Saturating Integral Control for Infinite-Dimensional Linear Systems

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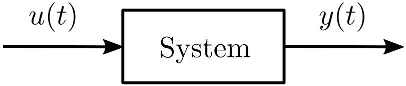
Introduction

Goal of the talk: Study the **output tracking problem** for linear systems.



Problem (Output Tracking)

Design a controller such that the output $y(t)$ of the system converges to a constant reference $y_{ref} \in \mathbb{R}$ i.e.,

$$|y(t) - y_{ref}| \to 0, \qquad \text{as} \quad t \to \infty.$$

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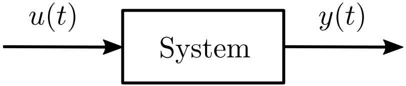

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In this talk:

- System is **linear**, **stable** and single-input-single-output (SISO)
- System is **infinite-dimensional**, usually arising from a PDE.

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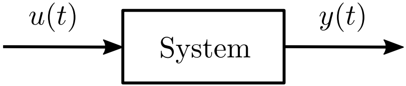

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- System is **linear**, **stable** and single-input-single-output (SISO)
- System is **infinite-dimensional**, usually arising from a PDE.
- Values $u(t) \in \mathbb{R}$ of the control input are **restricted** so that

$$u_{min} \leq u(t) \leq u_{max}, \qquad \text{for all } t \geq 0.$$

Applications

Output tracking for infinite-dimensional systems and PDEs:

- Temperature tracking control, e.g., in manufacturing processes
- Tracking control of flexible robotic manipulators, large-scale space structures, etc.

Reasons for the limitations $u_{min} \leq u(t) \leq u_{max}$

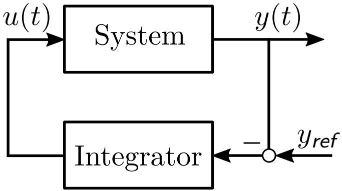
- Naturally arising from operational ranges of actuators:
  - Maximal torques of motors
  - Maximal voltages and currents of power supply units
  - Posivitity constraints: A heater cannot cool things
- A tool for energy efficiency: Avoid unnecessary use of power
- Can guarantee “anti-windup” by design
- Safety considerations: Avoid deadly voltages

Saturating Integral Control

For a stable SISO system, the integral controller

$$\dot{u}(t) = \kappa(y_{ref} - y(t)) \qquad \left(\text{i.e.} \qquad u(t) = \kappa \int_0^t (y_{ref} - y(s))ds\right)$$

can be used in achieving output tracking of the constant $y_{ref} \in \mathbb{R}$.



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Solution strategy: Define a differential equation where at each time $t \geq 0$

- $u(t)$ is allowed to change freely if $u_{min} < u(t) < u_{max}$
- $u(t)$ is not allowed to increase if $u(t) = u_{max}$
- $u(t)$ is not allowed to decrease if $u(t) = u_{min}$

The Saturating Integrator

$$\dot{u}(t) = \mathscr{S}(u(t), \kappa(y_{ref} - y(t)))$$

where

$$\mathscr{S}(u,y) = \begin{cases} \max\{y,0\} & \text{if } u \leq u_{min}, \\ y & \text{if } u \in (u_{min}, u_{max}), \\ \min\{y,0\} & \text{if } u \geq u_{max}. \end{cases}$$

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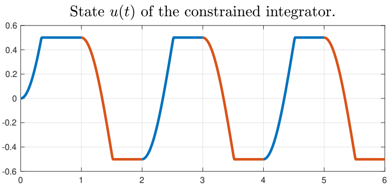

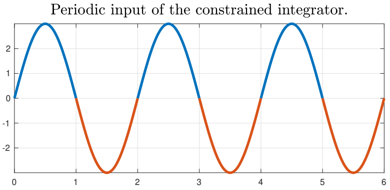

Background

The saturating integrator **can** be used for output tracking:

- Lorenzetti–Weiss 2023: Stable **nonlinear** SISO systems
- Matlab: "limited integrator"
- Also: Lorenzetti–Weiss 2022: Stable nonlinear MIMO systems

Other closely related designs:

- PI for nonlinear systems: Desoer–Lin '85, Konstantopoulos *et. al.* '16, Guiver *et. al.* '17, Simpson-Porco '21,
- PI for ∞-dim with saturation: Logemann–Ryan–Townley '98–'04.

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Novelty in our work:

- Proof that the saturating integrator works for linear but **infinite-dimensional** SISO systems.

Motivation: A Linear Finite-Dimensional System

Using the saturating integral controller for (A, B, C, D) leads to the closed-loop system

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t), & x(0) &= x_0 \in \mathbb{R}^n \\
y(t) &= Cx(t) + Du(t) & & \\
\dot{u}(t) &= \mathscr{S}(u(t), \kappa(y_{\mathit{ref}} - y(t))), & u(0) &= u_0 \in \mathbb{R}.
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Theorem (Lorenzetti–Weiss '23)

Assume that the system is stable and $P(0) = C(-A)^{-1}B + D > 0$. *There exists* $\kappa^* > 0$ *such that if* $\kappa \in (0, \kappa^*)$ *and if* $y_{\mathit{ref}} \in \mathbb{R}$ *satisfies*

$$u_{\mathit{min}} < \frac{y_{\mathit{ref}}}{P(0)} < u_{\mathit{max}},$$

then for some $\alpha > 0$ *we have*

$$e^{\alpha t}|y_{\mathit{ref}} - y(t)| \stackrel{t\to\infty}{\longrightarrow} 0 \qquad \forall x_0 \in \mathbb{R}^n,\ u_0 \in \mathbb{R}.$$

Main Result: A Well-Posed Linear System

Consider a "*Well-Posed Linear System*" (A, B, C, P) on an infinite-dimensional state space X,

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t), & x(0) &= x_0 \in X \\
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Theorem (Main Result)

Assume the system is exponentially stable and $P(0) > 0$. There exists $\kappa^* > 0$ such that if $\kappa \in (0, \kappa^*)$ and if $y_{\mathit{ref}} \in \mathbb{R}$ satisfies

$$u_{\mathit{min}} < \frac{y_{\mathit{ref}}}{P(0)} < u_{\mathit{max}},$$

then for some $\alpha > 0$ we have

$$\int_0^\infty e^{\alpha t} |y_{\mathit{ref}} - y(t)|^2 dt < \infty, \qquad \forall x_0 \in X,\ u_0 \in \mathbb{R}.$$

The Main Challenges

- The function $\mathscr{S}$ in the saturating integrator is discontinuous, and difficult to define for L^2-inputs.

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- For a well-posed linear system (A, B, C, P), the existence of solutions of the (nonlinear) closed-loop system need to be analysed carefully.

$\rightsquigarrow$ We use nonlinear feedback techniques for WPS.

- Global exponential closed-loop stability and convergence of the tracking error using Lyapunov methods.

$\rightsquigarrow$ The Lyapunov methods (inspired by singular perturbations) can be used on ∞-dim spaces, with certain technicalities.

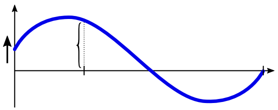
A Boundary Controlled Wave Equation

Consider a wave equation on $\xi \in [0,1]$,

$$\frac{\partial^2 w}{\partial t^2}(\xi,t) = \frac{\partial^2 w}{\partial \xi^2}(\xi,t) - \alpha(\xi)\frac{\partial w}{\partial t}(\xi,t),$$

$$\frac{\partial w}{\partial \xi}(0,t) = u(t), \qquad w(1,t) = 0,$$

$$y(t) = \frac{\partial w}{\partial \xi}(\xi_0,t),$$



where $\alpha \in C[0,1]$, $\alpha(\xi) \geq 0$, and $\alpha \neq 0$, and where ξ_0 is fixed.

- Can be represented as a *well-posed linear system* with state $x(t) = (w(\cdot,t), \partial_t w(\cdot,t))$ on $X = H_r^1(0,1) \times L^2(0,1)$.
- SISO and exponentially stable due to the damping with $\alpha \neq 0$, and $P(0) > 0$.

A Boundary Controlled Wave Equation

Result

There exists $\kappa^* > 0$ *such that if* $\kappa \in (0, \kappa^*)$ *and if* $y_{ref} \in \mathbb{R}$ *satisfies*

$$u_{min} < \frac{y_{ref}}{P(0)} < u_{max},$$

then

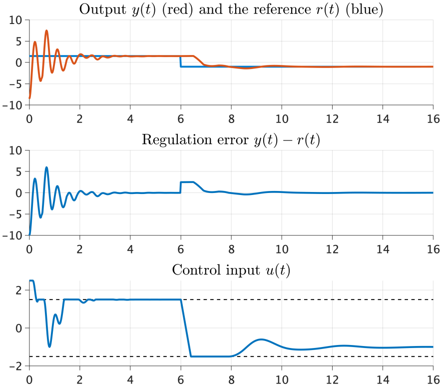
$$\int_0^\infty e^{\alpha t} \left| \frac{\partial w}{\partial \xi}(\xi_0, t) - y_{ref} \right|^2 dt < \infty,$$

for some $\alpha > 0$ *and for all* $x_0 \in X$ *and* $u_0 \in \mathbb{R}$.

In addition, for initial conditions $x_0 \in X$ and $u_0 \in \mathbb{R}$ satisfying the boundary conditions at $t = 0$, we have

$$e^{\alpha t} \left| \frac{\partial w}{\partial \xi}(\xi_0, t) - y_{ref} \right| \to 0, \qquad \text{as } t \to \infty.$$

Numerical Illustration



Additional property:

If u_0 **outside** $[u_{min}, u_{max}]$, then the integrator will bring $u(t)$ to the interval and keep it there.

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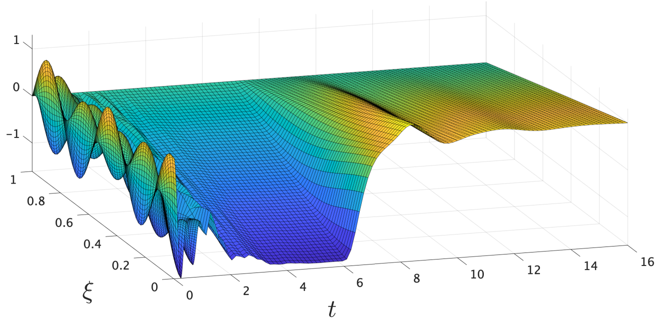

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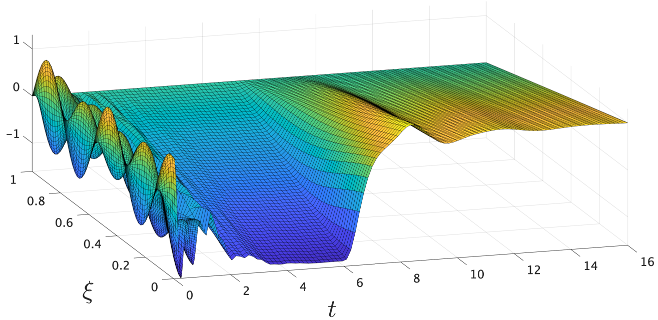

The Bottom Line:

The saturating integrator works for most stable PDE systems!

Thank you!