Robust Output Regulation for PDEs: The Abstract Approach

Lassi Paunonen and Jukka-Pekka Humaloja

Tampere University, Finland

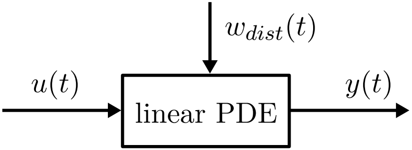
CDC 2022

Funded by Academy of Finland grant
349002 (2022–2026)

Introduction

Problem

*Study **robust output regulation** of linear PDE models.*



Output Regulation = Tracking + Disturbance Rejection:
Design a controller such that the output $y(t)$ of the system converges to a reference signal despite the disturbance $w_{dist}(t)$, i.e.,

$$\|y(t) - y_{ref}(t)\| \to 0, \qquad \text{as} \quad t \to \infty$$

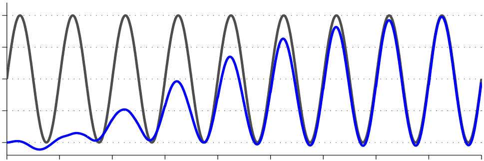
Robustness: The controller is required to tolerate uncertainty in the parameters of the system.

The Reference and Disturbance Signals

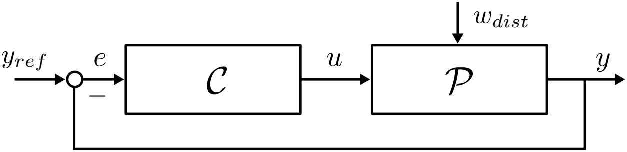
The reference and disturbance signals are of the form

$$y_{ref}(t) = \sum_{k=0}^{q} a_k \cos(\omega_k t + \theta_k)$$
$$w_{dist}(t) = \sum_{k=0}^{q} b_k \cos(\omega_k t + \varphi_k)$$

with **known frequencies** $0 = \omega_0 < \omega_1 < \cdots < \omega_q$ and unknown amplitudes and phases.



The Dynamic Error Feedback Controller



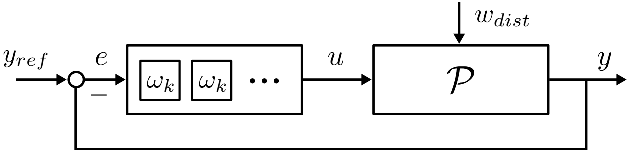
We consider a dynamic error feedback controller which is a (ideally finite-dimensional) linear system.

Result

The Robust Output Regulation Problem is solvable if the system

- *is stabilizable and detectable*
- *does not have transmission zeros at the frequencies* $\pm i\omega_k$ *of* $y_{\text{ref}}(t)$ *and* $w_{\text{dist}}(t)$.

The Internal Model Principle



Theorem (Francis–Wonham, Davison 1970's, ...)

The following are equivalent:

- *The controller solves the robust output regulation problem.*
- *Closed-loop system is stable and the controller has* ***an internal model*** *of the frequencies* $\{\omega_k\}_k$ *of* $w_{\text{dist}}(t)$ *and* $y_{\text{ref}}(t)$.

*"**Internal Model**": For every k, the complex frequencies $\pm i\omega_k$ must be eigenmodes of the controller dynamics with at least $p = \dim Y$ independent eigenvectors.*

Internal Model Based Controller Design

The robust output regulation problem can be solved in two parts:

- Step 1° Include a suitable internal model into the controller
- Step 2° Use the rest of the controller's parameters to stabilize the closed-loop system.

Internal model has fixed structure (easy), the closed-loop stability can be achieved in several ways (often the main challenge).

Internal Model Based Control for PDEs

Step 1° Include a suitable internal model into the controller

Step 2° Use the rest of the controller's parameters to stabilize the closed-loop system.

For controlling a PDE model, there are (at least) **two approaches:**

- "PDE-based": Construct a controller directly based on the individual PDE model.
  - Controller often has a "PDE part" which acts as an observer.
  - [W. Guo, BZ Guo, TT Meng et. al.]
- "Abstract": Represent the PDE as an infinite-dimensional linear system and use existing results for controller design.
  - Controller is an abstract system with PDE-type dynamics

PDE-Based vs. Abstract Routes

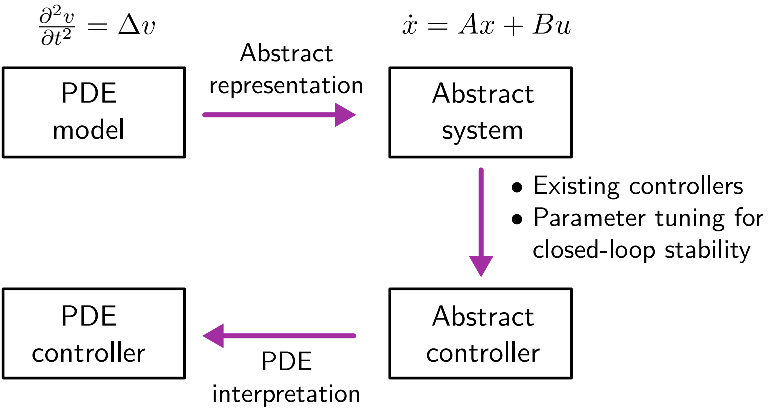
$$\frac{\partial^2 v}{\partial t^2} = \Delta v$$

PDE model

- IM-based structure
- Observer design
- Prove regulation+robustness

PDE controller

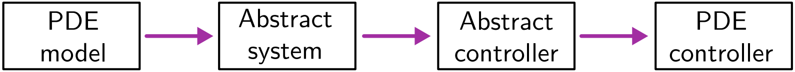
PDE-Based vs. Abstract Routes



Why Take the “Abstract Detour”?

Pros:

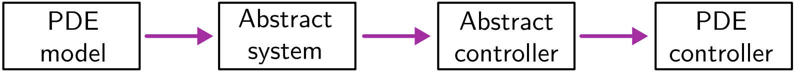
+ No need to prove regulation and robustness
+ Several existing controller design methods, used as black-box
+ Avoids repetition in the parts of the design which are common to all IM-based controllers (e.g., IM-structure, observer form), and “zooms in” on the parts that matter.
+ The Robust Output Regulation Problem has a lot of **structure** and the abstract approach reduces controller design to particular PDE stabilization problems.



Why Take the "Abstract Detour"?

Cons:

- Abstract representation can require deep knowledge of abstract systems, and can still be challenging
  - + Representation know for many important PDEs!
  - + Exact knowledge typically not required!
- PDE interpretation of the controller can require effort
  - + Making this easier is an important topic for future research!
  - + CDC 2022 Main Result



When to Take the Abstract Route?

When do the Pros outweigh the Cons?

- PDE has distributed inputs and outputs $\rightsquigarrow$ abstract representation and PDE interpretation are "easy".
- PDE is on 1D domain $\rightsquigarrow$ abstract representation often already known, and PDE interpretation is straightforward
- PDE is parabolic (on nD domain) $\rightsquigarrow$ can often be represented as "regular linear systems"
- PDE is exponentially stable $\rightsquigarrow$ A very simple ODE controller.

When to Take the Abstract Route?

When do the Pros outweigh the Cons?

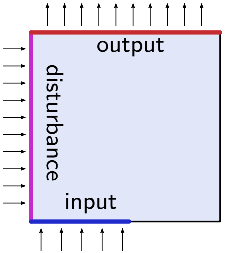
- PDE has distributed inputs and outputs $\rightsquigarrow$ abstract representation and PDE interpretation are "easy".
- PDE is on 1D domain $\rightsquigarrow$ abstract representation often already known, and PDE interpretation is straightforward
- PDE is parabolic (on nD domain) $\rightsquigarrow$ can often be represented as "regular linear systems"
- PDE is exponentially stable $\rightsquigarrow$ A very simple ODE controller.

When do the Cons outweigh the Pros?

- The PDE is on a multi-dimensional domain, has boundary inputs and outputs, and is not parabolic
  - $\rightsquigarrow$ Abstract representation can be challenging (if not known)
  - $\rightsquigarrow$ If the system is not externally well-posed, existing abstract designs may not be applicable.

An Example: 2D Boundary Controlled Heat Equation

$$x_t(\xi,t) = \Delta x(\xi,t), \quad x(\xi,0) = x_0(\xi)$$
$$\frac{\partial x}{\partial n}(\xi,t) = u(t), \qquad \xi \in \Gamma_1$$
$$\frac{\partial x}{\partial n}(\xi,t) = w_{dist}(t), \qquad \xi \in \Gamma_2$$
$$\frac{\partial x}{\partial n}(\xi,t) = 0, \qquad \xi \in \Gamma_0$$
$$y(t) = \int_{\Gamma_3} x(\xi,t)d\xi.$$



Theorem (Byrnes–Gilliam–Weiss 2002)

*The PDE can be represented abstractly as a **regular linear system***

$\rightsquigarrow$ can employ existing abstract control designs for RLS

Step 1: Abstract Representation of the PDE

Abstract system on $X = L^2(\Omega)$ has the form

$$\dot{x}(t) = Ax(t) + Bu(t) + B_d w_{dist}(t), \qquad x(0) = x_0 \in X$$
$$y(t) = C_\Lambda x(t).$$

Theorem (P. 2016, 2017)

The Robust Output Regulation Problem is solvable if the system

- *is stabilizable using state feedback and output injection*
- *does not have transmission zeros at the frequencies* $\pm i\omega_k$ *of* $y_{ref}(t)$ *and* $w_{dist}(t)$.

Good news: Our heat equation has both properties!

Step 2: Abstract Controller Design

Abstract system on $X = L^2(\Omega)$ has the form

$$\dot{x}(t) = Ax(t) + Bu(t) + B_d w_{dist}(t), \qquad x(0) = x_0 \in X$$
$$y(t) = C_\Lambda x(t).$$

Theorem (P. 2016)

The Robust Output Regulation Problem can be solved with controller

$$\dot{z}_1(t) = G_1 z_1(t) + G_2(y(t) - y_{ref}(t))$$
$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(\hat{y}(t) - y(t) + y_{ref}(t))$$
$$\hat{y}(t) = C_\Lambda \hat{x}(t)$$
$$u(t) = K_1 z_1(t) + K_2 \hat{x}(t)$$

with matrices G_1, G_2 and bounded operators L, K_1, and K_2.

Step 2: Abstract Controller Design

$$y_{ref}(t) = \sum_{k=0}^{q} a_k \cos(\omega_k t + \theta_k), \qquad w_{dist}(t) = \sum_{k=0}^{q} b_k \cos(\omega_k t + \varphi_k)$$

Matrices G_1, G_2: Explicit expressions based on $\{\omega_k\}_k$ and $\dim Y$.

Bounded operators L, K_1 and K_2: Chosen so that the semigroups generated by

$$A + LC \qquad \text{and} \qquad \begin{bmatrix} G_1 & G_2 C_\Lambda \\ 0 & A \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} \begin{bmatrix} K_1, K_2 \end{bmatrix}$$

are exponentially stable. Alternatives:

- Rewrite as a stabilization problem for a PDE-ODE cascade.
- Numerical approximations and LQR/LQG (Banks–Ito 1997)
- "Forwarding" (P. 2016) + numerics (P.–Humaloja 2022)

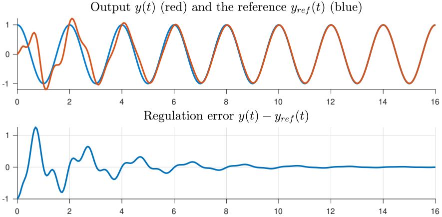
Step 3: From Abstract to PDE Controller

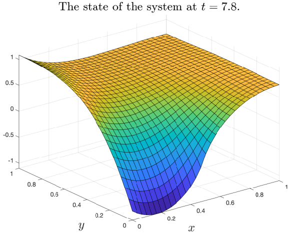
Theorem (P.–Humaloja CDC 2022)

The state of the controller is the weak solution of the ODE-PDE system

$$\begin{aligned}
\dot{z}_1(t) &= G_1 z_1(t) + G_2(y(t) - y_{ref}(t)) \\
\hat{x}_t(\xi,t) &= \Delta \hat{x}(\xi,t) + L(\xi)\Big(\int_{\Gamma_3} \hat{x}(\xi,t)d\xi - y(t) + y_{ref}(t)\Big) \\
\frac{\partial \hat{x}}{\partial n}(\xi,t) &= K_1 z_1(t) + K_2 \hat{x}(\cdot,t) \qquad \text{on } \Gamma_1, \\
\frac{\partial \hat{x}}{\partial n}(\xi,t) &= 0 \qquad \textit{elsewhere on } \partial\Omega \\
u(t) &= K_1 z_1(t) + K_2 \hat{x}(\cdot,t).
\end{aligned}$$

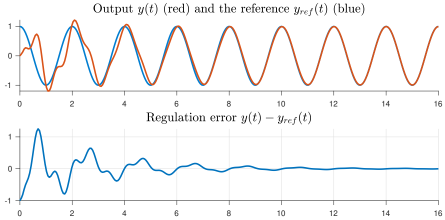
Example: Numerical Simulation

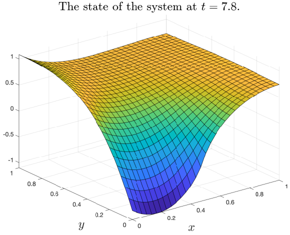




Simulation code available at `https://github.com/lassipau/`

Example: Numerical Simulation





RORPack – Matlab/Python libraries for Robust Output Regulation

- Routines for internal model based controller design
- Simulation and visualisation of results
- Several different types of PDE test cases implemented

Available at `https://github.com/lassipau/rorpack-matlab/`