# Partial Differential Equations in Mathematical Modeling

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# 1 Introduction

The term **Partial Differential Equation** (PDE) is a general name used to describe a differential equation whose solution depends on more than one variable and includes partial derivatives with respect to more than one of these variables. In particular, the following equations are examples of PDEs

$$\frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y) = 0, \qquad x,y \in [0,1] \tag{1}$$

$$\frac{\partial f}{\partial t}(x,t) = a(x)\frac{\partial^2 f}{\partial x^2}(x,t) + b(x)\frac{\partial f}{\partial x}(x,t), \qquad x \in [0,1],\ t \geq 0 \tag{2}$$

$$\frac{\partial f}{\partial t}(x,t) = f(x,t)\frac{\partial f}{\partial x}(x,t) - \frac{\partial^3 f}{\partial x^3}(x,t), \qquad x \in [0,1],\ t \geq 0 \tag{3}$$

The most typical variables in the partial differential equations used in mathematical modeling are spatial coordinates (usually denoted with '$x, y, z$') and time (denoted with "$t$").

In these lectures we will concentrate our attention to models that include time. Such models are used in describing **dynamical** behaviour, or behaviour evolving with time. Particular examples of phenomena that are modeled with partial differential equations include the following.

- Models of vibrations in materials and waves in liquids.
- Models of flows of liquids and gases.
- Models of electromagnetic fields and radiation via Maxwell's equations.
- Models of distribution of heat and diffusion of materials.

On the other hand, PDE models without the time variable are called **static**, and they are widely used especially in the study of mechanical structures.

We will also only study **linear** PDEs, which means that the equation does not contain products or powers of the unknown function $f$ or its derivatives. In the above examples the equations (1) and (2) are linear, and equation (3) is **nonlinear** (due to the first term on the right-hand side).

# 2 Terminology and Basic Properties of PDEs

The existence and uniqueness of solutions of partial differential equations are questions that are very relevant from a mathematical point of view, but especially in the case of nonlinear models finding precise answers to these questions may be difficult or even impossible. The solvability and properties of linear partial differential equations are understood quite well, and in many (but not all!) situations nonlinear equations may be approximated with sufficient accuracy with **linearized** models. In these lectures we will not consider the solvability properties of PDEs in detail, and we will also not consider the detailed properties of the solution function $f$. However, as a general rule you should note that if an equation depending on $x$ and $t$ contains a partial derivative with respect to $x$, then one should be able to compute this partial derivative for the solution $f(x,t)$ for any $x$ and $t$, and so on. The existence of the partial derivatives (for some $(x,t)$) also implies that the solution $f(x,t)$ is continuous (at $(x,t)$).

The **order** of the partial differential equation with respect to a variable refers to the order of the highest partial derivative with respect to that variable. In particular, in the examples in Section 1 we have that

- Equation (1) is of order $2$ with respect to both $x$ and $y$.
- Equation (2) is of order $1$ with respect to $t$ and of order $2$ with respect to $x$.
- Equation (3) is of order $1$ with respect to $t$ and of order $3$ with respect to $x$.

When the partial differential equation contains variables related to the spatial coordinates $x$, $y$, and $z$, the **(spatial) domain** of the PDE refers to the range of the variables $(x,y,z)$ where the PDE is defined. In particular, if we want to consider the distribution of heat in a room with dimensions $l > 0$ (width), $d > 0$ (depth) and $h > 0$ (height), we can define the partial differential equation for the variables $x, y, z$ so that

$$x \in [0,\ell], \qquad y \in [0,d], \qquad z \in [0,h],$$

and the domain of the heat equation model will be the three-dimensional region $[0,\ell] \times [0,d] \times [0,h] \subset \mathbb{R}^3$.

In equations depending on the time variable $t$ we consider the solution of the PDE for $t \geq t_0$, i.e., starting from some **initial time** $t_0 \in \mathbb{R}$. In this situation we add to the PDE a condition that specifies what the solution of the equation is at the initial time $t = t_0$, and this is called an **initial condition** for the PDE. For example, if $f(x,y,z,t)$ describes the temperature at the coordinates $(x,y,z)$ at the time $t$, and we want to study the behaviour of the temperature $f$ for times $t \geq t_0$, we specify an initial condition

$$f(x,y,z,t_0) = g(x,y,z), \qquad (x,y,z) \in [0,\ell] \times [0,d] \times [0,h]$$

where $g$ is a function that describes what are the temperatures in the coordinates $(x,y,z)$ inside the room at time $t = t_0$.

The definition of a PDE also requires so-called **boundary condition**, which determine the properties of the solution on the edges of the domain. There are different types of boundary conditions, and the two most typical ones are the following:

- **Dirichlet boundary conditions:** The solution of the PDE is required to have specific values on the edges of the domain. For example for the equation (1) we could require that

$$\begin{aligned} f(x,0) &= h_1(x), \quad f(x,1) = h_2(x), \qquad \forall x \in [0,1] \\ f(0,y) &= h_3(y), \quad f(1,y) = h_4(y), \qquad \forall y \in [0,1] \end{aligned}$$

  where $h_1$, $h_2$, $h_3$, and $h_4$ are functions defined on $[0,1]$.

- **Neumann boundary conditions:** The normal derivative

$$\frac{\partial f}{\partial \boldsymbol{n}}(x,y,z,t) = \nabla f(x,y,z,t) \cdot \boldsymbol{n}(x,y,z)$$

  of the solution $f(x,y,z,t)$ is required to have specific values on the boundary of the domain. Here $\nabla f(x,y,z,t) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})^T$ is the gradient of $f$ and $\boldsymbol{n}(x,y,z)$ is the **unit normal vector** of the boundary of the domain at the point $(x,y,z)$.

Both Dirichlet and Neumann boundary conditions are discussed in more detail in the following sections.

**Remark.** It is important to note that in order for the PDE to have a well-defined solution, the **initial condition must be in agreement with the boundary conditions**. That is, the function describing the solution of the equation at the initial time $t = t_0$ needs to satisfy the boundary conditions. There are also other (weaker) notions of solutions of PDEs where this is not required, but such **weak** and **mild solutions** outside the scope of this course.

## 2.1 Numerical Solutions and Visualization

Finding an explicit formula for the solution of a partial differential equation can be very difficult or even impossible, but the solutions can be approximated with numerical methods. Numerical approximation methods for PDEs include the following (in the case of two spatial variables $x$ and $y$ and time $t$).

- **Finite Differences**: The domain of the PDE is divided into a grid consisting of small rectangles, and the solution $f(x,y,t)$ is approximated at the corners of these rectangles (the “nodes” of the grid). This method is very simple to implement for rectangular domains.

- **Finite Element Method**: A bit more advanced method than Finite Differences. The domain of the PDE is divided into triangles instead of rectangles. The Finite Element Method works very well for domains of various shapes.

The above methods are only mentioned here as a reference, and they are not studied in detail in these lecture notes.

In approximating the solutions of partial differential equations with dependence on time $t$ it is common to only approximate the partial derivatives associated to the spatial variables $(x, y, z)$, and leave the partial derivatives with respect to $t$ in the equation. This way, the PDE can be reduced to a system of ordinary differential equations with the variable $t$. Such systems can be solved efficiently in Matlab using functions such as `ode45` and `ode15s`.

PDEs with only two variables, especially those depending on $t \geq t_0$ and one spatial variable $x \in [0, \ell]$, can also be solved with the Matlab function `pdepe`. The use of the `pdepe` function is illustrated in the Matlab codes associated to the lectures.

For solving and illustrating PDEs including two or three spatial variables (and the possible time $t$) Matlab includes the "PDE Toolbox" (initiated with the command `pdetool`). PDE Toolbox uses the Finite Element Method in approximating the solutions of the equations.

# 3 The One-Dimensional Heat Equation

In this section we study a partial differential equation called the **heat equation**. This equation is used to describe various processes involving diffusion of heat and substances. We concentrate on the situation where the equation has one spatial variable $x$ and all the physical parameters are constant (see Section 3.1 for a full list of the parameters). In this case the heat equation is of the form

$$\frac{\partial T}{\partial t}(x,t) = \alpha \frac{\partial^2 T}{\partial x^2}(x,t) + \frac{1}{c\rho} q(x,t), \qquad x \in [0,\ell], \quad t \geq 0 \tag{4a}$$

$$T(x,0) = T_{init}(x), \qquad x \in [0,\ell] \tag{4b}$$

$$T(0,t) = T_0, \quad T(\ell,t) = T_\ell. \tag{4c}$$

The physical parameters have the following meanings. We assume all parameters to be constant throughout the length of the rod.

- $K > 0$ is the **thermal conductivity** of the material.
- $c > 0$ is the **specific heat capacity** of the material, i.e., the amount of energy required to increase the temperature of a unit mass of material with one degree.
- $\rho > 0$ is the **mass density** of the material.
- $\alpha = \frac{K}{c\rho}$ is the **thermal diffusivity** of the material.

The term $\frac{1}{c\rho} q(x,t)$ is a **source term** which describes the heat generated or lost internally in the material, and the heat added or removed from the material due to external actions. The initial condition of the PDE is given in (4b) and the boundary condition (4c) indicate that for all times the solution has constant values $T_0$ and

$T_\ell$ at the endpoints of the domain $[0, \ell]$. It should be noted that also other boundary conditions would be possible depending on the physical situation, and these are discussed later in Section 3.3.

The physical parameters of actual materials can often be measured with suitable test setups. For example, the parameter $\alpha$ can be determined using Flash Laser Analysis (see https://en.wikipedia.org/wiki/Laser_flash_analysis). Estimating the parameters based on measurements is also a topic of one of the exercise problems in Section 5.

The equation (4) can be used to describe the evolution of the temperature profile in a thin metal rod of uniform material, as in Figure 1.

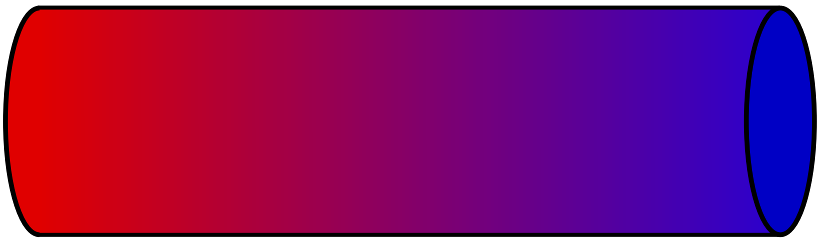


Figure 1: The Heat Distribution in a Metal Rod.

To be more precise, each of the assumptions "thin", "metal", and "uniform material" have the following implications:

- The rod should be "thin" in the sense that the spreading out of the heat happens mostly in the $x$ direction, and not as much in other directions.
- Rod should be metal or some other material that has good and even conductance of heat throughout its length.
- The material should be "uniform" in the sense that the spreading of the heat happens in a similar way in all parts of the rod. If this is not true, then some of the physical parameters $K$, $c$, and $\rho$ can depend on the variable $x$ instead, and the heat equation will have a slighly different form (see e.g., "Heat equation" in Wikipedia for more information).
- The rod should also be "insulated" from its surroundings in such a way that the amount of heat escaping from the rod through its sides is very small compared to the amount of heat flowing inside the rod itself.

If these conditions are (at least roughly) met, then the the heat equation (4) describes the evolution of the temperature in the rod so that

- $T(x, t)$ is the temperature of the rod at the position $x \in [0, \ell]$ for all times $t > 0$.

- The two endpoints of the rods are held in constant temperatures $T(0,t) = T_0$ and $T(\ell,t) = T_\ell$.
- Initially at time $t = 0$ the temperature distribution in the rod is given by the function $T_{init}(x)$.

## 3.1 Derivation of the Heat Equation

In this section we derive the PDE model (4) from the physical principles and conservation laws. For simplicity, we will only do this in the situation where the equation does not have a source term, i.e., $q(x,t) \equiv 0$. We consider a situation depicted in Figure 2.

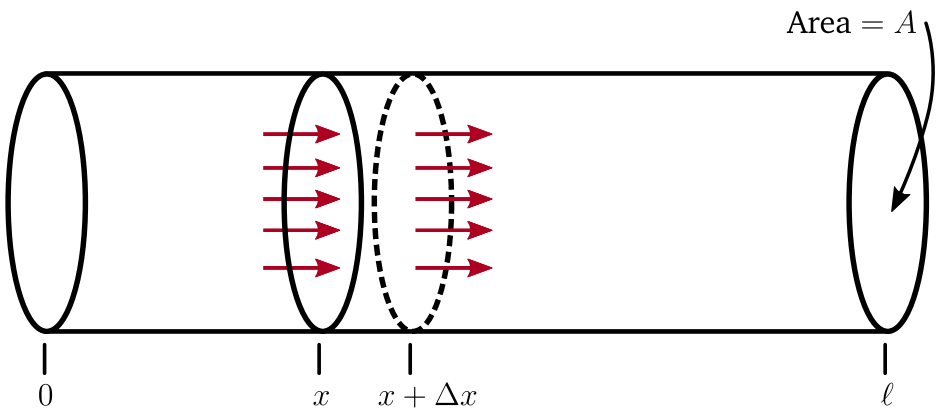


Figure 2: The slice $[x, x+\delta]$

Our aim is to show that at each time $t > 0$ and at each position $x \in [0,\ell]$ the partial differential equation (4a) describes the behaviour of the temperature in the rod in Figure 2. We denote by $T(x,t)$ the temperature of the rod at position $x \in [0,\ell]$ and at time $t > 0$. We use the following two properties:

(1) Fourier's law: Heat flows from areas of higher temperature to those of lower temperature in such a way that the "rate of heat flowing through a surface per unit area is propotional to the negative temperature gradient", i.e.,

$$\text{rate of heat flow} = -AK\frac{\partial T}{\partial x}(x,t)$$

where $A$ is the area of the surface and $K$ is the thermal conductivity.

(2) Conservation of energy: Without internal heat sources or sinks the change of energy in a given region of the rod is determined by the heat flowing through its boundaries.

Consider a piece of the rod beginning at $x$ and ending at $x + \Delta x$, where $\Delta x > 0$ is assumed to be so small that the temperature in the rod between $x$ and $\Delta x$ is $T(x,t)$

at time $t > 0$ (see Figure 2). At time $t > 0$ the amount of heat energy in this piece of the rod is given by

$$E_{heat}(t) = c \times \text{mass} \times \text{temperature at time } t,$$

where $c > 0$ is the specific heat capacity. Since the mass of the piece of rod is $\rho A \Delta x$ where $A$ is the area of its intersection (assumed to be constant), and its temperature is $T(x,t)$ at time $t > 0$, we can write

$$E_{heat}(t) = c \cdot \rho A \Delta x \cdot T(x,t). \tag{5}$$

Let $\Delta t > 0$ be small. Since there are no internal heat sources or sinks, the conservation of energy tells us that the change of energy in the piece of the rod on the time interval $[t, t+\Delta t]$ depends only on the amount of heat flowing in and out of the piece through its edges at $x$ and $x + \Delta x$, i.e.,

$$\text{Change of } E_{heat}(t) = (\text{heat flowing in at } x) \; - \; (\text{heat flowing out at } x + \Delta x)$$

Since $\Delta t$ is small, the amount of heat flowing in and out of the piece of rod is approximately the rate of the heat flow times $\Delta t$. The Fourier's law then tells us that

$$\begin{aligned}
\text{Change of } E_{heat}(t) &= (\text{heat flowing in at } x) \; - \; (\text{heat flowing out at } x + \Delta x) \\
&= \Delta t \cdot \left( -KA \frac{\partial T}{\partial x}(x,t) \right) - \Delta t \cdot \left( -KA \frac{\partial T}{\partial x}(x + \Delta x, t) \right) \\
&= KA \cdot \Delta t \left( \frac{\partial T}{\partial x}(x + \Delta x, t) - \frac{\partial T}{\partial x}(x,t) \right).
\end{aligned}$$

On the other hand, this change of $E_{heat}(t)$ on the interval is precisely $E_{heat}(t + \Delta t) - E_{heat}(t)$, and using the formula (5) we get an equation

$$c\rho A \Delta x \cdot T(x, t + \Delta t) - c\rho A \Delta x \cdot T(x,t) = KA \cdot \Delta t \left( \frac{\partial T}{\partial x}(x + \Delta x, t) - \frac{\partial T}{\partial x}(x,t) \right),$$

which is equivalent to

$$\frac{T(x, t + \Delta t) - T(x,t)}{\Delta t} = \frac{K}{c\rho} \cdot \frac{1}{\Delta x} \left( \frac{\partial T}{\partial x}(x + \Delta x, t) - \frac{\partial T}{\partial x}(x,t) \right).$$

Letting $\Delta x \to 0$ and $\Delta t \to 0$ we get a limit

$$\frac{\partial T}{\partial t}(x,t) = \frac{K}{c\rho} \cdot \frac{\partial}{\partial x} \left( \frac{\partial T}{\partial x}(x,t) \right) = \frac{K}{c\rho} \cdot \frac{\partial^2 T}{\partial x^2}(x,t),$$

which is precisely (4a) with $\alpha = \frac{K}{c\rho}$.

## 3.2 The Solution of the Heat Equation

The characteristic property of the heat equation (4) is that the heat in the system diffuses from the areas of high temperature to those of low temperature. Over time, without any additional sources of heat, the heat in the system distributes evenly.

**Explicit Solution of the Homogeneous Equation**

The heat equation (4) is **homogeneous** if the source term is identically zero, i.e., $q(x,t) \equiv 0$. In this case the solution of the heat equation with Dirichlet boundary conditions

$$u(0,t) = 0, \qquad u(\ell,t) = 0, \tag{6}$$

can be determined using the method called **separation of variables**. In this method we "separate" the dependence of the variables $x$ and $t$ in $T(x,t)$ by searching for a solution of the form

$$T(x,t) = \phi(t)T_0(x).$$

If we substitute this function to (4a), we get

$$\phi'(t)T_0(x) = \alpha\phi(t)T_0''(x) \qquad \Leftrightarrow \qquad \frac{1}{\alpha}\frac{\phi'(t)}{\phi(t)} = \frac{T_0''(x)}{T_0(x)}.$$

Since in the latter expression, the left-hand side depends only on $t$ and the right-hand side only on $x$, both of the quantities must be equal to a constant value $\lambda \in \mathbb{R}$. Because of this, the functions $\phi(t)$ and $T_0(t)$ must satisfy the ordinary differential equations

$$\phi'(t) = \alpha\lambda\phi(t), \qquad T_0''(x) = \lambda T_0(x).$$

The first equation has a unique solution

$$\phi(t) = e^{\alpha\lambda t}\phi(0).$$

If $\lambda = 0$, the second equation has a unique solution $T_0(x) = ax + b$. However, in this case the boundary conditions (6) can only be satisfied if $a = b = 0$. This corresponds to the trivial solution $T(x,t) \equiv 0$. If $\lambda \neq 0$, the second equation has a general solution given by

$$T_0(x) = A_\lambda e^{\sqrt{-\lambda}x} + B_\lambda e^{-\sqrt{-\lambda}x}, \qquad x \in [0,\ell].$$

The values of $A_\lambda$ and $B_\lambda$ are determined by the boundary conditions (6). In particular, $T(0,t) = 0$ requires that $A_\lambda + B_\lambda = T_0(0) = 0$, and thus

$$T_0(x) = A_\lambda(e^{\sqrt{-\lambda}x} - e^{-\sqrt{-\lambda}x}), \qquad x \in [0,\ell].$$

If $\lambda < 0$, then $e^{\sqrt{-\lambda}\ell} \neq e^{-\sqrt{-\lambda}\ell}$, and therefore the boundary condition $T(\ell,t) = 0$ can only be satisfied if $A_\lambda = 0$, which again leads to the trivial solution $T(x,t) \equiv 0$. Finally, if $\lambda > 0$, then the boundary condition $T(\ell,t) = 0$ is satisfied if and only if

$$e^{\sqrt{-\lambda}\ell} = e^{-\sqrt{-\lambda}\ell} \qquad \Leftrightarrow \qquad e^{2i\sqrt{\lambda}\ell} = 1 \qquad \Leftrightarrow \qquad 2\sqrt{\lambda}\ell = 2\pi n, \quad n \in \mathbb{N}$$
$$\Leftrightarrow \qquad \lambda = \frac{n^2\pi^2}{\ell^2}, \quad n \in \mathbb{N}.$$

Combining with the formula $\phi(t) = e^{\alpha\lambda t}\phi(0)$ this means that for every $n \in \mathbb{N}$ the function

$$T_n(x,t) = \tilde{C}_n e^{-\alpha\frac{n^2\pi^2}{\ell^2}t}\left(e^{i\frac{n\pi}{\ell}x} - e^{-i\frac{n\pi}{\ell}x}\right) = C_n e^{-\alpha\frac{n^2\pi^2}{\ell^2}t}\sin\left(\frac{n\pi}{\ell}x\right) \tag{7}$$

is a solution of the heat equation for any constant $C_n \in \mathbb{C}$. Note that if $T_1(x,t)$ and $T_2(x,t)$ are two solutions of the heat equation (4a) with boundary conditions (6), then also $T_1(x,t) + T_2(x,t)$ is a solution satisfying the same boundary conditions. Moreover, it turns out that the general solution of the heat equation (4a) with the boundary conditions (6) can be represented as an infinite linear combination of the terms of the form (7),

$$T(x,t) = \sum_{n=1}^{\infty} C_n e^{-\alpha\frac{n^2\pi^2}{\ell^2}t}\sin\left(\frac{n\pi}{\ell}x\right).$$

It remains to determine the coefficients $C_n$ based on the initial condition $T(x,0) = T_{init}(x)$. Setting $t = 0$ in the above formula we get $T(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{\ell}x\right)$. Then the theory of Fourier series tells us that the coefficients $\{C_n\}_n$ can be computed with the formulas

$$C_n = \frac{2}{\ell}\int_0^\ell T_{init}(x)\sin\left(\frac{n\pi}{\ell}x\right)dx, \qquad n \in \mathbb{N}.$$

### Effect of the Physical Parameters and the Source Term

The higher parameter $\alpha > 0$ makes the diffusion process faster. As we saw in the previous section, $\alpha = \frac{K}{c\rho}$, where $K$ is the **thermal conductivity**, $c$ is the **specific heat capacity**, and $\rho$ the **density** of the material. This means that the higher thermal conductivity speeds up the diffusion, whereas higher specific heat capacity and higher density slow it down. Figure 3 shows the solution of the heat equation for three different parameters $\alpha$. The boundary conditions and the initial condition of the equation are chosen as

$$T(0,t) = 1, \qquad T(1,t) = 0, \qquad T(x,0) = 10x(1-x)^2 + 1 - x, \quad f(x,t) \equiv 0.$$

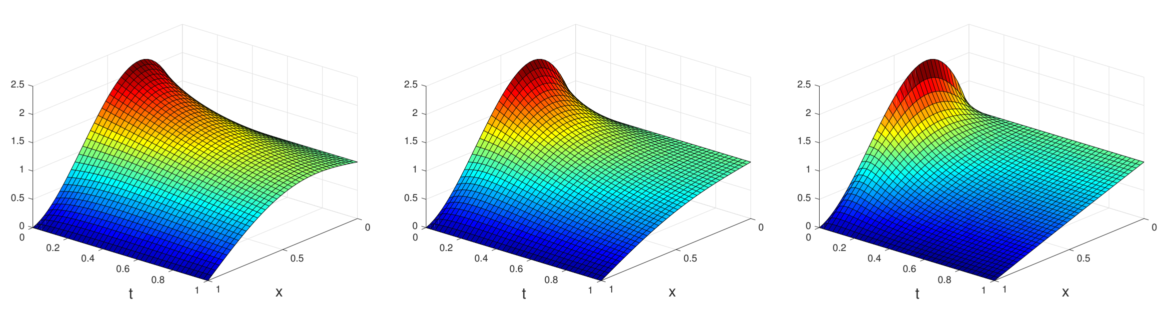


Figure 3: Heat diffusion for 3 different parameters $\alpha \in \{0.1, 0.2, 0.5\}$ (left to right).

The source term $\frac{1}{c\rho}q(x,t)$ can increase or decrease the temperature in the heat equation. Figure 4 shows the solution of the heat equation for three different functions

$$q_1(x,t) = \chi_{[0,1/2]}(x), \qquad \text{where} \quad \chi_{[0,1/2]}(x) = \begin{cases} 1 & x \in [0,1/2] \\ 0 & \text{otherwise} \end{cases}$$

$$q_2(x,t) = \chi_{[1/4,3/4]}(x)\frac{1}{1+t^3}$$

$$q_3(x,t) = 3(1-x)^3 \sin(4t).$$

In these three examples, the dependence on $x$ in the function $q(x,t)$ affects in which parts of the spatial interval $x \in [0,1]$ the heat is added or removed at each time. The dependence on $t$ on the other hand affects how the amount of energy added or removed depends on time. For example in the case of $q_1(x,t)$, the heat is added to the equation only on the subinterval $[0,1/2]$, and the amount of added heat is constant with respect to time. On the other hand, in the case of $q_2(x,t)$ a fair amount of heat is added initially, and the amount of added heat decreases over time. Finally, in the case of $q_3(x,t)$ the amount of added and removed heat changes periodically due to the periodic function $\sin(4t)$.

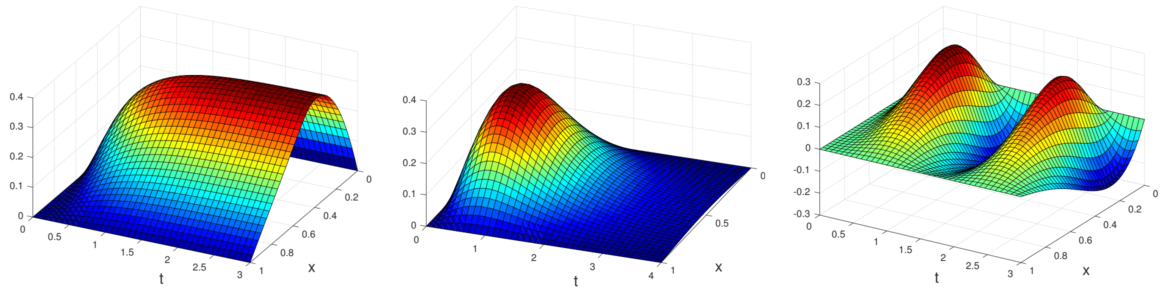


Figure 4: Heat diffusion for the source terms with $q_1$, $q_2$, and $q_3$ (left to right).

## 3.3 The Boundary Conditions of the Heat Equation

The boundary conditions (4c) of the heat equation are of the Dirichlet type, where the solution $T(x,t)$ is required to have constant value on the endpoints $x = 0$ and $x = \ell$ of the domain $[0,1]$. As mentioned before, this corresponds to the situation where the temperature of the two ends of the metal rod are held at constant temperatures $T_0$ and $T_\ell$.

In an alternative situation the ends of the metal rod could be insulated in such a way that heat energy cannot flow in or out at the boundaries $x = 0$ and $x = \ell$. Mathematically this corresponds to the derivatives of $T(x,t)$ with respect to $x$ being zero $x = 0$ and $x = \ell$. If we modify the boundary conditions (4c) we arrive at the heat equation

$$\frac{\partial T}{\partial t}(x,t) = \alpha \frac{\partial^2 T}{\partial x^2}(x,t), \qquad x \in [0,\ell], \quad t \geq 0 \tag{8a}$$

$$T(x,0) = T_{init}(x), \qquad x \in [0,\ell] \tag{8b}$$

$$\frac{\partial T}{\partial x}(0,t) = 0, \quad \frac{\partial T}{\partial x}(\ell,t) = 0. \tag{8c}$$

The boundary conditions (8c) are Neumann boundary conditions.

It is also possible to combine Dirichlet and Neumann boundary conditions for a single equation, as long as each boundary point $x = 0$ and $x = \ell$ only have one boundary condition. Indeed, in a situation where one end of the metal rod is held at a constant temperature and the other end is insulated, the boundary conditions of the equations (8) can be replaced with

$$T(0,t) = T_0, \qquad \text{and} \qquad \frac{\partial T}{\partial x}(\ell,t) = 0$$

or

$$\frac{\partial T}{\partial x}(0,t) = 0, \qquad \text{and} \qquad T(\ell,t) = T_\ell.$$

# 4 The Wave Equation

The one-dimensional wave equation studied in this section can in particular be used to describe the motion of a vibrating string of length $\ell > 0$. If the density of the string is the same throughout its length, and if the string is held fixed at both ends, at $x = 0$ and $x = \ell$, then the partial differential equation describing the motion is given by

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t), \qquad x \in [0,\ell], \quad t \geq 0 \tag{9a}$$

$$u(x,0) = u_{init}(x), \quad \frac{\partial u}{\partial t}(x,0) = v_{init}(x) \qquad x \in [0,\ell] \tag{9b}$$

$$u(0,t) = 0, \qquad u(\ell,t) = 0. \tag{9c}$$

Here $u(x,t)$ is the vertical position of the string at the coordinate $x \in (0,\ell)$ and at the time instant $t > 0$, as in Figure 5.

The quantity $c > 0$ is the **wave speed**, and it depends on the density and the tension of the string. The boundary conditions (9c) describe that the both ends of the string are held at vertical position zero at all times.

## 4.1 Derivation of the Wave Equation

We will now derive the wave equation from the Newton's Second Law $F = ma$. We denote the vertical position of the string with $u(x,t)$ at time $t \geq 0$ and at the

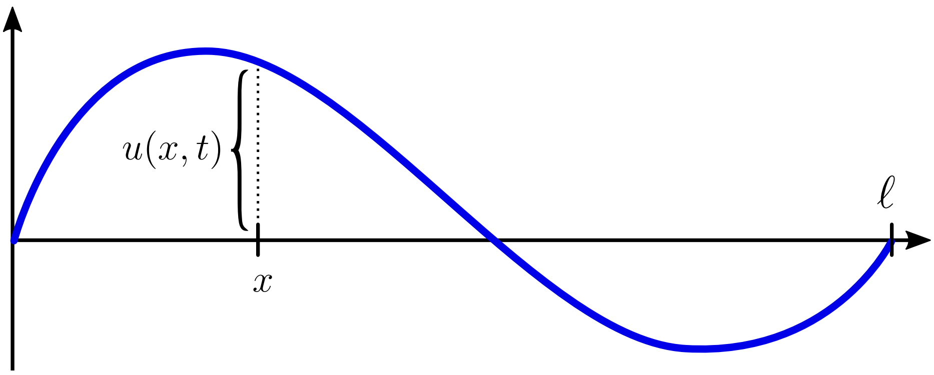


Figure 5: The Motion of a String.

horizontal position $x \in (0, \ell)$. We assume that the situation is simple in the sense that the following hold.

- The string has constant mass density
- The weight of the string is relatively small, so we can ignore the effect of gravity.
- The considered oscillations are small so that the length of the string stays approximately constant, the slope of the string at every point is relatively small, and the at each point the string moves mainly in the vertical direction.
- The string does not resist the bending, and the tension is constant in the horizontal direction.

We use the following notation

- $\rho$ is the constant mass density of the string (units of mass per length of a string).
- $\theta(x,t)$ is the angle of the string at a point $x \in (0, \ell)$ and at a time $t \geq 0$, so that

$$\frac{\partial u}{\partial x}(x,t) \;=\; \text{“slope of the string at } x \in (0,\ell)\text{”} \;=\; \tan\theta(x,t).$$

- $\tau(x,t)$ is the total tension at a point $x \in (0, \ell)$ and at a time $t \geq 0$.
- $\tau_{hor}$ is the constant tension in the horizontal direction.

Consider the situation in Figure 6 where we study to motion of particles of the string on the small segment $[x, x + \Delta x]$.

Newton's Second Law $F = ma$ tells us that the vertical displacement $u(x,t)$ on this segment satisfies

$$F = ma = \rho \Delta x \frac{\partial^2 u}{\partial t^2}, \tag{10}$$

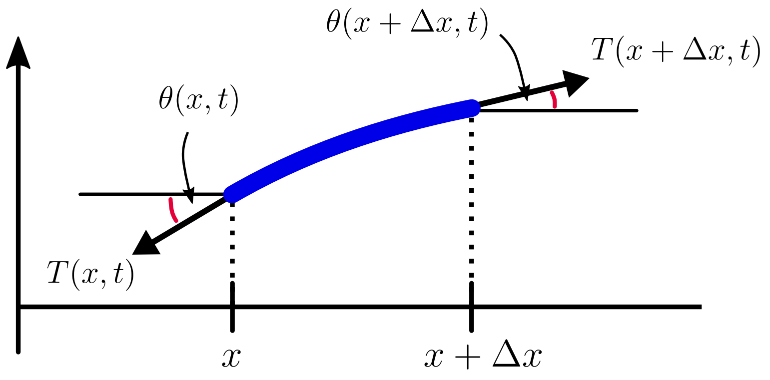


Figure 6: The one-dimensional wave equation

where $F$ is the total force affecting the points on the segment that arises from the differences in the vertical components of the tensions at $x$ and $x + \Delta x$. Taking into account the signs (positive direction is upwards), we have that the vertical components of the tensions add to

$$F = \tau(x + \Delta x, t) \sin\theta(x + \Delta x, t) - \tau(x, t) \sin\theta(x, t).$$

However, we assume that the horizontal tension is constant on the string, i.e., $\tau(x, t) \cos\theta(x, t) = \tau(x + \Delta x, t) \cos\theta(x + \Delta x, t) = \tau_{hor}$, and we can therefore use $\tan\theta = \frac{\sin\theta}{\cos\theta}$ to compute

$$\begin{aligned} F &= \tau(x + \Delta x, t) \sin\theta(x + \Delta x, t) - \tau(x, t) \sin\theta(x, t) \\ &= \tau(x + \Delta x, t) \cos\theta(x + \Delta x, t) \tan\theta(x + \Delta x, t) - \tau(x, t) \cos\theta(x, t) \tan\theta(x, t) \\ &= \tau_{hor} \left(\tan\theta(x + \Delta x, t) - \tan\theta(x, t)\right) \\ &= \tau_{hor} \left( \frac{\partial u}{\partial x}(x + \Delta x, t) - \frac{\partial u}{\partial x}(x, t) \right), \end{aligned}$$

where we have used the fact that $\frac{\partial u}{\partial x}$ is the slope of the string at $x$, which is alternatively given by $\tan\theta(x, t)$. Combining this expression for $F$ with (10) and dividing both sides with $\rho\Delta x$ leads to

$$\frac{\partial^2 u}{\partial t^2} = \frac{\tau_{hor}}{\rho} \cdot \frac{1}{\Delta x} \left( \frac{\partial u}{\partial x}(x + \Delta x, t) - \frac{\partial u}{\partial x}(x, t) \right).$$

If we let $\Delta x \to 0$, the difference on the right-hand side converges to $\frac{\partial^2 u}{\partial x^2}$, and we obtain the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

where $c^2 = \tau_{hor}/\rho$.

## 4.2 The Solution of the Wave Equation

### Explicit Solution of the Wave Equation

If we for simplicity ignore the effect of the boundary conditions, and instead consider the wave equation (9a) on the spatial domain $x \in \mathbb{R}$, then the general solution of the wave equation (9a) can be derived using the **method of characteristics**. If we consider new variables $s = x - ct$ and $r = x + ct$, then the chain rule implies that

$$\frac{\partial}{\partial x} = \frac{\partial s}{\partial x}\frac{\partial}{\partial s} + \frac{\partial r}{\partial x}\frac{\partial}{\partial r} = \frac{\partial}{\partial s} + \frac{\partial}{\partial r}$$

$$\frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial s} + \frac{\partial}{\partial r}\right)\left(\frac{\partial}{\partial s} + \frac{\partial}{\partial r}\right) = \frac{\partial^2}{\partial s^2} + 2\frac{\partial^2}{\partial s \partial r} + \frac{\partial^2}{\partial r^2}$$

$$\frac{\partial}{\partial t} = \frac{\partial s}{\partial t}\frac{\partial}{\partial s} + \frac{\partial r}{\partial t}\frac{\partial}{\partial r} = -c\frac{\partial}{\partial s} + c\frac{\partial}{\partial r}$$

$$\frac{\partial^2}{\partial t^2} = \left(-c\frac{\partial}{\partial s} + c\frac{\partial}{\partial r}\right)\left(-c\frac{\partial}{\partial s} + c\frac{\partial}{\partial r}\right) = c^2\frac{\partial^2}{\partial s^2} - 2c^2\frac{\partial^2}{\partial s \partial r} + c^2\frac{\partial^2}{\partial r^2}.$$

Substituting $\frac{\partial^2}{\partial x^2}$ and $\frac{\partial^2}{\partial t^2}$ to the equation (9a) we get

$$c^2\frac{\partial^2 u}{\partial s^2} - 2c^2\frac{\partial^2 u}{\partial s \partial r} + c^2\frac{\partial^2 u}{\partial r^2} = c^2\left(\frac{\partial^2 u}{\partial s^2} + 2\frac{\partial^2 u}{\partial s \partial r} + \frac{\partial^2 u}{\partial r^2}\right)$$

$$\Leftrightarrow \qquad \frac{\partial^2 u}{\partial s \partial r} = 0.$$

The general solution of the last equation is $u(s, r) = F(s) + G(r)$, where $F$ and $G$ can be arbitrary (suitably differentiable) functions. If we change back to the original variables $x$ and $t$, we can see that the solution of the wave equation (9a) is

$$u(x, t) = F(x - ct) + G(x + ct).$$

The two components $F(x - ct)$ and $G(x + ct)$ correspond to two waves travelling to the right and to the left, respectively. Finally, the functions $F$ and $G$ can be determined based on the initial conditions (9b). This way, the solution of the wave equation with the initial conditions (9b) is given by the **d'Alembert's formula**

$$u(x, t) = \frac{u_{init}(x - ct) + u_{init}(x + ct)}{2} + \int_{x-ct}^{x+ct} v_{init}(z)dz \qquad (11)$$

for all $x \in \mathbb{R}$ and $t \geq 0$.

**Behaviour of Solutions of the Wave Equation**

As we observed in the previous section, the general solutions of the wave equation consist of two parts, one moving to the right and one to the left. In particular, the solutions include "wave pulses" that travel only in a single direction, such as the one pictured in Figure 7. The parameter $c > 0$ is the **wave speed**, which literally determines the speed of propagation of the two components of the solution. Solutions that start out with a single wave pulse travelling to the right or to the left can be constructed by choosing suitable initial conditions for the wave equation. If we aim at a solution $u(x, t) = F(x - ct)$, then the appropriate initial conditions can be determined by simply substituting $t = 0$ into $u$ and its time-derivative. This way we can see that the initial conditions corresponding to the solution $u(x,t) = F(x - ct)$ are given by

$$u_{init}(x) = u(x,0) = F(x), \qquad v_{init}(x) = \frac{\partial u}{\partial t}(x,0) = -cF'(x),$$

where the coefficient $-c$ results from differentiating $F(x - ct)$ with respect to $t$.

When a pulse travelling to the right hits a boundary with a Dirichlet boundary condition $u(\ell, t) = 0$, it is **reflected** off the boundary and begins to move left. The pulse eventually returns to the reflection of its initial profile. This is illustrated in Figure 8.

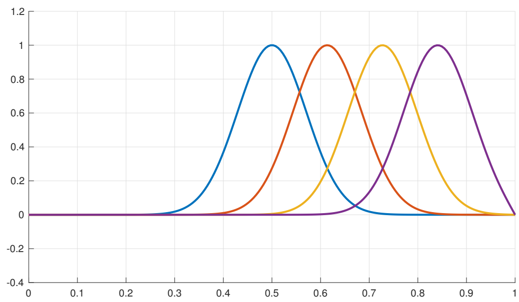


Figure 7: A wave pulse travelling to the right.

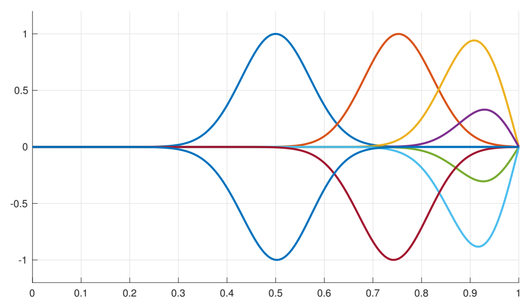


Figure 8: A wave pulse reflecting at the boundary.

Figure 9 shows a single wave pulse splitting into two parts, one moving to the left and one to the right. This kind of solution corresponds to an initial displacement of the string with zero intial velocity. In Figure 9 the initial displacement and velocity are given by

$$u_{init}(x) = e^{-100(x-1/2)^2}, \qquad v_{init}(x) \equiv 0.$$

The solutions of the wave equation also includes **standing waves** such as the one shown in Figure 10.

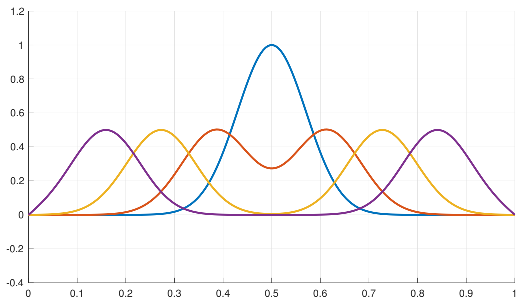


Figure 9: A splitting wave.

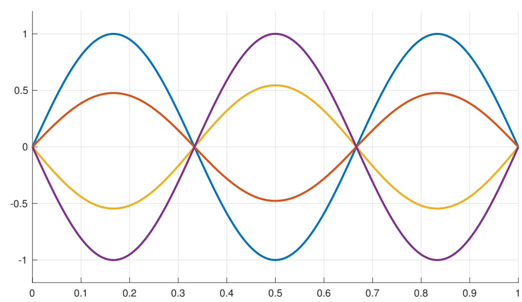


Figure 10: A standing wave.

## 4.3 Boundary Conditions of the Wave Equation

Similarly as in the case of the heat equation, also the wave equation can be studied for different types of boundary conditions. In particular, the Neumann boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = 0, \qquad \text{and} \qquad \frac{\partial u}{\partial x}(\ell,t) = 0,$$

are used to model a situation where the endpoints of the string can move freely in the vertical direction, but the ends of the string are held at a constant angles so that the tangents of the string at $x = 0$ and $x = \ell$ stay horizontal at all times.

# 5 Exercise Problems

You should complete and document three exercise problems: Problems 1 and 2, and either 3 or 4.

1. Modify the derivation of the solution $T(x,t)$ using the separation of variables to find the solution of the heat equation (4a) with the boundary conditions

$$T(0,t) = 0 \qquad \text{and} \qquad \frac{\partial T}{\partial x}(\ell,t) = 0.$$

2. In this problem we will look for **standing waves** of the wave equation (9a).

   (a) Look for a solution of the wave equation in the form $u_k(x,t) = \cos(\omega_k t)\psi_k(x)$ where $\omega_k \in \mathbb{R}$ (we are again using "separation of variables") for some index $k \in \mathbb{Z}$. Substitute $u_k(x,t)$ into the wave equation and derive an ordinary differential equation for $\psi_k(x)$.

   (b) Find $\psi_k(x)$ as a solution of the differential equation in part (a). Note that the equation has a general solution for any "frequency" $\omega_k$, but both of the boundary conditions can be satisfied only for certain specific values of $\omega_k \in \mathbb{R}$. Find these frequencies $\omega_k$.

   (c) Illustrate the standing wave solutions $u_k(x,t)$ for different $\omega_k$ and describe the role of the frequency $\omega_k$.

3. [**Matlab**] Download the Matlab codes for the simulations of the heat equation and the wave equation from the course homepage.

   (a) Experiment with different values of the parameters $K, c, \rho$ (in the heat equation) and test different source functions $q(x,t)$. Briefly describe the effects of these changes to the behaviour of the solutions. Repeat the same study for the wave equation.

(b) In this problem we will try to estimate the specific heat capacity $c > 0$ and the thermal conductivity $K > 0$ of the heat equation based on measurements. Assume the metal rod has a known mass density $\rho = 1.2$ and length $\ell = 1.0$. Use the Matlab code to find the values of $K$ and $c$ (with 2 or 3 decimal point accuracy) based on the following information:

   - With the intial heat profile $T_0(x) = 20x(1-x)^5$ and zero source function $q(x,t) \equiv 0$ the temperature at the point $x_0 = 0.75$ at time $t_0 = 0.40$ is $T(x_0, t_0) = 0.160$.
   - With the intial heat profile $T_0(x) = x(1-x)$ and source function $q(x,t) = 8\chi_{[0,1/5]}(x)$ the temperature at the point $x_0 = 0.60$ at time $t_0 = 3.35$ is $T(x_0, t_0) = 0.109$.

Describe in detail how you arrived at the solution. Comment on what factors influence the estimated values of $c$ and $K$.
Hint: You can find the values of $\alpha$ and $c$ with trial and error using simulations. In order to find the temperature $T(x_0, t_0)$, you should modify the variables `xmesh` and `tspan` so that they include the points you are interested in.

4. In this problem we study the wave 1D wave equation (9a) with Dirichlet boundary conditions.

   (a) Show that the d'Alembert's formula in equation (11) satisfies the wave equation and the initial conditions. Use the formula to find the solution of the wave equation (9a) with $\ell = 1$ and $c = 1$ for $u_{init}(x) = \sin(2\pi x)$ and $v_{init}(x) \equiv 0$. Simplify the formula as much as possible.

   (b) How do the Dirichlet boundary conditions at $x = 0$ and $x = \ell$ relate the functions $F(\cdot)$ and $G(\cdot)$ in the solution of the wave equation?

   (c) Describe how the wave speed $c > 0$ in the wave equation could be estimated based on measurements from the solution of the wave equation.

# References


[1] Matthew J. Hancock "Linear partial differential equations: The 1–D Heat Equation", lecture notes, Massachusetts Institute of Technology, 2006. Available at `http://ocw.mit.edu/courses/mathematics/18-303-linear-partial-differential-equations-fall-2006/lecture-notes/heateqni.pdf` (cited Tuesday 3rd October, 2017).

[2] Matthew J. Hancock "Linear partial differential equations: The 1–D Wave Equation", lecture notes, Massachusetts Institute of Technology, 2006. Available at `http://ocw.mit.edu/courses/mathematics/18-303-linear-partial-differential-equations-fall-2006/lecture-notes/waveeqni.pdf` (cited Tuesday 3rd October, 2017).