Robust stability of observers

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Structure of the presentation

1. Problem formulation
2. Solution
3. A simple example

System $\Sigma(A, B, C)$

Let X, U and Y be Hilbert spaces. Consider the distributed parameter system $\Sigma(A, B, C)$,

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

Assumptions:

- $A : X \supset \mathcal{D}(A) \to X$ generates a C_0-semigroup on X
- $B \in \mathcal{L}(U, X)$ and $C \in \mathcal{L}(X, Y)$
- (A, B) exponentially stabilizable, (A, C) exponentially detectable
- $\widetilde{A}$ is an operator on X such that $\widetilde{A} - A$ is an A-bounded operator.

Main Problem

Consider the stabilization of $\Sigma(A, B, C)$ with an observer,

$$\begin{aligned} \dot{x} &= Ax + BK\hat{x} \\ \dot{\hat{x}} &= A\hat{x} + BK\hat{x} + LC(\hat{x} - x) \end{aligned}$$

where $K \in \mathcal{L}(X, U), L \in \mathcal{L}(Y, X)$ are chosen such that $A + BK$ and $A + LC$ are exponentially stable.

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Replace the operator A with $\widetilde{A}$ in the observer, i.e.

$$\begin{aligned} \dot{x} &= Ax + BK\hat{x} \\ \dot{\hat{x}} &= \widetilde{A}\hat{x} + BK\hat{x} + LC(\hat{x} - x) \end{aligned}$$

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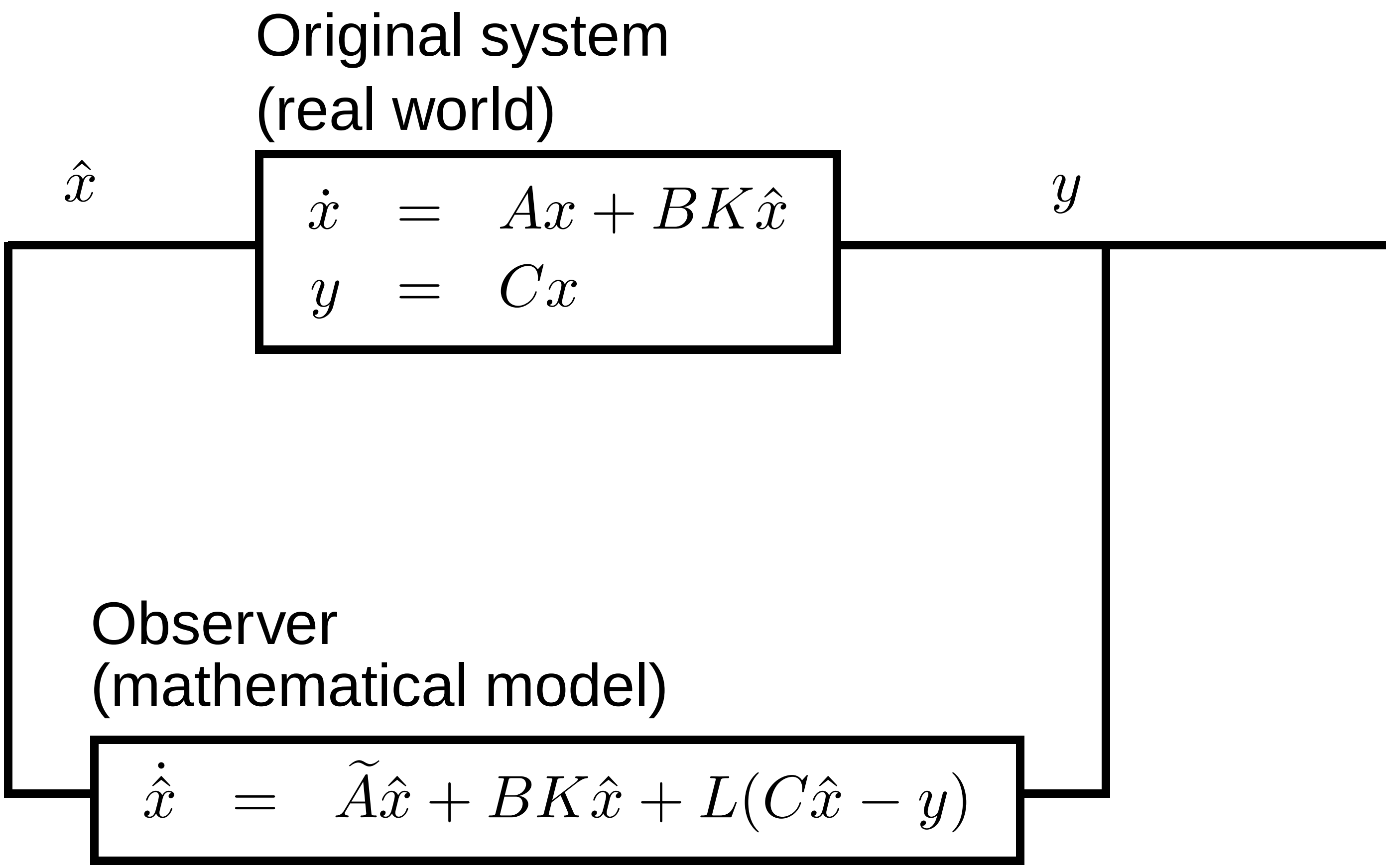
Under what conditions is the new closed-loop system exponentially stable?

Motivation


Original system
(real world)
$\hat{x}$
$\dot{x} = Ax + BK\hat{x}$
$y = Cx$
y
Observer
(mathematical model)
$\dot{\hat{x}} = \widetilde{A}\hat{x} + BK\hat{x} + L(C\hat{x} - y)$

The closed-loop system is stable if the composite operator

$$\widetilde{A}_c = \begin{bmatrix} A + BK & BK \\ \widetilde{A} - A & \widetilde{A} + LC \end{bmatrix}$$

generates an exponentially stable C_0-semigroup on $X \times X$.

The closed-loop system is stable if the composite operator

$$\widetilde{A}_c = \begin{bmatrix} A+BK & BK \\ \widetilde{A}-A & \widetilde{A}+LC \end{bmatrix}$$

generates an exponentially stable C_0-semigroup on $X \times X$.

On the other hand

$$\begin{bmatrix} A+BK & BK \\ \widetilde{A}-A & \widetilde{A}+LC \end{bmatrix} = \underbrace{\begin{bmatrix} A+BK & BK \\ 0 & A+LC \end{bmatrix}}_{=:\, A_c} + \begin{bmatrix} 0 & 0 \\ \widetilde{A}-A & \widetilde{A}-A \end{bmatrix}$$

and A_c generates an exponentially stable C_0-semigroup $S(t)$ on $X \times X$.

Divide the problem into two parts:

- Under what conditions does the operator $\widetilde{A}_c$ generate a C_0-semigroup on $X \times X$? (The new closed-loop system is well-posed)
- What additional conditions are needed for this C_0-semigroup to be exponentially stable?

Divide the problem into two parts:

- Under what conditions does the operator $\widetilde{A}_c$ generate a C_0-semigroup on $X \times X$? (The new closed-loop system is well-posed)
- What additional conditions are needed for this C_0-semigroup to be exponentially stable?

Conditions in terms of

- Resolvent operators $R(\lambda, A + BK)$ and $R(\lambda, A + LC)$,
- C_0-semigroups generated by $A + BK$ and $A + LC$.

Conditions on the resolvent operators

The resolvent operator of

$$A_c = \begin{bmatrix} A+BK & BK \\ 0 & A+LC \end{bmatrix}$$

is given by

$$R(\lambda, A_c) = \begin{bmatrix} R(\lambda, A+BK) & R(\lambda, A+BK)BKR(\lambda, A+LC) \\ 0 & R(\lambda, A+LC) \end{bmatrix}.$$

for all $\lambda \in \rho(A+BK) \cap \rho(A+LC)$ (in particular if $\lambda \in \mathbb{C}^+$).

C_0-semigroup generation:

Theorem 1 (Kaiser & Weis 2003)**.** *Let* $\mathcal{A}$ *generate a* C_0*-semigroup on a Hilbert space* X *and let* $\mathcal{B}$ *be a closed operator on* X *with domain* $\mathcal{D}(\mathcal{B}) \supset \mathcal{D}(\mathcal{A})$*. If there exist constants* $0 < q < 1$ *and* $\lambda_0 \in \mathbb{R}^+$ *s.t. for all* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda \geq \lambda_0$

$$\begin{aligned} \|\mathcal{B}R(\lambda,\mathcal{A})\| &\leq q \\ \|R(\lambda,\mathcal{A})\mathcal{B}x\| &\leq q\|x\| \qquad \forall x \in \mathcal{D}(\mathcal{B}), \end{aligned}$$

then the operator $\mathcal{A}+\mathcal{B}$ *generates a* C_0*-semigroup on* X*.*

Applied to our problem:

Proposition 2. *The closed-loop system is operator generates a C_0-semigroup on $X \times X$ if $\widetilde{A} - A$ with domain $\mathcal{D}(\widetilde{A} - A) = \mathcal{D}(A)$ is a closed operator and there exist constants $\lambda_0 \in \mathbb{R}$ and $0 < q < 1/\sqrt{2}$ such that*

$$\begin{aligned} \|(\widetilde{A} - A)R(\lambda, A + BK)\| &\leq q \\ \|(\widetilde{A} - A)R(\lambda, A + LC)\| &\leq q \\ \|R(\lambda, A + LC)(\widetilde{A} - A)x\| &\leq q\|x\| \qquad \forall x \in \mathcal{D}(A) \end{aligned}$$

for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq \lambda_0$.

Stability:

Lemma 3. *Let* $\mathcal{A}$ *generate an exponentially stable* C_0*-semigroup on a Hilbert space* X *and* $\mathcal{B}$ *be an* A*-bounded operator such that* $\mathcal{A}+\mathcal{B}$ *generates a* C_0*-semigroup on* X*. If there exists a constant* $0<q<1$ *s.t.*

$$\|\mathcal{B}R(\lambda,\mathcal{A})\| \leq q \qquad \forall\lambda\in\mathbb{C}^+,$$

then the C_0*-semigroup generated by* $\mathcal{A}+\mathcal{B}$ *is exponentially stable.*

Stability:

Lemma 4. *Let $\mathcal{A}$ generate an exponentially stable C_0-semigroup on a Hilbert space X and $\mathcal{B}$ be an A-bounded operator such that $\mathcal{A}+\mathcal{B}$ generates a C_0-semigroup on X. If there exists a constant $0 < q < 1$ s.t.*

$$\|\mathcal{B}R(\lambda,\mathcal{A})\| \leq q \qquad \forall \lambda \in \mathbb{C}^+,$$

then the C_0-semigroup generated by $\mathcal{A}+\mathcal{B}$ is exponentially stable.

This follows from

$$\mathcal{A}+\mathcal{B} \text{ exp. stable} \quad \Leftrightarrow \quad \sup_{\lambda\in\mathbb{C}^+} \|R(\lambda,\mathcal{A}+\mathcal{B})\| < \infty$$

and

$$(\lambda I - \mathcal{A} - \mathcal{B})x = (I - \mathcal{B}R(\lambda,\mathcal{A}))(\lambda I - \mathcal{A})x \quad \forall x \in \mathcal{D}(\mathcal{A}).$$

Applied to our problem:

Proposition 5. *If the new closed-loop system operator generates a C_0-semigroup on $X \times X$, then the new closed-loop system is stable if there exist a constant $0 < q < 1/\sqrt{2}$ such that*

$$\begin{aligned} \|(\widetilde{A} - A)R(\lambda, A + BK)\| &\leq q \\ \|(\widetilde{A} - A)R(\lambda, A + BK)\|\|BK\|\|R(\lambda, A + LC)\| & \\ +\|(\widetilde{A} - A)R(\lambda, A + LC)\| &\leq q \end{aligned}$$

for all $\lambda \in \mathbb{C}^+$.

Conclusion:

Corollary 6. *The new closed-loop system is exponentially stable if* $\widetilde{A} - A$ *with domain* $\mathcal{D}(\widetilde{A} - A) = \mathcal{D}(A)$ *is a closed operator and there exist constants* $\lambda_0 \in \mathbb{R}$ *and* $0 < q < 1/\sqrt{2}$ *such that*

$$\begin{aligned} \|(\widetilde{A} - A)R(\lambda, A + BK)\| &\leq q \\ \|(\widetilde{A} - A)R(\lambda, A + BK)\|\|BK\|\|R(\lambda, A + LC)\| & \\ + \|(\widetilde{A} - A)R(\lambda, A + LC)\| &\leq q \end{aligned}$$

for all $\lambda \in \mathbb{C}^+$ *and*

$$\|R(\lambda, A + LC)(\widetilde{A} - A)x\| \leq q\|x\| \qquad \forall x \in \mathcal{D}(A)$$

for all $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda \geq \lambda_0$.

Conditions on the C_0-semigroups

The C_0-semigroup generated by

$$A_c = \begin{bmatrix} A+BK & BK \\ 0 & A+LC \end{bmatrix}$$

is given by

$$S(t) = \begin{bmatrix} T_1(t) & T_3(t) \\ 0 & T_2(t) \end{bmatrix},$$

where $T_1(t)$ is the C_0-semigroup generated by $A+BK$, $T_2(t)$ is the C_0-semigroup generated by $A+LC$ and

$$T_3(t)x = \int_0^t T_1(t-s)BKT_2(s)x\,ds \quad \text{for all } x \in X.$$

C_0-semigroup Generation:

Theorem 7 (Miyadera)**.** *Let* $\mathcal{A}$ *generate a* C_0*-semigroup* $T(t)$ *on a Banach space* X *and let* $\mathcal{B}$ *be an* $\mathcal{A}$*-bounded operator. If there exist constants* $0 < q < 1$ *and* $t_0 > 0$ *s.t.*

$$\int_0^{t_0} \|\mathcal{B}T(t)x\| dt \leq q\|x\| \qquad \forall x \in \mathcal{D}(\mathcal{A}),$$

then the operator $\mathcal{A} + \mathcal{B}$ *generates a* C_0*-semigroup on* X*.*

Applied to our problem:

Proposition 8. *The new closed-loop system operator generates a C_0-semigroup on $X \times X$ if there exist constants $t_0 > 0$ and q_j for $j = 1, 2, 3$ such that*

$$\int_0^{t_0} \|(\widetilde{A} - A)T_j(t)x\| dt \leq q_j \|x\| \qquad \forall x \in \mathcal{D}(A)$$

and

$$\max\{q_1, q_2 + q_3\} < 1/\sqrt{2}.$$

Stability:

Theorem 9 (Pandolfi & Zwart 1991)**.** *Let $\mathcal{A}$ generate an exponentially stable C_0-semigroup $T(t)$ on a Hilbert space X and let $\mathcal{B}$ be an $\mathcal{A}$-bounded perturbation such that the operator $\mathcal{A}+\mathcal{B}$ generates a C_0-semigroup on X. Define $L \geq 0$ and $M \geq 0$ by*

$$L^2 = \sup_{x\in\mathcal{D}(\mathcal{A}), \|x\|=1} \int_0^\infty \|\mathcal{B}T(t)x\|^2 dt$$

$$M^2 = \sup_{x\in X, \|x\|=1} \int_0^\infty \|T(t)x\|^2 dt.$$

If L is finite and

$$L < \frac{1}{2M},$$

then the C_0-semigroup generated by $\mathcal{A}+\mathcal{B}$ is exponentially stable.

Applied to our problem:

Proposition 10. *If the operator* $\widetilde{A}_c$ *generates a* C_0*-semigroup on* $X \times X$*, then the new closed-loop system is stable if there exist constants* $k_j \geq 0$ *for* $j = 1, 2, 3$ *such that*

$$\int_0^\infty \|(\widetilde{A} - A)T_j(t)x\|^2 dt \leq k_j^2 \|x\|^2 \qquad \forall x \in \mathcal{D}(A)$$

and $k_1 + k_2 + k_3 < \frac{1}{2M}$ *where* $M \geq 0$ *is any constant such that*

$$\int_0^\infty \|S(t)x\|^2 dt \leq M^2 \|x\|^2 \qquad \textit{for all} \;\; x \in X.$$

Conclusion:

Corollary 11. *The new closed-loop system is stable if the following hold:*

- *There exist constants* $t_0 > 0$ *and* q_j *for* $j = 1, 2, 3$ *such that*

$$\int_0^{t_0} \|(\widetilde{A} - A)T_j(t)x\| dt \leq q_j \|x\| \qquad \forall x \in \mathcal{D}(A)$$

and $\max\{q_1, q_2 + q_3\} < 1/\sqrt{2}$.

- *There exist constants* $k_j \geq 0$ *for* $j = 1, 2, 3$ *such that*

$$\int_0^{\infty} \|(\widetilde{A} - A)T_j(t)x\|^2 dt \leq k_j^2 \|x\|^2 \qquad \forall x \in \mathcal{D}(A)$$

and $k_1 + k_2 + k_3 < \frac{1}{2M}$ *where* $\int_0^{\infty} \|S(t)x\|^2 dt \leq M^2 \|x\|^2$ *for all* $x \in X$.

Example

Consider a system $\Sigma(A,B,C)$ on $X = L^2([0,1])$ with system operator

$$\begin{aligned} Ax &= \alpha \frac{d^2x}{dz^2} - \beta \frac{dx}{dz} - \gamma x \\ \mathcal{D}(A) &= \left\{ x \in X \;\middle|\; x, \frac{dx}{dz} \text{ abs. cont. } \frac{d^2x}{dz^2} \in X,\ x(0) = x(1) = 0 \right\} \\ Bu(t) &= 5u(t)\chi_{[0.6,0.8]}(z) \\ Cx &= 5\langle x, \chi_{[0.2,0.4]} \rangle \end{aligned}$$

Example

Consider a system $\Sigma(A,B,C)$ on $X = L^2([0,1])$ with system operator

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Bu(t) &= 5u(t)\chi_{[0.6,0.8]}(z) \\
Cx &= 5\langle x, \chi_{[0.2,0.4]} \rangle \\
\widetilde{A}x &= \widetilde{\alpha} \frac{d^2x}{dz^2} - \widetilde{\beta} \frac{dx}{dz} - \widetilde{\gamma} x \\
\mathcal{D}(\widetilde{A}) &= \mathcal{D}(A)
\end{aligned}
$$

The resolvent conditions are satisfied for

$$\alpha = 5.1 \quad \beta = 0.9 \quad \gamma = -55.1$$
$$\widetilde{\alpha} = 5 \quad \widetilde{\beta} = 1 \quad \widetilde{\gamma} = -55$$

$$Kx \approx -83\langle x, e^{-\frac{\beta}{2\alpha}\cdot} \sin(\pi\cdot)\rangle \quad \forall x \in X$$

$$Ly \approx -76ye^{-\frac{\beta}{2\alpha}\cdot} \sin(\pi\cdot)$$

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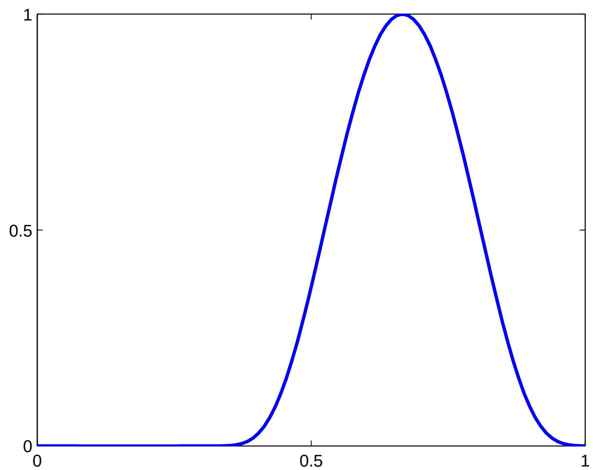
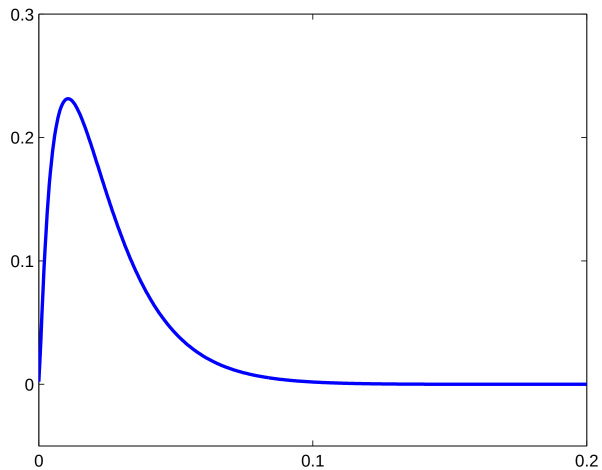
$$Ly \approx -76 y e^{-\frac{\beta\,\cdot}{2\alpha}} \sin(\pi\cdot)$$



x_0



$y(t)$

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When the original system is known:

$$\begin{bmatrix} A+BK & BK \\ \widetilde{A}-A & \widetilde{A}+LC \end{bmatrix} = \begin{bmatrix} A+BK & BK \\ 0 & A+LC \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \widetilde{A}-A & \widetilde{A}-A \end{bmatrix}$$

When the approximate system is known:

$$\begin{bmatrix} A+BK & BK \\ \widetilde{A}-A & \widetilde{A}+LC \end{bmatrix} = \begin{bmatrix} \widetilde{A}+BK & BK \\ 0 & \widetilde{A}+LC \end{bmatrix} + \begin{bmatrix} A-\widetilde{A} & 0 \\ \widetilde{A}-A & 0 \end{bmatrix}$$