(Non-uniform) Stability of Coupled PDEs: A Systems Theory Approach

Lassi Paunonen

Tampere University, Finland

February 27th, 2020

Supported by Academy of Finland grants
298182 (2016-2019) and 310489 (2017-2021)

Main Objectives

Problem

*Consider different types of **coupled PDE systems** from the point of view of **control theory**.*

Focus on **non-uniform stability**, but exponential stability included.

Main Objectives

Problem

*Consider different types of **coupled PDE systems** from the point of view of **control theory**.*

Focus on **non-uniform stability**, but exponential stability included.

Motivation:

- Coupling of stable and unstable PDEs and ODEs often leads to rational decay of energy, i.e., polynomial stability.

Main results:

- Discussion and a (hopefully new) viewpoint.
- New stability results for coupled PDEs.
- Disclaimer: Will not solve all your problems!

Outline

(1) "Passive feedback structures" in coupled PDE systems

- Highlight parallels in coupled PDEs and linear systems

(2) Conversion from coupled PDEs to coupled systems

- Examples on "how to set it up".

(3) What do we get?

- General conditions for polynomial and nonuniform stability of coupled PDEs and systems.

Coupled PDE-PDE and PDE-ODE systems appear in models of

- Fluid-structure interactions
- Thermo-elasticity
- Mechanical systems, e.g., beams with tip masses
- Magnetohydrodynamics
- Acoustics

Especially:

- Networks of 1D PDEs of mixed types (wave/heat/beam/ transport) with coupling BCs at vertices.

Coupled PDE-PDE and PDE-ODE systems appear in models of

- Fluid-structure interactions
- Thermo-elasticity
- Mechanical systems, e.g., beams with tip masses
- Magnetohydrodynamics
- Acoustics

Especially:

- Networks of 1D PDEs of mixed types (wave/heat/beam/ transport) with coupling BCs at vertices.

Couplings may either be

- Through the **boundary** (Fluid-structure, acoustics), or
- **inside a shared domain** (Thermo-elasticity, MHD)

We will focus on couplings that are **passive** (details later).

Example: Coupled Wave–Heat Systems

Models for fluid–structure and heat–structure interactions:

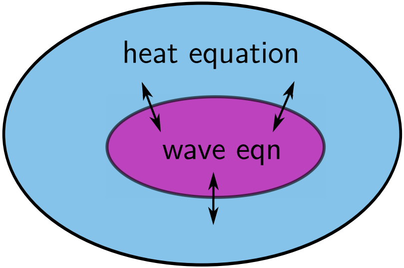
$$\frac{\partial^2 u}{\partial t^2}(x,t) = \Delta u(x,t)$$

$$\updownarrow$$

coupling BCs

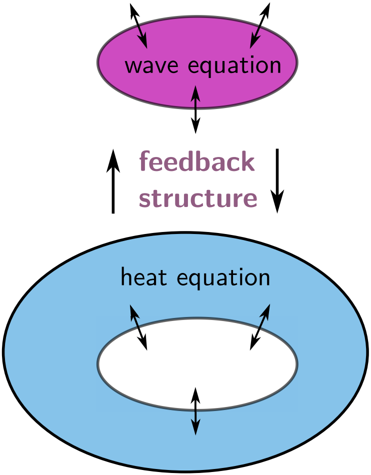
$$\updownarrow$$

$$\frac{\partial w}{\partial t}(x,t) = \Delta w(x,t)$$

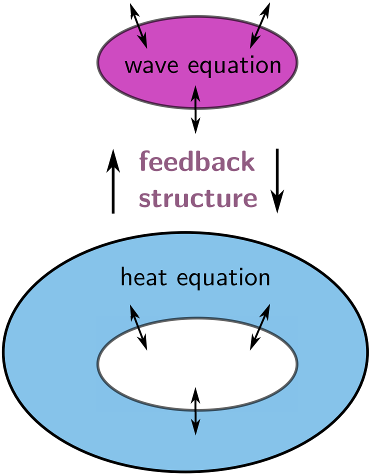


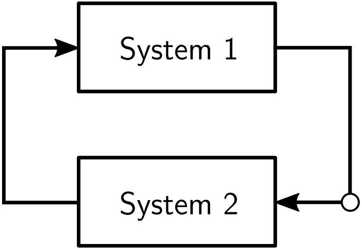
References: Avalos & Triggiani, Duyckaerts, Mercier, Nicaise, Ammari, Zuazua, Guo, Ng & Seifert, and many others.

Coupled Wave–Heat Systems

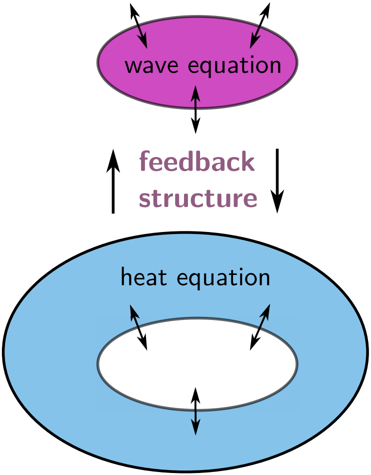


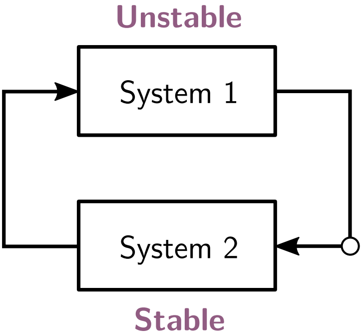
Coupled Wave–Heat Systems



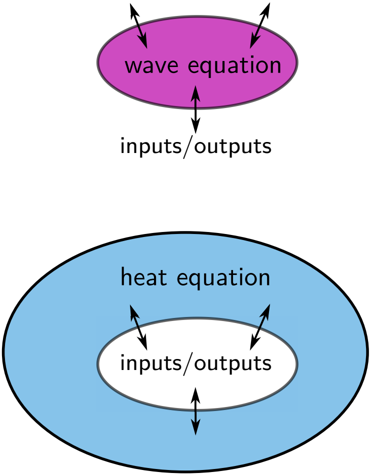


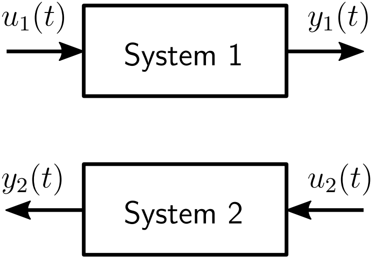
Coupled Wave–Heat Systems





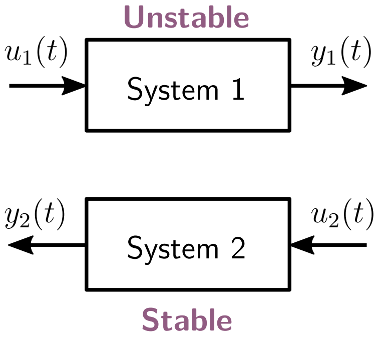
Inputs and Outputs



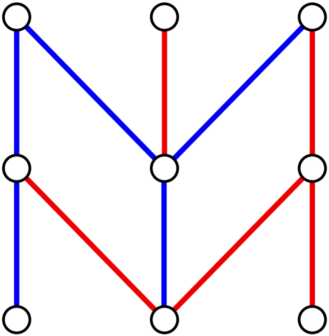


Problem

Use the properties of the two systems to deduce stability of the coupled PDE.

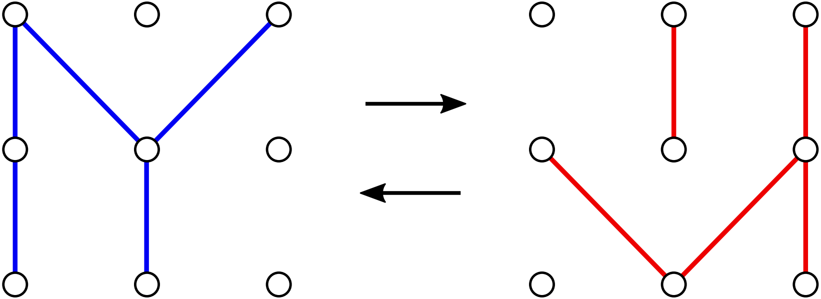


Analysis of Networks of PDEs of mixed types



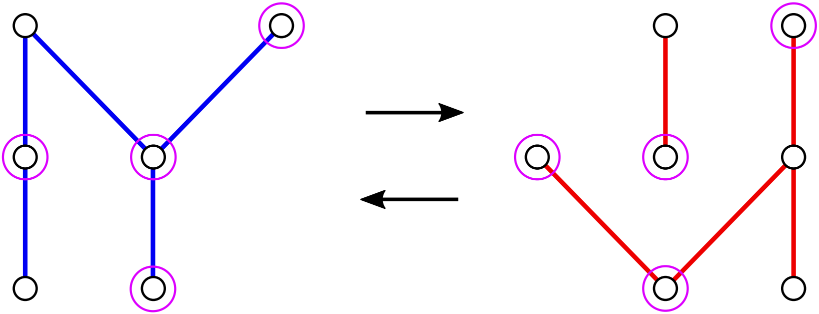
Types of PDEs: “UNSTABLE” and “STABLE”

Analysis of Networks of PDEs of mixed types



"UNSTABLE" system vs. "STABLE" system

Analysis of Networks of PDEs of mixed types



Inputs and ouputs defined at the previously shared vertices.

Linear Control Systems

Consider the **linear control system**:

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0 \in X$$
$$y(t) = Cx(t) + Du(t)$$

where X is Hilbert, A generates a semigroup, and B and C are either bounded or unbounded.

"Passive" Systems

To keep things simple, we only focus on systems

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0 \in X$$
$$y(t) = B^* x(t)$$

where X is Hilbert, A generates a **contraction semigroup**, and $B \in \mathcal{L}(U, V^*)$ for some suitable spaces U and $V^* \supseteq X$.

"Passive" Systems

To keep things simple, we only focus on systems

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0 \in X$$
$$y(t) = B^* x(t)$$

where X is Hilbert, A generates a **contraction semigroup**, and $B \in \mathcal{L}(U, V^*)$ for some suitable spaces U and $V^* \supseteq X$.

Such systems are "**impedance passive**", which in particular means they have "**no internal sources of energy**",

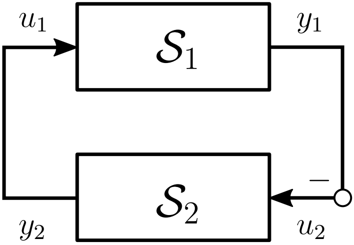
$$\frac{d}{dt}\|x(t)\|^2 \leq 2\,\mathrm{Re}\langle u(t), y(t)\rangle_Y$$

Examples:

- Many mechanical systems, RLC circuits, ...

Feedback Theory of Passive Systems

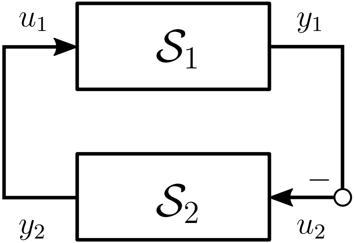
Property: "Power-preserving interconnection" preserves passivity!



$\Rightarrow$ Closed-loop semigroup contractive on Hilbert $X_1 \times X_2$.

Feedback Theory of Passive Systems

Property: "Power-preserving interconnection" preserves passivity!



$\Rightarrow$ Closed-loop semigroup contractive on Hilbert $X_1 \times X_2$.

Some results exist on exponential stability, here we focus on **non-uniform stability** $\rightarrow$ decay rates for total energy.

Coupled Passive Systems

If for $k = 1, 2$ we let

$$\begin{aligned} \dot{x}_k(t) &= A_k x_k(t) + B_k u_k(t), \qquad x_k(0) \in X_k \\ y_k(t) &= B_k^* x_k(t), \end{aligned}$$

then the "**power-preserving interconnection**" leads to

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A_1 & B_1 B_2^* \\ -B_2 B_1^* & A_2 \end{bmatrix}}_{=:\, A} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Question

How to choose U, B_1, and B_2?

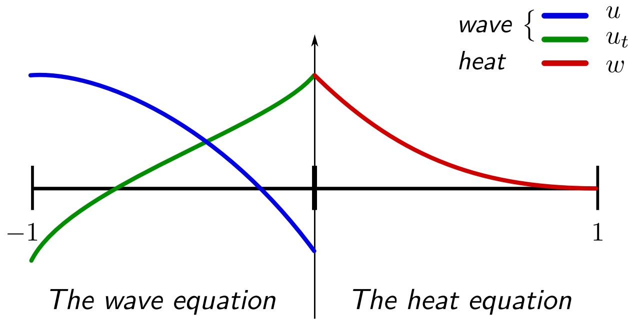
Example: 1D Wave–Heat Model

$$\begin{cases} v_{tt}(\xi,t) = v_{\xi\xi}(\xi,t), & \xi \in (-1,0),\ t>0, \\ w_t(\xi,t) = w_{\xi\xi}(\xi,t), & \xi \in (0,1),\ t>0, \\ v_\xi(0,t) = w_\xi(0,t), \quad v_t(0,t) = w(0,t), & t>0, \end{cases}$$

- [Xu Zhang & Zuazua, Batty, Paunonen & Seifert, (2D version: Avalos, Triggiani & Lasiecka)]
- Known: Non-uniform stability with $\alpha = 1/2$.

Example: 1D Wave–Heat Model

$$\begin{cases} v_{tt}(\xi,t) = v_{\xi\xi}(\xi,t), & \xi \in (-1,0),\ t>0, \\ w_t(\xi,t) = w_{\xi\xi}(\xi,t), & \xi \in (0,1),\ t>0, \\ v_\xi(0,t) = w_\xi(0,t), \quad v_t(0,t) = w(0,t), & t>0, \end{cases}$$



Example: 1D Wave-Heat — Open-Loop Splitting

Wave system on $(-1,0)$:

$$v_{tt}(\xi,t) = v_{\xi\xi}(\xi,t)$$
$$y_1(t) = v_\xi(0,t)$$
$$u_1(t) = v_t(0,t)$$

Unstable

Heat system on $(0,1)$:

$$w_t(\xi,t) = w_{\xi\xi}(\xi,t)$$
$$y_2(t) = w(0,t)$$
$$u_2(t) = -w_\xi(0,t)$$

Stable

The systems **are** impedance passive. We have $U = \mathbb{C}$ and B_1 and B_2 are unbounded.

Polynomial and Non-Uniform Stability

Theorem (Borichev & Tomilov '10)

Let $T(t)$ be a uniformly bounded C_0-semigroup on a Hilbert space X. Let A be the generator of $T(t)$ and $\sigma(A) \cap i\mathbb{R} = \emptyset$.

For any constant $\alpha > 0$, the following are equivalent:

$$\|T(t)x_0\| \leq \frac{M}{t^{1/\alpha}} \|Ax_0\| \qquad \text{for some } M > 0$$

$$\|(is - A)^{-1}\| \leq M_R(1 + |s|^\alpha), \qquad \text{for some } M_R > 0$$

General: Batty & Duyckaerts '08, Rozendaal, Seifert & Stahn '17.

Application: $E(t) \sim \|T(t)x_0\|^2$ for many PDE systems.

Polynomial and Non-Uniform Stability

Since our coupled systems are contractive by default,

*"Non-uniform stability **only** requires a resolvent estimate"*

Polynomial and Non-Uniform Stability

Since our coupled systems are contractive by default,

"Non-uniform stability ***only*** *requires a resolvent estimate"*

Limit case: Bounded resolvent

$$\|(is - A)^{-1}\| \leq M_R, \qquad s \in \mathbb{R}$$

implies **exponential stability**, i.e., $\exists M, \omega > 0$ such that

$$\|T(t)x_0\| \leq Me^{-\omega t}\|x_0\|, \qquad x_0 \in X.$$

Problem

Derive a resolvent estimate for

$$A := \begin{bmatrix} A_1 & B_1 B_2^* \\ -B_2 B_1^* & A_2 \end{bmatrix}$$

in terms of the properties of

- (A_1, B_1, B_1^*) [**Unstable**]
- (A_2, B_2, B_2^*) [**Stable**]

Problem

Derive a resolvent estimate for

$$A := \begin{bmatrix} A_1 & B_1 B_2^* \\ -B_2 B_1^* & A_2 \end{bmatrix}$$

- (A_1, B_1, B_1^*) [**Unstable**] *and* (A_2, B_2, B_2^*) [**Stable**]

Summary of results:

$$\|(is - A)^{-1}\| \lesssim M_1(|s|) M_2(|s|)$$

Problem

Derive a resolvent estimate for

$$A := \begin{bmatrix} A_1 & B_1 B_2^* \\ -B_2 B_1^* & A_2 \end{bmatrix}$$

- (A_1, B_1, B_1^*) [**Unstable**] *and* (A_2, B_2, B_2^*) [**Stable**]

Summary of results:

$$\|(is - A)^{-1}\| \lesssim M_1(|s|) M_2(|s|)$$

- $M_1(\cdot)$ increasing when (B_1^*, A_1) is not "exactly observable" (limit case $M_1(\cdot) \equiv$ const. if exactly observable)

Problem

Derive a resolvent estimate for

$$A := \begin{bmatrix} A_1 & B_1 B_2^* \\ -B_2 B_1^* & A_2 \end{bmatrix}$$

- (A_1, B_1, B_1^*) [**Unstable**] *and* (A_2, B_2, B_2^*) [**Stable**]

Summary of results:

$$\|(is - A)^{-1}\| \lesssim M_1(|s|) M_2(|s|)$$

- $M_1(\cdot)$ increasing when (B_1^*, A_1) is not "exactly observable" (limit case $M_1(\cdot) \equiv$ const. if exactly observable)
- When $P_2(is) = B_2^*(is - A_2)^{-1} B_2$ is the "transfer function",

$$M_2(s) \sim \|\left[\operatorname{Re} P_2(is)\right]^{-1}\| \qquad \left(M_2(s) \sim \frac{1}{\operatorname{Re} P_2(is)} \right)$$

Important Special Case

Consider $T(t)$ generated by

$$A := \begin{bmatrix} A_1 & B_1 B_2^* \\ -B_2 B_1^* & A_2 \end{bmatrix}$$

Proposition

- *Assume* (B_1^*, A_1) *is exactly observable*
- *Assume there exists* $\alpha \geq 0$ *such that*

$$\operatorname{Re} P_2(is) \gtrsim \frac{1}{1 + |s|^\alpha}$$

Then $T(t)$ *is polynomially stable,* $\|(is - A)^{-1}\| \lesssim 1 + |s|^\alpha$*,*

$$\|T(t)x_0\| \leq \frac{M}{t^{1/\alpha}} \|Ax_0\|, \qquad x_0 \in \mathcal{D}(A).$$

The Observability Condition on (B_1^*, A_1)

Proposition

- *Assume* (B_1^*, A_1) *is exactly observable*

Technical definition: $\exists \tau, \kappa > 0$ such that

$$\int_0^\tau \|B_1^* T_1(t)x\|^2 dt \geq \kappa \|x\|^2, \qquad x \in \mathcal{D}(A).$$

Skew-adjointness of A_1: Equivalent to

$A_1 - B_1 B_1^*$ generates an exponentially stable semigroup

and the system (A_1, B_1, B_1^*) is "*stabilized exponentially by negative output feedback* $u(t) = -y(t)$".

Comments:

- Theorem requires some admissibility and well-posedness assumptions (swept under the carpet here). Limits 2D-nD BC.
- The more general version details the effect of the lack of exact observability of (B_1^*, A_1).

Optimality

- Obtained rate is not always optimal, especially if
  - A_1 has no spectral gap (2D, nD waves)
- A nice way of getting (possibly) suboptimal rates easily.

Comments:

- Theorem requires some admissibility and well-posedness assumptions (swept under the carpet here). Limits 2D-nD BC.
- The more general version details the effect of the lack of exact observability of (B_1^*, A_1).

Optimality

- Obtained rate is not always optimal, especially if
  - A_1 has no spectral gap (2D, nD waves)
- A nice way of getting (possibly) suboptimal rates easily.

References:

- Paunonen (SIAM J. Control Optim. 2019)

Example: 1D Wave-Heat

Wave system on $(-1,0)$:

$$\begin{aligned}\rho(\xi)v_{tt}(\xi,t) &= (T(\xi)v_\xi)_\xi(\xi,t)\\ u_1(t) &= v_t(0,t)\\ y_1(t) &= T(0)v_\xi(0,t)\end{aligned}$$

Heat system on $(0,1)$:

$$\begin{aligned}w_t(\xi,t) &= w_{\xi\xi}(\xi,t)\\ y_2(t) &= w(0,t)\\ u_2(t) &= -w_\xi(0,t)\end{aligned}$$

- A_1 skew-adjoint, (B_1^*, A_1) exactly observable.
- $P_2(is) = B_2^*(is - A_2)^{-1}B_2$ satisfies $\operatorname{Re} P_2(is) \sim |s|^{-1/2}$.

Thus the closed-loop system is polynomially stable,

$$\|(is - A)^{-1}\| \lesssim 1 + |s|^{1/2} \qquad \text{and} \qquad \|T(t)x_0\| \le \frac{M}{t^2}\|Ax_0\|.$$

Generalises results of [Zhang-Zuazua, Batty-Paunonen-Seifert].

Example: Wave equation with an Acoustic BC

Consider a wave equation with a dynamic BC at $\xi = 1$:

$$\begin{aligned} \rho(\xi)v_{tt}(\xi,t) &= (T(\xi)v_\xi(\xi,t))_\xi, \qquad 0<\xi<1, \\ m\delta_{tt}(t) &= -d\delta_t(t) - k\delta(t) - \beta v_t(1,t) \\ v_\xi(1,t) &= \delta_t(t), \qquad v_t(0,t) = 0. \end{aligned}$$

Studied by Beale, and Muños Rivera & Qin.

Example: Wave equation with an Acoustic BC

Wave system on $(0,1)$:

$$\rho(\xi)v_{tt}(\xi,t) = (T(\xi)v_\xi)_\xi(\xi,t)$$
$$y_1(t) = T(1)v_\xi(1,t)$$
$$u_1(t) = v_t(1,t)$$

The ODE part

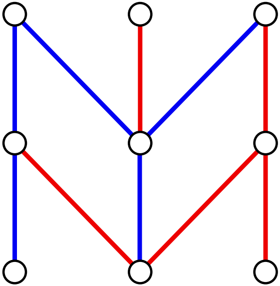
$$m\ddot{\delta}(t) + k\delta(t) + d\dot{\delta}(t) = \beta u_c(t)$$
$$y_c(t) = T(1)\dot{\delta}(t).$$

- A_1 skew-adjoint, (B_1^*, A_1) exactly observable.
- $P_2(is) = B_2^*(is - A_2)^{-1}B_2$ satisfies $\operatorname{Re} P_2(is) \sim s^{-2}$.

Thus the closed-loop system is polynomially stable,

$$\|(is - A)^{-1}\| \lesssim 1 + s^2 \qquad \text{and} \qquad \|T(t)x_0\| \leq \frac{M}{\sqrt{t}}\|Ax_0\|.$$

Networks of PDEs of mixed types



TODO:

Results should be applicable for wave and heat equations on networks.

Conclusions

In this presentation:

- Discussion of coupled PDE and PDE-ODE systems from the viewpoint of systems theory
- General conditions for non-uniform and polynomial stability of coupled systems.

LP, "Stability and Robust Regulation of Passive Linear Systems" SIAM J. Control Optim. 2019, http://arxiv.org/abs/1706.03224

LP, "On polynomial stability of coupled partial differential equations in 1D" Proceedings of SOTA 2018 https://arxiv.org/abs/1911.06715