On Perturbation of Strongly Stable Riesz-Spectral Operators

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Main Problem

Problem

Assume

- $A : \mathcal{D}(A) \subset X \to X$ *generates a strongly stable* C_0*-semigroup*
- $B = \langle \cdot, g \rangle b \in \mathcal{L}(X)$.

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Special case: $\sigma(A) \subset \mathbb{C}^-$ *and* $\operatorname{Re} \lambda \to 0$ *only as* $|\operatorname{Im} \lambda| \to \infty$.

Motivation

Robust output regulation & an infinite-dimensional signal generator.

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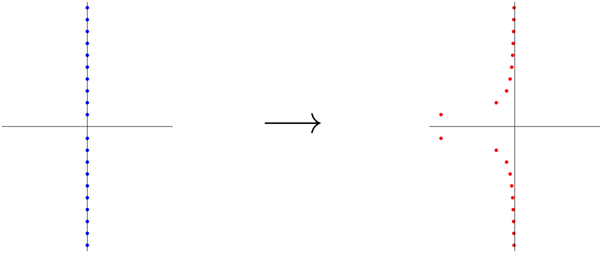
Strong stabilization of the internal model leads to an operator with spectrum approaching $i\mathbb{R}$ asymptotically.



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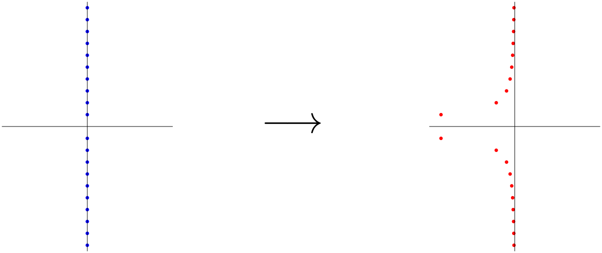

Robustness properties?

Main Problem

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Consider an operator

$$A = \sum_{k \in \mathbb{Z}} \lambda_k \langle \cdot, \phi_k \rangle \phi_k, \quad \mathcal{D}(A) = \left\{ x \mid \sum_{k \in \mathbb{Z}} |\lambda_k|^2 |\langle x, \phi_k \rangle|^2 < \infty \right\}$$

- $\{\phi_k\}$ *is an orthonormal basis of a Hilbert space* X
- $\{\lambda_k\} \subset \mathbb{C}^-$ *and has no finite accumulation points*
- $\operatorname{Re} \lambda_k \to 0$ *as* $|\operatorname{Im} \lambda_k| \to \infty$.

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Determine sets $M, N \subset X$ *such that if* $b \in M$ *and* $g \in N$, *then*

a) $\sigma(A + \langle \cdot, g\rangle b) \subset \mathbb{C}^-$

b) $A + \langle \cdot, g\rangle b$ *generates a strongly stable* C_0*-semigroup.*

Earlier Work

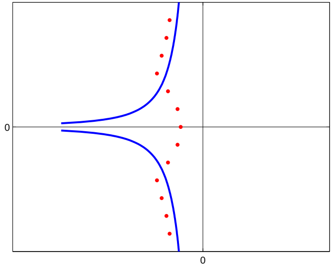
- G. M. Sklyar & A. V. Rezounenko (Robustness of a strongly stabilizing feedback)
- S. Caraman (Robustness of strong stability of compact C_0-semigroups)
- Pole placement (Sun Shun-Hua, etc.)

Asymptotic behaviour of $\sigma(A)$

Assumption (Geometric assumption on $\sigma(A)$)

There exist constants $c, \alpha > 0$ *and* $y_0 > 0$ *such that*

$$\operatorname{Re}\lambda_k \leq -\frac{c}{|\operatorname{Im}\lambda_k|^\alpha} \qquad \textit{if} \quad |\operatorname{Im}\lambda_k| \geq y_0$$



Main Results

Definition

For $\beta \geq 0$ define a Hilbert space $(M_\beta, \|\cdot\|_\beta)$ such that

$$M_\beta = \left\{ x \in X \mid \sum_{k \in \mathbb{Z}} |\lambda_k|^{2\beta} |\langle x, \phi_k \rangle|^2 < \infty \right\}$$

$$\|\cdot\|_\beta = \left(\sum_{k \in \mathbb{Z}} |\lambda_k|^{2\beta} |\langle \cdot, \phi_k \rangle|^2 \right)^{\frac{1}{2}}$$

For $n \in \mathbb{N}_0$ we have $M_n = \mathcal{D}(A^n)$ and $\|x\|_n = \|A^n x\|$.

Perturbation of the Spectrum

Proposition

Let $\sigma(A)$ satisfy the geometric assumption for some $\alpha > 0$.

If $b \in M_\beta$ and $g \in M_\gamma$ for some $\beta, \gamma \geq 0$ such that $\beta + \gamma \geq \alpha$, then there exist $\delta_\beta, \delta_\gamma > 0$ such that

$$\sigma(A + \langle \cdot, g \rangle b) \subset \mathbb{C}^-$$

whenever $\|b\|_\beta < \delta_\beta$ and $\|g\|_\gamma < \delta_\gamma$.

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Proof.

Using the first Weinstein-Aronszajn formula. □

Corollary

Let $\sigma(A)$ satisfy the geometric assumption for some $\alpha > 0$.

If $b \in \mathcal{D}(A^n)$ and $g \in \mathcal{D}(A^m)$ for some $n, m \in \mathbb{N}_0$ such that $n + m \geq \alpha$, then there exist $\delta_n, \delta_m > 0$ such that

$$\sigma(A + \langle \cdot, g \rangle b) \subset \mathbb{C}^-$$

whenever $\|A^n b\| < \delta_n$ and $\|A^m g\| < \delta_m$.

Preservation of Uniform Boundedness

Proposition

Let $\sigma(A)$ satisfy the geometric assumption for some $\alpha > 0$.

Denote by $T_B(t)$ the C_0-semigroup generated by $A + \langle \cdot, g \rangle b$. Then

- *If $b \in M_\alpha$, there exits $\delta > 0$ such that $T_B(t)$ is uniformly bounded whenever $\|b\|_\alpha < \delta$;*
- *If $g \in M_\alpha$, there exits $\delta > 0$ such that $T_B(t)$ is uniformly bounded whenever $\|g\|_\alpha < \delta$.*

Outline of the Proof

Lemma (Casarino & Piazzera)

Let A generate a uniformly bounded C_0-semigroup $T(t)$ on X and let $B \in \mathcal{L}(X)$. If there exists $0 < C < 1$ such that for all $t \geq 0$ and $x \in X$

$$\int_0^t \|BT(s)x\| ds \leq C\|x\|,$$

then the semigroup generated by $A + B$ is uniformly bounded.

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Apply this to $A + B$ and $A^* + B^*$.

Preservation of Strong Stability

Proposition

Let $\sigma(A)$ satisfy the geometric assumption for some $\alpha > 0$.

If $b \in M_\beta$ and $g \in M_\gamma$ for some $\beta, \gamma \geq 0$ such that $\beta \geq \alpha$ or $\gamma \geq \alpha$, then there exist $\delta_\beta, \delta_\gamma > 0$ such that

$$A + \langle \cdot, g \rangle b$$

generates a strongly stable C_0-semigroup whenever $\|b\|_\beta < \delta_\beta$ and $\|g\|_\gamma < \delta_\gamma$.

Extensions

- $A = \sum \lambda_k \langle \cdot, \psi_k \rangle \phi_k$ is a Riesz spectral operator
  - Straightforward
  - Separate spaces $b \in M_\beta$ and $g \in N_\gamma$ corresponding to $\{\psi_k\}$ and $\{\phi_k\}$, respectively.

Extensions

- $A = \sum \lambda_k \langle \cdot, \psi_k \rangle \phi_k$ is a Riesz spectral operator
    - Straightforward
    - Separate spaces $b \in M_\beta$ and $g \in N_\gamma$ corresponding to $\{\psi_k\}$ and $\{\phi_k\}$, respectively.
- A general finite-rank perturbation $B = \sum_{j=1}^{m} \langle \cdot, g_j \rangle b_j$
    - Using Weinstein-Aronszajn determinant

Strongly Stabilized Wave Equation

Consider a wave equation on $(0,1)$:

$$\frac{\partial^2 w}{\partial t^2}(z,t) = \frac{\partial^2 w}{\partial z^2}(z,t) + b_0(z)u(t)$$
$$w(0,t) = w(1,t) = 0$$

where $b_0(z) = \sqrt{3}(1-z)$.

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Define $x = \begin{bmatrix} w \\ \frac{\partial w}{\partial t} \end{bmatrix}$ and $A_0 = -\frac{\partial^2}{\partial z^2}$ with domain

$$\mathcal{D}(A_0) = \Big\{ w \in L^2(0,1) \;\Big|\; w, w' \text{ abs. cont. } w'' \in L^2(0,1), \\ w(0) = w(1) = 0 \;\Big\}$$

Linear system

$$\dot{x} = Ax + Bu, \qquad x(0) = x_0$$

on a Hilbert space $X = \mathcal{D}(A_0^{\frac{1}{2}}) \times L^2(0,1)$. Here

$$A = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}, \qquad B = b = \begin{bmatrix} 0 \\ b_0 \end{bmatrix}$$

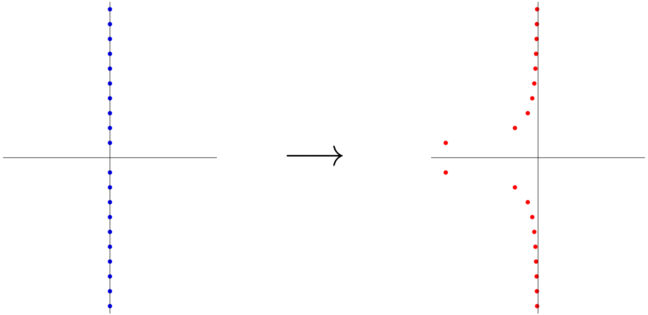
and

$$A = \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} in\pi \langle \cdot, \phi_n \rangle \phi_n,$$

where $\{\phi_n\}_n$ is an orthonormal basis of X.

Strong Stabilization

Find feedback K such that $\sigma(A+BK) = \{-\frac{\pi}{|n|^2} + in\pi\}_{\substack{n\in\mathbb{Z}\\ n\neq 0}}$



Now $A + BK$ is a Riesz spectral operator with eigenvalues

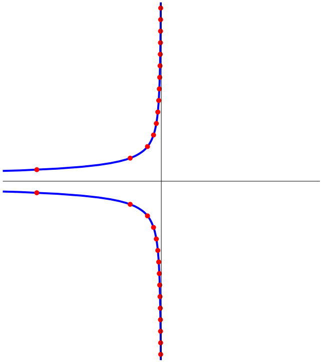
$$\mu_n = -\frac{\pi}{|n|^2} + in\pi$$

and eigenfunctions

$$\varphi_n = \frac{\mu_n - \lambda_n}{\langle b, \phi_n \rangle} R(\mu_n, A) b = \sum_{k \neq 0} \frac{n}{k - in^2 k(n-k)} \phi_k$$

Geometric Assumption

The spectrum of $A + BK$ satisfies the geometric assumption for $\alpha = 2$, $c = \pi^3$ and $y_0 = \pi$.



Perturbations

Consider perturbations

$$\frac{\partial^2 w}{\partial t^2}(z,t) = \frac{\partial^2 w}{\partial z^2}(z,t) + b_0(z)u(t) + d_0\left(\langle w, g_1 \rangle_{L^2} + \langle \frac{\partial w}{\partial t}, g_2 \rangle_{L^2} \right)$$

where $d_0, g_2 \in \mathcal{D}(A_0)$ and $g_1 \in L^2(0,1)$

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where $d_0, g_2 \in \mathcal{D}(A_0)$ and $g_1 \in L^2(0,1)$, i.e.

$$\dot{x} = (A+BK)x + d\langle x, g\rangle_X$$

where

$$d = \begin{bmatrix} 0 \\ d_0 \end{bmatrix}, \qquad g = \begin{bmatrix} A_0^{-1} g_1 \\ g_2 \end{bmatrix}$$

and $d, g \in \mathcal{D}(A_0) \times \mathcal{D}(A_0^{\frac{1}{2}}) = \mathcal{D}(A+BK)$.

Spectrum

Now $\alpha = 2$, compute $\delta > 0$ such that $\sigma(A + BK + \langle \cdot, g \rangle d) \subset \mathbb{C}^-$ if

$$\|(A + BK)g\| < \delta \quad \text{and} \quad \|(A + BK)d\| < \delta.$$

We can use $\delta = \frac{1}{4}$ and $\sigma(A + BK + \langle \cdot, g \rangle d) \subset \mathbb{C}^-$ whenever

$$\|d_0'\|_{L^2} + 21\|d_0\|_{L^2} < \frac{1}{4}$$

$$\sqrt{\|g_1\|_{L^2}^2 + \|g_2'\|_{L^2}^2} < \frac{1}{88}$$

$$\sqrt{\frac{1}{\pi}\|g_1\|_{L^2}^2 + \|g_2\|_{L^2}^2} < \frac{1}{88}$$

Uniform Boundedness

- Since $\alpha = 2$, we need that d or g is in $\mathcal{D}(A+BK)^2$;
- This is a difficult requirement since $\mathcal{R}(B) \not\subset \mathcal{D}(A)$;
- If we choose a smoother b_0, we need a larger α;
- Other conditions for preservation of uniform boundedness?
  - Riesz-spectral operators?

Conclusions

- Conditions for the preservation of the property $\sigma(A) \subset \mathbb{C}^-$ and the strong stability of a semigroup
- Application to a wave equation

Thank You!