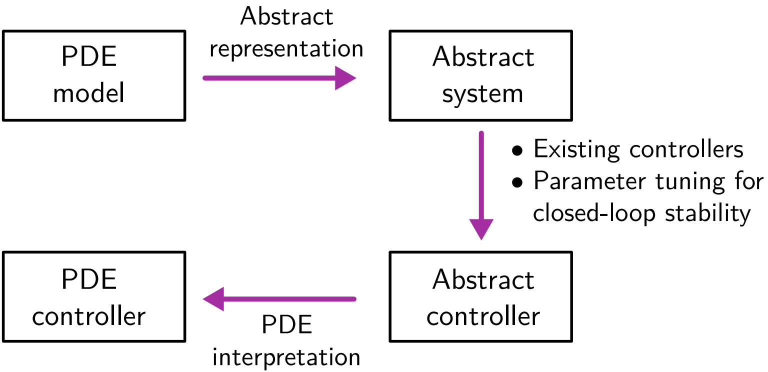
PDE-Based vs. Abstract Routes

PDE model

- IM-based structure
- Observer design
- Prove regulation+robustness

PDE controller

PDE-Based vs. Abstract Routes



Why Take the "Abstract Detour"?

Pros:

- \+ No need to prove regulation and robustness
- \+ Several existing controller design methods, used as black-box
- \+ Avoids repetition in the parts of the design which are common to all IM-based controllers (e.g., IM-structure, observer form), and "zooms in" on the parts that matter.
- \+ The Robust Output Regulation Problem has a lot of **structure** and the abstract approach reduces controller design to particular PDE stabilization problems.

PDE model → Abstract system → Abstract controller → PDE controller

Why Take the "Abstract Detour"?

Cons:

- Abstract representation can require deep knowledge of abstract systems, and can still be challenging
  + Representation know for many important PDEs!
  + Exact knowledge typically not required!
- PDE interpretation of the controller can require effort
  + Making this easier is an important topic for future research!

$$\boxed{\text{PDE model}} \longrightarrow \boxed{\text{Abstract system}} \longrightarrow \boxed{\text{Abstract controller}} \longrightarrow \boxed{\text{PDE controller}}$$

When to Take the Abstract Route?

When do the Pros outweigh the Cons?

- PDE has distributed inputs and outputs $\rightsquigarrow$ abstract representation and PDE interpretation are “easy”.
- PDE is on 1D domain $\rightsquigarrow$ abstract representation often already known, and PDE interpretation is straightforward
- PDE is parabolic (on nD domain) $\rightsquigarrow$ can often be represented as “regular linear systems”
- PDE is exponentially stable $\rightsquigarrow$ A very simple ODE controller.

When to Take the Abstract Route?

When do the Pros outweigh the Cons?

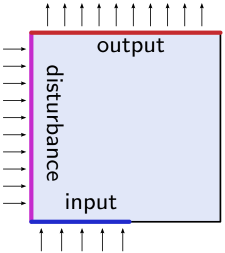
- PDE has distributed inputs and outputs $\rightsquigarrow$ abstract representation and PDE interpretation are "easy".
- PDE is on 1D domain $\rightsquigarrow$ abstract representation often already known, and PDE interpretation is straightforward
- PDE is parabolic (on nD domain) $\rightsquigarrow$ can often be represented as "regular linear systems"
- PDE is exponentially stable $\rightsquigarrow$ A very simple ODE controller.

When do the Cons outweigh the Pros?

- The PDE is on a multi-dimensional domain, has boundary inputs and outputs, and is not parabolic
  - $\rightsquigarrow$ Abstract representation can be challenging (if not known)
  - $\rightsquigarrow$ If the system is not externally well-posed, existing abstract designs may not be applicable.

An Example: 2D Boundary Controlled Heat Equation

$$
\begin{aligned}
x_t(\xi,t) &= \Delta x(\xi,t), \quad x(\xi,0) = x_0(\xi) \\
\frac{\partial x}{\partial n}(\xi,t) &= u(t), \qquad \xi \in \Gamma_1 \\
\frac{\partial x}{\partial n}(\xi,t) &= w_{dist}(t), \qquad \xi \in \Gamma_2 \\
\frac{\partial x}{\partial n}(\xi,t) &= 0, \qquad \xi \in \Gamma_0 \\
y(t) &= \int_{\Gamma_3} x(\xi,t)d\xi.
\end{aligned}
$$



Theorem (Byrnes–Gilliam–Weiss 2002)

The PDE can be represented abstractly as a ***regular linear system***

$\rightsquigarrow$ can employ existing abstract control designs for RLS

Step 1: Abstract Representation of the PDE

Abstract system on $X = L^2(\Omega)$ has the form

$$\dot{x}(t) = Ax(t) + Bu(t) + B_d w_{\mathit{dist}}(t), \qquad x(0) = x_0 \in X$$
$$y(t) = C_\Lambda x(t).$$

Theorem (P. 2016, 2017)

The Robust Output Regulation Problem is solvable if the system

- *is stabilizable using state feedback and output injection*
- *does not have transmission zeros at the frequencies* $\pm i\omega_k$ *of* $y_{\mathit{ref}}(t)$ *and* $w_{\mathit{dist}}(t)$.

Good news: Our heat equation has both properties!

Step 2: Abstract Controller Design

Abstract system on $X = L^2(\Omega)$ has the form

$$\dot{x}(t) = Ax(t) + Bu(t) + B_d w_{\mathit{dist}}(t), \qquad x(0) = x_0 \in X$$
$$y(t) = C_\Lambda x(t).$$

Theorem (P. 2016)

The Robust Output Regulation Problem can be solved with controller

$$\dot{z}_1(t) = G_1 z_1(t) + G_2(y(t) - y_{\mathit{ref}}(t))$$
$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(\hat{y}(t) - y(t) + y_{\mathit{ref}}(t))$$
$$\hat{y}(t) = C_\Lambda \hat{x}(t)$$
$$u(t) = K_1 z_1(t) + K_2 \hat{x}(t)$$

with matrices G_1, G_2 and bounded operators L, K_1, and K_2.

Step 2: Abstract Controller Design

$$y_{\mathit{ref}}(t) = \sum_{k=0}^{q} a_k \cos(\omega_k t + \theta_k), \qquad w_{\mathit{dist}}(t) = \sum_{k=0}^{q} b_k \cos(\omega_k t + \varphi_k)$$

Matrices G_1, G_2: Explicit expressions based on $\{\omega_k\}_k$ and $\dim Y$.

Bounded operators L, K_1 and K_2: Chosen so that the semigroups generated by

$$A + LC \qquad \text{and} \qquad \begin{bmatrix} G_1 & G_2 C_\Lambda \\ 0 & A \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} \begin{bmatrix} K_1, K_2 \end{bmatrix}$$

are exponentially stable. Alternatives:

- Rewrite as a stabilization problem for a PDE-ODE cascade.
- Numerical approximations and LQR/LQG (Banks–Ito 1997)
- "Forwarding" (P. 2016) + numerics (P.–Humaloja 2022)

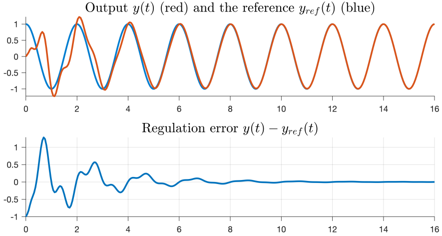
Step 3: From Abstract to PDE Controller

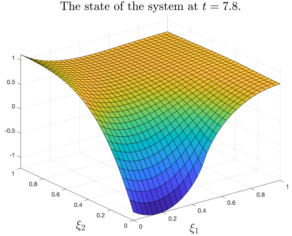
Theorem (P.–Humaloja CDC 2022)

The state of the controller is the weak solution of the ODE-PDE system

$$
\begin{aligned}
\dot{z}_1(t) &= G_1 z_1(t) + G_2(y(t) - y_{\mathit{ref}}(t)) \\
\hat{x}_t(\xi,t) &= \Delta \hat{x}(\xi,t) + L(\xi)\Big(\int_{\Gamma_3} \hat{x}(\xi,t)d\xi - y(t) + y_{\mathit{ref}}(t)\Big) \\
\frac{\partial \hat{x}}{\partial n}(\xi,t) &= K_1 z_1(t) + K_2 \hat{x}(\cdot,t) \qquad \text{on } \Gamma_1, \\
\frac{\partial \hat{x}}{\partial n}(\xi,t) &= 0 \qquad \textit{elsewhere on } \partial\Omega \\
u(t) &= K_1 z_1(t) + K_2 \hat{x}(\cdot,t).
\end{aligned}
$$

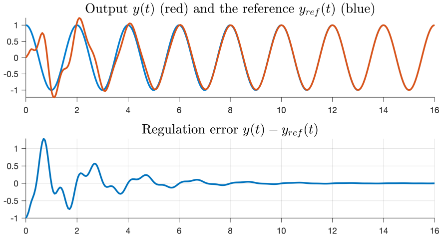
Example: Numerical Simulation

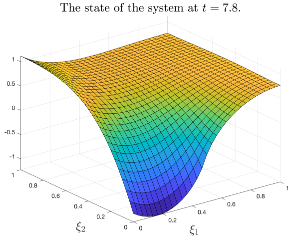




Simulation code available at `https://github.com/lassipau/`

Example: Numerical Simulation





RORPack – Matlab/Python libraries for Robust Output Regulation

- Routines for internal model based controller design
- Simulation and visualisation of results
- Several different types of PDE test cases implemented

Available at `https://github.com/lassipau/rorpack-matlab/`

Part II: Reduced-Order Controllers for Parabolic PDEs

The Parabolic Linear System

$$\dot{x}(t) = Ax(t) + Bu(t) + B_d w_{\mathit{dist}}(t), \qquad x(0) = x_0 \in X$$
$$y(t) = Cx(t) + Du(t)$$

- A generates an **analytic** semigroup on the Hilbert space X
  - Convection-diffusion-reaction PDEs
  - Beams and plates with Kelvin-Voigt damping
- $u(t) \in U$ input, $y(t) \in Y$ output, $w_{\mathit{dist}}(t) \in U_d$ disturbance
- $B \in \mathcal{L}(U, X)$, $C \in \mathcal{L}(X, Y)$, $B_d \in \mathcal{L}(U_d, X)$ with U, U_d, Y finite-dimensional.

The Parabolic Linear System

$$\dot{x}(t) = Ax(t) + Bu(t) + B_d w_{\textit{dist}}(t), \qquad x(0) = x_0 \in X$$
$$y(t) = Cx(t) + Du(t)$$

- A generates an **analytic** semigroup on the Hilbert space X
  - Convection-diffusion-reaction PDEs
  - Beams and plates with Kelvin-Voigt damping
- $u(t) \in U$ input, $y(t) \in Y$ output, $w_{\textit{dist}}(t) \in U_d$ disturbance
- $B \in \mathcal{L}(U, X)$, $C \in \mathcal{L}(X, Y)$, $B_d \in \mathcal{L}(U_d, X)$ with U, U_d, Y finite-dimensional.

Problem

Achieve Robust Output Regulation $\|y(t) - y_{\textit{ref}}(t)\| \to 0$ *as* $t \to \infty$ *using a* ***finite-dimensional (ODE) controller****.*

Earlier Work

System unstable, **finite-dimensional** controller:

- Schumacher 1983, Curtain 1983 (quite strict conditions)
- Deutscher 2011, 2013 (no robustness, “spillover” possible)
- Lhachemi–Prieur 2022 (uses eigenmodes)

Earlier Work

System unstable, **finite-dimensional** controller:

- Schumacher 1983, Curtain 1983 (quite strict conditions)
- Deutscher 2011, 2013 (no robustness, "spillover" possible)
- Lhachemi–Prieur 2022 (uses eigenmodes)

Main Result (Paunonen-Phan IEEE TAC 2020)

Our controller design:

- *Internal model based robust design*
- *For general parabolic systems*
- *Direct: Uses Galerkin approximation and LQR/LQG methods*
- *Model reduction step ensures low order*

Extension to PDEs with boundary control in Phan–Paunonen 2021.

Galerkin Approximations – Assumptions

$$\dot{x}(t) = Ax(t) + Bu(t) + B_d w_{dist}(t), \qquad x(0) = x_0 \in X$$
$$y(t) = Cx(t) + Du(t)$$

Assumption

There exists a sesquilinear form $a(\cdot,\cdot) : V \times V \to \mathbb{C}$ *such that*

$$\langle -A\phi, \psi \rangle = a(\phi, \psi), \qquad \forall \phi \in D(A), \psi \in V$$

and $a(\cdot,\cdot)$ *bounded and coercive, i.e.,* $\exists c_1, c_2, \lambda_0 > 0$ *s.t.*

$$|a(\phi,\psi)| \leq c_1 \|\phi\|_V \|\psi\|_V \qquad \text{(boundedness)}$$
$$\operatorname{Re} a(\phi,\phi) \geq c_2 \|\phi\|_V^2 - \lambda_0 \|\phi\|_X^2 \qquad \text{(coercivity)}$$

- $\Rightarrow A - \lambda_0 I$ generates an analytic semigroup on X
- Typical for nD convection-diffusion-reaction equations

Galerkin Approximations

Assumption

$$\langle -A\phi, \psi \rangle = a(\phi, \psi), \qquad \forall \phi \in D(A), \psi \in V$$

Assumption (Approximating Subspaces V_N)

There are subspaces $(V_N)_N \subset V$, $\dim V_N < \infty$, such that any $\phi \in V$ can be approximated by $\phi_N \in V_N$ in the sense

$$\|\phi - \phi_N\|_V \to 0, \qquad \text{as} \quad N \to \infty.$$

V_N define **Galerkin approximations** (A^N, B^N, C^N) of (A, B, C),

$$\begin{aligned}
\langle -A^N \phi, \psi \rangle &= a(\phi, \psi), \qquad && \forall \phi, \psi \in V_N, \\
\langle B^N u, \phi \rangle &= \langle u, B^* \phi \rangle, \qquad && \forall u \in U, \phi \in V_N, \\
C^N &= C|_{V_N}
\end{aligned}$$

The Low Order Robust Controller

Aim: Use the approximations (A^N, B^N, C^N) in controller design.

The Low Order Robust Controller

Aim: Use the approximations (A^N, B^N, C^N) in controller design.

The end result: A finite-dimensional robust controller

$$\begin{aligned}
\dot z_1(t) &= G_1 z_1(t) + G_2(y(t) - y_{ref}(t))\\
\dot z_2(t) &= A_L^r z_2(t) + B_L^r u(t) + L^r(y_{ref}(t) - y(t))\\
u(t) &= K_1^N z_1(t) + K_2^r z_2(t)
\end{aligned}$$

and a design algorithm for $(G_1, G_2, A_L^r, B_L^r, K_1^N, K_2^r, L^r)$ based on:

- The frequencies $\{\omega_k\}_k$ of $w_{dist}(t)$ and $y_{ref}(t)$.
- The Galerkin approximation (A^N, B^N, C^N, D) **(no** B_d!).
- A dimension parameter $r \in \mathbb{N}$ for the model reduction step.

The Low Order Robust Controller

Aim: Use the approximations (A^N, B^N, C^N) in controller design.

The end result: A finite-dimensional robust controller

$$\begin{aligned}
\dot z_1(t) &= G_1 z_1(t) + G_2(y(t) - y_{\mathit{ref}}(t))\\
\dot z_2(t) &= A_L^r z_2(t) + B_L^r u(t) + L^r(y_{\mathit{ref}}(t) - y(t))\\
u(t) &= K_1^N z_1(t) + K_2^r z_2(t)
\end{aligned}$$

The Design Algorithm for $(G_1, G_2, A_L^r, B_L^r, K_1^N, K_2^r, L^r)$:

- Explicit formulas for G_1, G_2 based on $\{\omega_k\}_k$
- $(A_L^r, B_L^r, K_1^N, K_2^r, L^r)$ obtained from (A^N, B^N, C^N, D) by
  1. Solving finite-dimensional LQR/LQG problems, and
  2. Balanced Truncation for a stable ODE system

The Main Result

$$\begin{aligned}
\dot{z}_1(t) &= G_1 z_1(t) + G_2(y(t) - y_{ref}(t)) \\
\dot{z}_2(t) &= A_L^r z_2(t) + B_L^r u(t) + L^r(y_{ref}(t) - y(t)) \\
u(t) &= K_1^N z_1(t) + K_2^r z_2(t)
\end{aligned}$$

Theorem

Assume (A,B,C,D) is exponentially stabilizable and detectable and its transmission zeros do not coincide with $\{\pm i\omega_k\}_k \subset i\mathbb{R}$.

If $N \in \mathbb{N}$ and $r \leq N$ are sufficiently large, the controller solves the robust output regulation problem. There exist $M, \alpha > 0$ such that

$$\|y(t) - y_{ref}(t)\| \leq Me^{-\alpha t}\left(\|x(0)\| + \|z(0)\| + \|y_{ref}\|_\infty + \|w_{dist}\|_\infty\right)$$

for all initial states $x(0) \in X$ and $z(0) \in Z$ and $y_{ref}(t)$, $w_{dist}(t)$.

Discussion

Theorem

If $N \in \mathbb{N}$ and $r \leq N$ are sufficiently large, the controller solves the robust output regulation problem. There exist $M, \alpha > 0$ such that

$$\|y(t) - y_{ref}(t)\| \leq M e^{-\alpha t} (\|x(0)\| + \|z(0)\| + \|y_{ref}\|_\infty + \|w_{dist}\|_\infty)$$

for all initial states $x(0) \in X$ and $z(0) \in Z$ and $y_{ref}(t)$, $w_{dist}(t)$.

- Uniform exponential convergence of $\|y(t) - y_{ref}(t)\| \to 0$, can achieve rate $\alpha > 0$ whenever $(A + \alpha I, B, C)$ is stab/detect.
- "N and r sufficiently large" inconvenient $\rightsquigarrow$ further research
- Required size of r depends (roughly) on decay of the Hankel singular values of (A^N, B^N, C^N) $\rightsquigarrow$ A fair amount of model reduction is typically possible for PDEs.

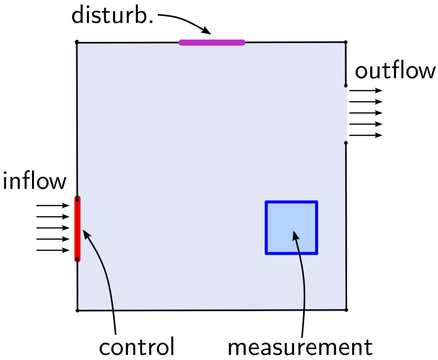
Control of Room Temperature Models

Problem

Achieve robust tracking of ***temperature*** *and* ***flow velocity*** *in a 2D room with inlets and outlets.*

The PDE model determines the temperature and velocity field of the incompressible fluid/air.

Thesis work by Konsta Huhtala (with Weiwei Hu).



Control of Room Temperature Models

Result 1: Temperature tracking only [Huhtala–P.–Hu '20]

- Temperature modelled by convection-diffusion equation, where velocity field solved from steady-state Navier–Stokes
- Distributed control and observation, boundary disturbance

Control of Room Temperature Models

Result 1: Temperature tracking only [Huhtala–P.–Hu '20]

- Temperature modelled by convection-diffusion equation, where velocity field solved from steady-state Navier–Stokes
- Distributed control and observation, boundary disturbance

Result 2: Temperature and velocity tracking [Huhtala–P.–Hu '22]

- Temperature and velocity modelled by linearised Boussinesq equations with boundary control and observation
- Actuators and sensors modelled with ODE systems $\rightsquigarrow$ the full system has bounded input and output operators

Control of Room Temperature Models

Result 1: Temperature tracking only [Huhtala–P.–Hu '20]

- Temperature modelled by convection-diffusion equation, where velocity field solved from steady-state Navier–Stokes
- Distributed control and observation, boundary disturbance

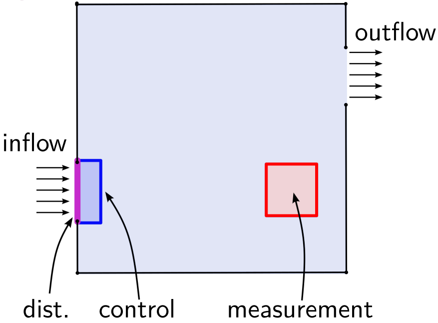
Result 2: Temperature and velocity tracking [Huhtala–P.–Hu '22]

- Temperature and velocity modelled by linearised Boussinesq equations with boundary control and observation
- Actuators and sensors modelled with ODE systems $\rightsquigarrow$ the full system has bounded input and output operators

Result 3: Flow tracking only [Huhtala–P. '21]

- Model for nonlinear fluid flow only (no temperature)
- Nonlinear controller design based on Natarajan–Bentsman '16

Simulation example: Room Temperature Control



Temperature tracking:

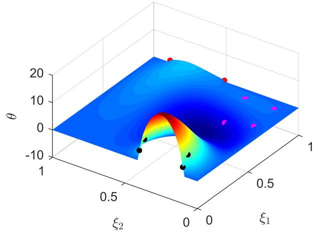
$$y_{ref}(t) = \sin(t) + 2\cos(2t),$$
$$w_{dist}(t) = 1.5\cos(3t).$$

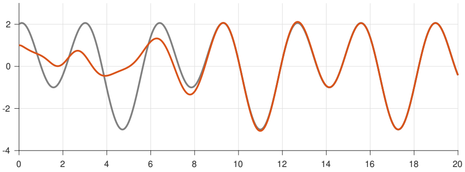
Reduced order controller design:

- Frequencies $\{\omega_k\}_k = \{1,2,3\}$, $\dim Y = 1$
- FEM approximation with 2nd order basis $\rightsquigarrow N = 1549$
- Controller design using Matlab (`are`, `balred` / **RORPack**)
- Controller dimension $6 + 10 = 16$ (IM dim + r)

Simulation

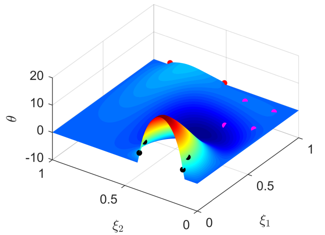
- Higher order approximation ($N_{high} = 6297$) for the plant
- Exponential convergence of the output with $\alpha \approx 0.5$

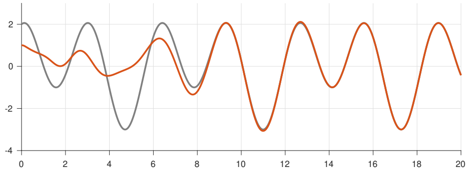




Simulation

- Higher order approximation ($N_{high} = 6297$) for the plant
- Exponential convergence of the output with $\alpha \approx 0.5$



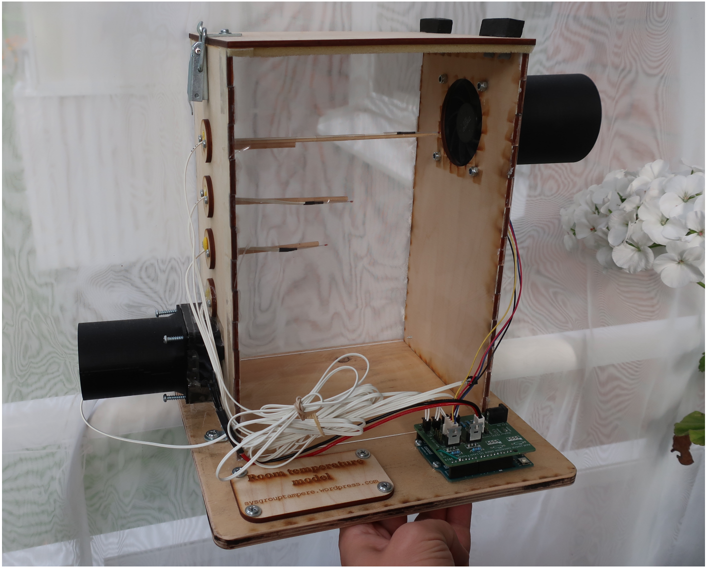


- The original PDE model is **stable**: Controller achieves radically faster convergence than "minimal order" controllers:

$$\textbf{Low-Gain controller:} \quad \dim = 6, \quad \max\ \alpha \le 0.1,$$

$$\textbf{Our controller:} \quad \dim = 16, \quad \alpha \approx 0.5 \qquad \text{(by design)}$$

A Curiosity/Hobby: An Experimental Setup



Summary

Part I:

- Introduction to robust output regulation for PDEs
- Comments on abstract control design methods

Part II:

- A low-order robust controller for parabolic systems based on Galerkin approximations
- Application to tracking of thermal fluid flows

References and Links

- LP and S. Pohjolainen, "The internal model principle for systems with unbounded control and observation", SICON 2014.
- LP, "Controller design for robust output regulation of regular linear systems", IEEE TAC 2016.
- LP and J.-P. Humaloja, "On robust regulation of PDEs: from abstract methods to PDE controllers", CDC 2022, arXiv:2203.09871
- LP and D. Phan, "Reduced order controller design for robust output regulation", IEEE TAC 2020.
- K. Huhtala, LP, and W. Hu, "Robust output regulation of the linearized Boussinesq equations with boundary control and observation", MCSS 2022.

RORPack: `https://github.com/lassipau/rorpack-matlab/`
Preprints: `https://sysgrouptampere.wordpress.com/`