Tools for Polynomial Stability Analysis of PDE Networks

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Main Objectives

Problem

Study dynamics of networks consisting of Partial Differential Equations (PDEs), with focus on ***long-time behaviour of the energy****.*

Motivation:

- Dynamics of networked systems are interesting!
- Both network structure and component dynamics contribute
- We focus on what happens when $t \to \infty$

What are PDE networks?

Throughout the talk a “**PDE network**” refers to

A collection of Partial Differential Equation models whose the connectivity can be described by a graph.

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Perhaps some examples will illustrate the concept. . .

Beam structures

- A truss bridge — beams which are joined at their ends.
- Deflection of each beam described by a PDE — **beam equation**
- Can study either dynamic behaviour or the steady state.

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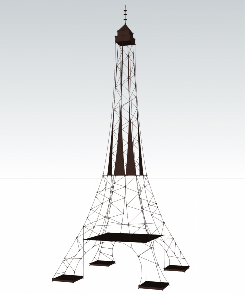
Truss structures

Structure determines a graph where **the beams are the edges and the meeting points are the nodes**.

$\rightsquigarrow$



We focus on such PDE networks, i.e., where

PDE $\leftrightarrow$ edge   connection $\leftrightarrow$ node/vertex

Pipe Networks

Pipe delivery networks for gas, water, and petrol can be modelled as PDE networks. PDEs are **transport equations**.

- Waterflow in channels: **wave equations**
- Heat or material diffusion in structures: **heat or convection-diffusion equations**

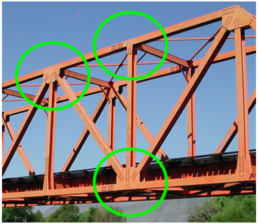
Couplings

Question

How do the PDEs in the network communicate?

Each PDE in the network involves:

- The actual differential equation (deflection along the beam)
- **Boundary conditions** (deflection, slope, strain etc **at the endpoints**)



$\rightsquigarrow$ The beam models interact with each other at the network nodes via their **boundary conditions**.

Examples of Boundary Couplings

Physical system: Gas pipe network

- PDE type: Transport equation
- Coupling at nodes: Incoming material = outgoing mater.

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Physical system: Truss bridge

- PDE type: Beam equation
- Coupling: Deflections are equal **and** forces sum to 0 **and**

All expressed in terms of boundary conditions, and the number depends on the PDE type.

Dynamics and Stability

Definition (Rough, but close enough)

We say the PDE network is **stable** if all its solutions converge to a steady state as $t \to \infty$.

- **Linear case** $\rightsquigarrow$ zero is the unique steady state
- Stability is often equivalent to the property that the **energy** of the solutions satisfies $E(t) \to 0$ as $t \to \infty$.

Goal

We are especially interested in the ***rate*** *of the convergence.*

Mixed-Type PDE Networks

Goal

*We focus on studying **stability** of networks which involve **two different types of PDEs**.*

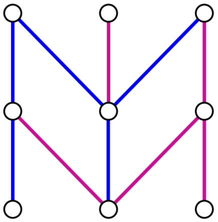
- Beams and strings (suspension bridge).
- Wave and heat equations (fluid-structure interactions?)
- Damped and undamped strings or beams.
- (PDE networks with dynamic couplings)

Stability of Mixed-Type PDE Networks

Goal

We focus on studying ***stability*** *of networks which involve* ***two different types of PDEs****.*

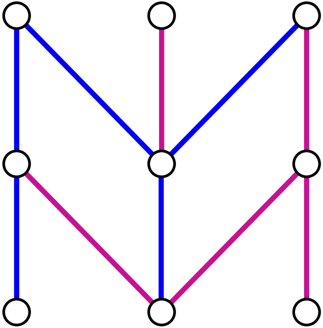
- Stability of mixed-type networks is particularly interesting!
- Convergence of solutions often *slower than exponential* $\rightsquigarrow$ "**polynomial stability**"
- Motivates deriving accurate (rational) convergence rates for $E(t) \to 0$



Part II

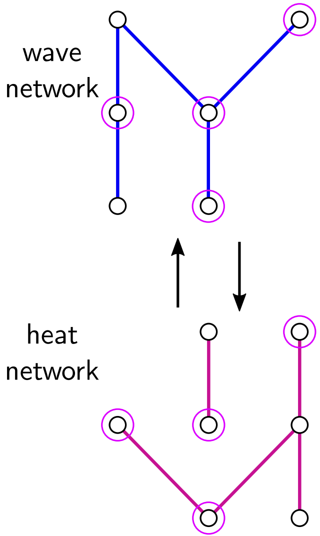
Stability Analysis

Dynamics of Mixed-Type Networks



Wave equations and heat equations

Wave–Heat Networks – Decoupled



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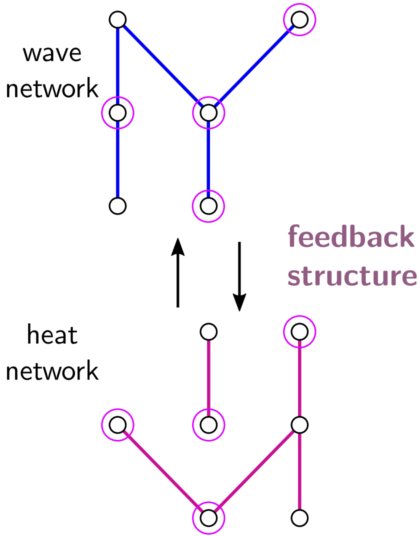
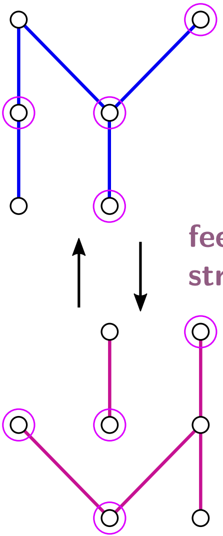


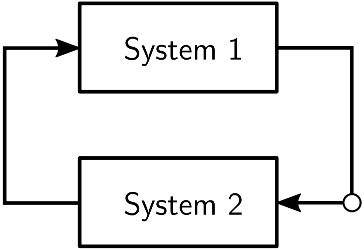
Wave–Heat Networks – Decoupled

wave
network

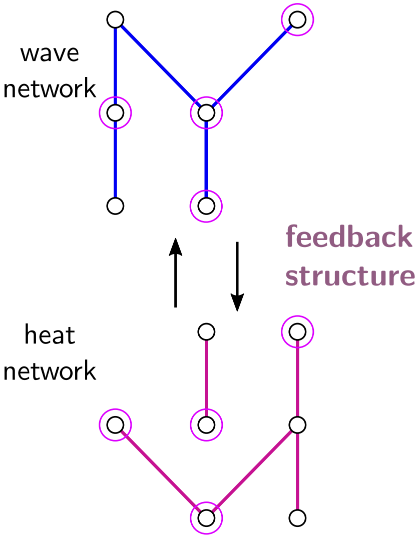


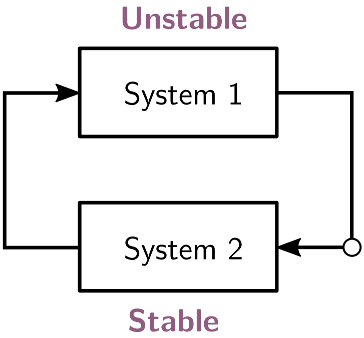
feedback structure

heat
network

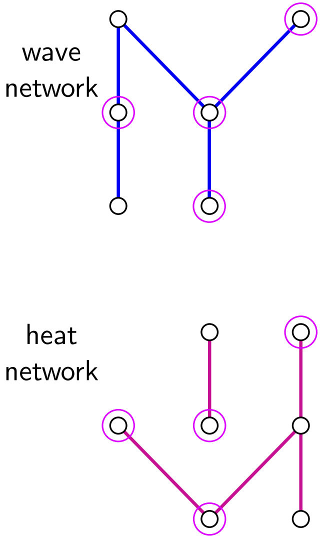


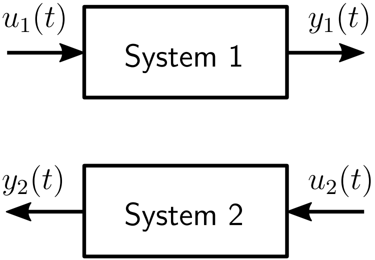
Wave–Heat Networks – Decoupled





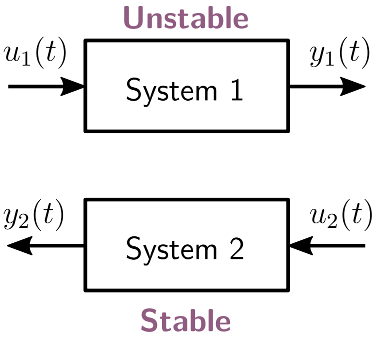
Inputs and Outputs





Problem

Use the properties of the two systems to deduce stability of the coupled PDE.

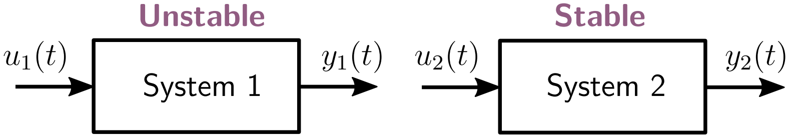


Benefits:

- "*Divide and conquer*"
- Reduce to well-known parts
- Modularity

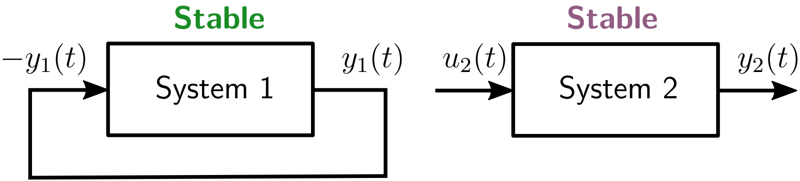
Assumption

- *The two systems are* ***impedance passive***
  - *They have no "internal sources of energy"*



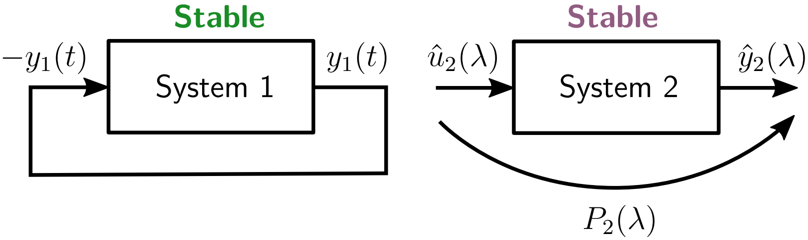
Assumption

- *The two systems are* ***impedance passive***
  - *They have no "internal sources of energy"*
- ***System 1*** *is stable under* ***(neg) output feedback***
  - *This often represents addition of* ***damping*** *to System 1.*



Assumption

- *The two systems are* ***impedance passive***
  - *They have no "internal sources of energy"*
- ***System 1*** *is stable under* ***(neg) output feedback***
  - *This often represents addition of* ***damping*** *to System 1.*
- *Knowledge of the* ***transfer function*** $P_2(\lambda)$ *of* ***System 2***
  - $P_2(\lambda)$ *describes how "Laplace transform of* u_2 *is mapped to Laplace transform of* y_2*", i.e.,* $\hat{y}_2(\lambda) = P_2(\lambda)\hat{u}_2(\lambda)$.



Main Stability Result

Theorem

Assume the following:

- *The two systems are* ***impedance passive***
- *System 1 becomes stable under* ***(neg) output feedback*** *and we know the energy decay rate* $E_1(t) \to 0$
- *We have an estimate for the* ***transfer function*** $P_2(\lambda)$ *of System 2 on the imaginary axis*

Then:

The network is stable and we have
a very good estimate *for the decay rate of* $E(t) \to 0$.

A Quick Mathematical Overview (WG1 Theme)

- The abstract model for the mixed-type networks is a coupled system of two **boundary control systems**,

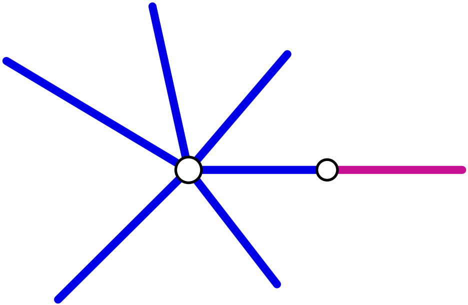
$$\begin{aligned} \dot{z}_1(t) &= L_1 z_1(t), & t &\geq 0, \\ \dot{z}_2(t) &= L_2 z_2(t), & t &\geq 0, \\ G_1 z_1(t) &= K_2 z_2(t), & t &\geq 0, \\ G_2 z_2(t) &= -K_1 z_1(t), & t &\geq 0 \end{aligned}$$

- Has a contraction semigroup [Aalto–Malinen 2013]
- Our main results establish **resolvent estimates**

$$\|(is - A)^{-1}\| \leq M_1(s) M_2(s), \qquad \forall s \in \mathbb{R},$$

and these lead to **polynomial** or **semi-uniform stability**.

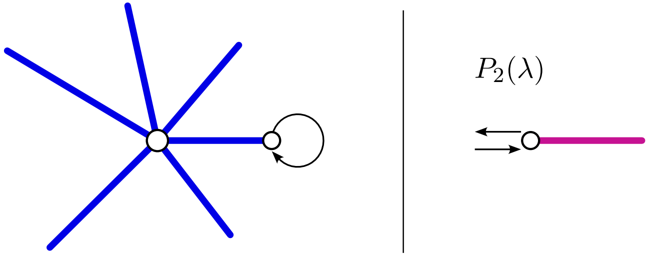
Application to Wave-Heat Networks



Question

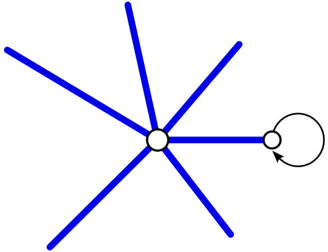
Decay rate for energy $E(t)$ in a network of N wave equations and 1 heat equation?

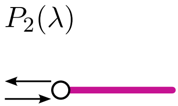
Wave/Heat Decoupling



- In the wave network (System 1) the "output feedback" is equivalent to damping at a single exterior node.
  $\rightsquigarrow$ Stability results in the literature
- Transfer function $P_2(\lambda)$ of the heat equation (System 2) can be computed explicitly.

Wave/Heat Decoupling



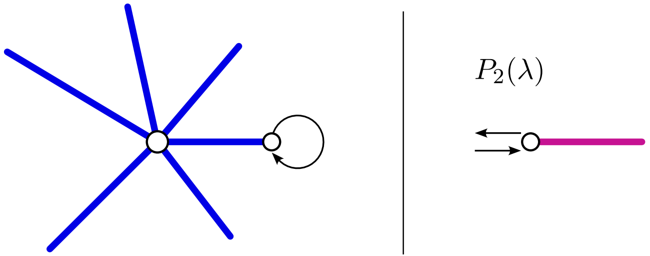


Theorem

If decay rate for classical solutions in the ***decoupled*** *wave-network is* $E_1(t) \sim t^{-\alpha}$, $\alpha > 0$, *then the energy* $E(t)$ *of classical solutions of the* ***wave-heat network*** *has decay*

$$E(t) \sim t^{-\frac{4\alpha}{4+\alpha}}, \qquad t \to \infty. \qquad \textit{(slower)}$$

Wave/Heat Decoupling



Theorem (Using Valein–Zuazua '09)

*Assume that **decoupled** wave-network is **star-shaped** with same wave speeds, and ratios of lengths of the waves are irrational numbers "of constant type". Then energy of classical solutions has decay*

$$E(t) \sim t^{-\frac{4}{4N-3}}, \qquad t \to \infty.$$

A Quick Literature Survey

- PDE networks — single-type
  - Valein–Zuazua, Ammari–Jellouli, Ammari–Shel, Nicaise–Valein, Kramar-Fijavž et. al., Augner, ...
- Coupled PDEs — two equations or simple network structures (3–4 PDEs)
  - Zhang–Zuazua, Duyackerts, Li-Wang, Augner, Avalos–Lasiecka–Triggiani, Rao–Zhang, Avalos–Geredeli, ...
- Coupled abstract systems
  - Ben Ait Hassi et. al., Ammari et. al., Feng–Guo, LP, Boulouz–Bounit–Hadd, Dell'Oro–LP–Seifert, ...

Conclusions

In this talk:

- Study of dynamics and stability of networks of partial differential equations of mixed type.
- Main result on energy decay rate for the full network in terms of the properties of simpler networks.

S. Nicaise, LP, D. Seifert "Stability of Abstract Boundary-Coupled Systems", arXiv:2403.15253

LP, "On polynomial stability of coupled partial differential equations in 1D" Proceedings of SOTA 2018, arXiv:1911.06715

Thank You!