Linear Systems

MAT-62506

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Contents

1. Introduction to Linear Systems

1.1 Introduction

The purpose of this course is to give an introduction to the properties and control of linear systems. In particular, we consider a system with a *control input* $u(t)$, *measured output* $y(t)$ and possible *disturbance signal* $w(t)$ affecting the system.

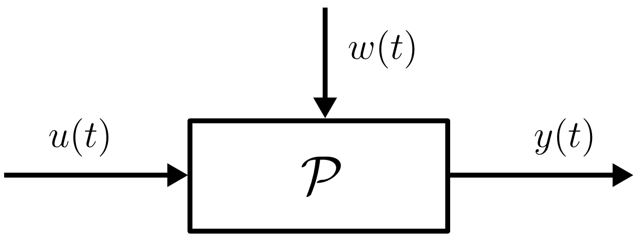

Figure 1.1: The control system.

The general idea in control theory is usually to design and implement a control input $u(t)$ such that the output $y(t)$ of the system behaves in a desired way despite the external disturbance signals $w(t)$.

On this course we concentrate on the control of *linear systems* that are described by differential equations of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0 \in X \tag{1.1a}$$
$$y(t) = Cx(t) + Du(t). \tag{1.1b}$$

Here $x(\cdot) : [0, \infty) \to X$ is a vector-valued function called the *state* of the system (1.1) and $\dot{x}(t)$ denotes the time-derivative of $x(t)$. The control input $u(\cdot) : [0, \infty) \to U$ and the measured output $y(\cdot) : [0, \infty) \to Y$ are either scalar or vector-valued functions depending on the situation. The spaces U and Y are called the *input space* and the *output space*, respectively.

With suitable choices of the *state space* X, and operators or matrices A, B, C and D it is possible to study and control several different types of systems. In this first introductory chapter we will consider some basic concepts related to systems theory and study formulate different types of mathematical models in the form (1.1).

Definition 1.1.1. In a situation where we choose $X = \mathbb{R}^n$ or $X = \mathbb{C}^n$ for some $n \in \mathbb{N}$, and A, B, C, and D are matrices of suitable sizes, the system (1.1) is a *finite-dimensional linear system*.

For a finite-dimensional linear system the solution of the differential equation (1.1a) can be given using the *matrix exponential function* e^{tA} associated to the square matrix $A \in \mathbb{C}^{n\times n}$. In particular, for a given input $u(\cdot) \in L^1_{\text{loc}}(0,\infty;U)$ the solution $x(t)$ of the equation (1.1a) is then given by the familiar "variation of parameters formula"

$$x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A}Bu(s)ds,$$

and substituting this expression into (1.1b) gives a formula

$$y(t) = Ce^{tA}x_0 + C\int_0^t e^{(t-s)A}Bu(s)ds + Du(t)$$

for the measured output $y(t)$ of the system.

Besides finite-dimensional linear systems, we will also study systems that are formulated on infinite-dimensional state spaces X.

Definition 1.1.2. In a situation where X is a Banach or a Hilbert space, and where $A : \mathcal{D}(A) \subset X \to X$, $B : U \to X$, $C : \mathcal{D}(C) \subset X \to Y$, and $D : U \to Y$ are linear operators, the system (1.1) is an *infinite-dimensional linear system*.

In this situation the solvability and obtaining the solution of the infinite-dimensional differential equation (1.1a) becomes more complicated. However, under suitable assumptions the state of the system (1.1) can be expressed using a *strongly continuous semigroup* $T(t)$ generated by the operator A. In fact, the strongly continuous semigroups generalize the matrix exponential functions to situations where X is infinite-dimensional and where $A : \mathcal{D}(A) \subset X \to X$ is a bounded or an unbounded operator. Infinite-dimensional linear systems and the theory of semigroups are studied in greater detail in Chapter 4.

1.2 Common Concepts in Systems Theory

In this section we outline some concepts related to control systems on a very general level. We will also come back to many of these concepts and study them in greater detail in the later chapters.

1.2.1 Stability of a System

One of the key concepts in systems theory is the *stability* of the system (1.1) to be controlled. Often the goal in the control is to design a control $u(t)$ to make the system (1.1) become stable, or alternatively, the stability of the system may be a prerequisite for a proposed control scheme to function properly.

There are many different ways to define *stability* for a system, and the appropriate choice of a definition usually depends on the situation at hand. In addition, some of the concepts are equivalent for certain subclasses of systems, such as the finite-dimensional linear systems, but become distinct in the case of infinite-dimensional systems.

The first two stability types defined here concern the "internal stability" of the system as they are defined in terms of the behaviour of the state $x(t)$ of the system.

Definition 1.2.1. The system (1.1) is called *asymptotically stable*, if in the case of the constant zero input $u(t) \equiv 0$ the state of the system (1.1) satisfies $x(t) \to 0$ as $t \to \infty$ for all $x_0 \in X$.

In the second stability type it is in addition required that the decay of the solutions $x(t)$ of (1.1a) decays at a uniform exponential rate.

Definition 1.2.2. The system (1.1) is called *exponentially stable*, if there exist $\omega > 0$ and $M \geq 1$ such that in the case of the constant zero input $u(t) \equiv 0$ the state of the system (1.1) satisfies

$$\|x(t)\| \leq Me^{-\omega t}\|x_0\|, \qquad \forall t \geq 0, \quad x_0 \in X.$$

Even though exponential stability is a strictly stronger definition than asymptotic stability, these two concepts coincide for finite-dimensional linear systems. In addition, the stability of the system can in this case be determined directly from the locations of the eigenvalues $\sigma(A)$ of the matrix A.

Theorem 1.2.3. *If $X = \mathbb{C}^n$, then the following are equivalent.*

(i) *The system* (1.1) *is asymptotically stable.*

(ii) *The system* (1.1) *is exponentially stable.*

(iii) $\operatorname{Re}\lambda < 0$ *for every* $\lambda \in \sigma(A)$.

Proof. See Theorem 2.2.1. □

On the other hand, we will see that even for simple infinite-dimensional systems the asymptotic stability and exponential stability become two distinct concepts. In particular, for infinite-dimensional systems the solutions $x(t)$ can decay to zero at rates that are strictly slower than exponential as $t \to \infty$. Moreover, for infinite-dimensional systems the stability of the system can only very rarely be determined only from the location of the spectrum $\sigma(A)$ of the operator A.

Finally, the next stability concept is an example of “external stability” — a stability type that is not concerned with the state of the system but instead on how the input affects the output of the system.

Definition 1.2.4. The system (1.1) is called *input-output stable*, if a “stable input” $u(t)$ to the system produces a “stable output” $y(t)$.

There are several variants of input-output stability, the most common ones are

L^2-input-output stability: If $u(\cdot) \in L^2(0,\infty;U)$, then $y(\cdot) \in L^2(0,\infty;Y)$

L^∞-input-output stability: If $u(\cdot) \in L^\infty(0,\infty;U)$, then $y(\cdot) \in L^\infty(0,\infty;Y)$, i.e., a bounded input results in a bounded output.

1.2.2 Controllability and Observability

Also the questions of *controllability* and *observability* deal with the very essential control theoretic properties of a linear systems (1.1). In particular, controllability is related to the question of *how much and how accurately can the state of the system be affected with the control input*, and observability is related to *whether or not all changes in the state of the system are affect the measured output of the system*. The controllability of the system can be formulated in the following way:

Definition 1.2.5. The system (1.1) is *controllable* (in time $\tau > 0$) if for every initial state $x_0 \in X$ and for every target state $x_1 \in X$ there exists a control input $u(\cdot) \in L^1_{\text{loc}}(0, \tau; U)$ such that at time $\tau > 0$ the state of the system is $x(\tau) = x_1$.

The above definition requires that the state of the system can be steered from any initial state x_0 to any final state x_1 in the finite time $\tau > 0$ with an appropriate control input. It turns out that for infinite-dimensional linear systems this is rarely the case, and thus the above definition is usually too strict a requirement. For this reason, a number of alternative weaker concepts have been defined for system on infinite-dimensional spaces [9, Ch. 4].

The controllability of a system does not depend on the operators or matrices C and D of the system (1.1). For finite-dimensional systems there are well-known criteria for testing the controllability of a system using the properties of the matrices A and B, such as the *Popov–Belevitch–Hautus Test* (or simply *PBH Test*) [15].

The observability of a system means that the knowledge of the input $u(t)$ and the output $y(t)$ of the system on a time-inverval $[0, \tau]$ uniquely determines the state of the system on this interval. In mathematical terms this can be formulated in the following way.

Definition 1.2.6. The system (1.1) is *observable* (in time $\tau > 0$) if there exists $k_\tau > 0$ such that

$$\int_0^\tau \|Cx(t)\|^2 dt \geq k_\tau^2 \|x_0\|^2.$$

What the above definition actually requires that the linear map from the initial state x_0 to the output with zero input $Cx(\cdot)$, i.e.,

$$x_0 \in X \mapsto Cx(\cdot) \in L^2(0, \tau; Y)$$

is bounded from below. In particular this means that the given output on the inverval $[0, \tau]$ determines the initial state x_0 uniquely. The state on the full inverval $[0, \tau]$ is then determined by the evolution of the state of the system (1.1).

The concept of observability again only depends on the matrices or operators A and C of the system (1.1). In addition, the controllability and the observability of a system are *dual concepts* of each other, which rougly means that the controllability (observability) of a system (A, B, C, D) is equivalent to the observability (controllability) of its *dual system* (A^*, C^*, B^*, D^*). This is true especially for finite-dimensional linear systems. The detailed definition of duality for infinite-dimensional systems requires a more careful consideration, and the result depends on the precise versions of controllability and observability that are employed, but in general the duality of the concepts is also true for infinite-dimensional systems [9, Ch. 4], [20, Ch. 11].

1.2.3 Feedback

In many situations it is beneficial to choose the input $u(t)$ that is dependent on either the state $x(t)$ or the output $y(t)$ of the system itself. This results in *feedback*, that is commonly encountered in control applications. Feedback can in particular be used to make the system stable.

Definition 1.2.7. In *state feedback* the output $u(t)$ of the system is chosen to depend on the state $x(t)$ in such a way that $u(t) = Kx(t) + \tilde{u}(t)$, where $K : X \to U$ is a linear operator and $\tilde{u}(\cdot)$ is the new input to the system.

A direct substitution of $u(t) = Kx(t) + \tilde{u}(t)$ to the equations (1.1) shows that after the state feedback the system becomes

$$\begin{aligned} \dot{x}(t) &= (A + BK)x(t) + B\tilde{u}(t), \qquad x(0) = x_0 \in X \\ y(t) &= (C + DK)x(t) + D\tilde{u}(t). \end{aligned}$$

State feedback is a powerful tool in control, but in many situations the state $x(t)$ of the system is not known, and it cannot therefore be used in designing the control input $u(t)$. Indeed, in many cases it is only possible to obtain indirect knowledge of the system via the measured output $y(t)$.

Definition 1.2.8. In *output feedback* the input $u(t)$ of the system is chosen in such a way that $u(t) = Ky(t) + \tilde{u}(t)$, where $K : Y \to U$ is a linear operator and $\tilde{u}(\cdot)$ is the new input to the system.

The output feedback scheme is depicted in Figure 1.2.

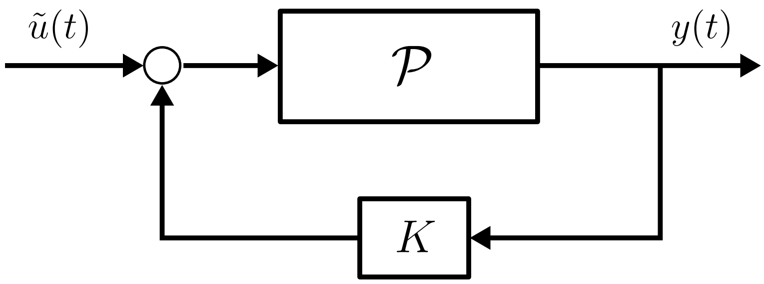

Figure 1.2: The system with output feedback.

If the operator $I - DK$ is boundedly invertible, then we can derive equations for the controlled system after application of output feedback. Indeed, if we substitute $u(t) = DKy(t) + \tilde{u}(t)$ to the equation (1.1b), we get

$$\begin{aligned} & y(t) = Cx(t) + Du(t) = Cx(t) + DKy(t) + D\tilde{u}(t) \\ \Leftrightarrow \quad & (I - DK)y(t) = Cx(t) + D\tilde{u}(t) \\ \Leftrightarrow \quad & y(t) = (I - DK)^{-1}Cx(t) + (I - DK)^{-1}D\tilde{u}(t). \end{aligned}$$

Substituting this into (1.1a) yields

$$\begin{aligned}\dot{x}(t) &= Ax(t) + BKy(t) + B\tilde{u}(t) \\ &= (A + BK(I-DK)^{-1}C)x(t) + BK(I-DK)^{-1}D\tilde{u}(t) + B\tilde{u}(t) \\ &= (A + BK(I-DK)^{-1}C)x(t) + B\left[K(I-DK)^{-1}D + I\right]\tilde{u}(t) \\ &= (A + BK(I-DK)^{-1}C)x(t) + B(I-KD)^{-1}\tilde{u}(t)\end{aligned}$$

since

$$\begin{aligned}K(I-DK)^{-1}D + I = (I-KD)^{-1}KD + I &= (I-KD)^{-1}(KD - I + I) + I \\ &= -I + (I-KD)^{-1} + I = (I-KD)^{-1}.\end{aligned}$$

Combining these we see that the system with the output feedback becomes

$$\begin{aligned}\dot{x}(t) &= (A + BK(I-DK)^{-1}C)x(t) + B(I-KD)^{-1}\tilde{u}(t) \qquad x(0) = x_0 \in X \\ y(t) &= (I-DK)^{-1}Cx(t) + (I-DK)^{-1}D\tilde{u}(t).\end{aligned}$$

This system is again a linear system of the form (1.1), but with the operators of the system have changed in the following way:

$$\begin{aligned}A &\to (A + BK(I-DK)^{-1}C) \\ B &\to B(I-KD)^{-1} \\ C &\to (I-DK)^{-1}C \\ D &\to (I-DK)^{-1}D.\end{aligned}$$

1.2.4 Output Tracking

One of the control problems that we consider on this course are concerned with *ouput tracking and disturbance rejection*, where the aim is to make the output of the plant converge to a given reference signal $y_{ref}(\cdot)$ as $t \to \infty$.

Definition 1.2.9. Let $y_{ref}(\cdot) : [0,\infty) \to Y$ is a given function. In *output tracking* the aim is to choose the input $u(t)$ of the system in such a way that

$$\|y(t) - y_{ref}(t)\| \to 0 \qquad \text{as} \quad t \to \infty.$$

Usually the reference signal is a linear combination of trigonometric functions. With such functions it is possible to approximate, for example, continuous periodic functions by truncating their Fourier series.

1.2.5 Robustness and Robust Control

The term *robustness* refers to a property that makes the control tolerant to changes and uncertainties in the parameters (A,B,C,D) of the controlled system (1.1). There is no one universal definition for "robustness", but instead its use and meaning depend on the situation at hand. For example, the controller could be required to achieve its goal even if the parameters (A,B,C,D) of the system (1.1) are replaced with

$$A + \Delta_A, \quad B + \Delta_B, \quad C + \Delta_C, \quad D + \Delta_D,$$

respectively, where $\Delta_A, \Delta_B, \Delta_C, \Delta_D$ are matrices or bounded linear operators satisfying $\|\Delta_A\| < \delta$, $\|\Delta_B\| < \delta$, $\|\Delta_C\| < \delta$, and $\|\Delta_D\| < \delta$ for some fixed $\delta > 0$.

Robustness is clearly a desirable property when designing control laws for real world systems due to the fact that any mathematical model can only describe the actual physical system with certain limited accuracy. Indeed, the difference between the real world control system and the mathematical model can be seen as a level of "uncertainty", and the designed controller must function properly despite it. We will later learn that incorporating feedback into the control is essential to achieving robustness.

1.2.6 Frequency Domain Theory and Transfer Functions*

Besides studying the behaviour of the control system (1.1) by considering the solution $x(t)$ of the differential equation (1.1a), we could alternatively only study the way how the input $u(t)$ affects the output of the system $y(t)$. One very convenient way to do this is to use instead study the Laplace transforms $\hat{u}$ and $\hat{y}$ of the functions u and y.

If we assume that B, C, and D are bounded linear operators and $\sigma(A) \subset \mathbb{C}_\beta^- = \{\, \lambda \mid \operatorname{Re}\lambda < \beta \,\} \subset \rho(A)$ for some $\beta \in \mathbb{R}$, then if $\gamma > \beta$ is such that $e^{-\gamma \cdot} x(\cdot) \in L^1(0,\infty;X)$, $e^{-\gamma \cdot} u(\cdot) \in L^1(0,\infty;U)$ and $e^{-\gamma \cdot} y(\cdot) \in L^1(0,\infty;Y)$, we can take Laplace transforms from the equations (1.1) and evaluate them at $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \gamma$. The Laplace transform of the time-derivative $\dot{x}(t)$ is equal to $\mathcal{L}(\dot{x})(\lambda) = \lambda\hat{x}(\lambda) - x(0)$, where we have denoted $\mathcal{L}(x) = \hat{x}$. The transformed equation (1.1) has the form

$$\lambda\hat{x}(\lambda) - x(0) = A\hat{x}(\lambda) + B\hat{u}(\lambda)$$

and using $\lambda \in \rho(A) = \mathbb{C} \setminus \sigma(A)$ (which implies that $\lambda - A$ is boundedly invertible) and $x(0) = x_0$ imply

$$\begin{aligned} & \lambda\hat{x}(\lambda) - x(0) = A\hat{x}(\lambda) + B\hat{u}(\lambda) \\ \Leftrightarrow \quad & (\lambda - A)\hat{x}(\lambda) = x_0 + B\hat{u}(\lambda) \\ \Leftrightarrow \quad & \hat{x}(\lambda) = (\lambda - A)^{-1}x_0 + (\lambda - A)^{-1}B\hat{u}(\lambda). \end{aligned}$$

We can similarly take the Laplace transforms of the equation (1.1b) that determines the output of the system to obtain

$$\hat{y}(\lambda) = C\hat{x}(\lambda) + D\hat{u}(\lambda) = C(\lambda - A)^{-1}x_0 + \left[C(\lambda - A)^{-1}B + D\right]\hat{u}(\lambda).$$

The first term in the expression for $\hat{y}(\lambda)$ depends only on the initial state x_0 of the system, and the second one depends only on the input $\hat{u}(\lambda)$. In particular, if we ignore the effect of the intial state, or equivalently consider the case $x(0) = x_0 = 0$, we then get an expression

$$\hat{y}(\lambda) = \left[C(\lambda - A)^{-1}B + D\right]\hat{u}(\lambda) = P(\lambda)\hat{u}(\lambda)$$

for the output $\hat{y}$ in terms of the input $\hat{u}$. The operator-valued function $P(\cdot)$ that maps the input $\hat{u}$ to the output $\hat{y}$ has a special name.

Definition 1.2.10. For $\lambda \in \rho(A) = \mathbb{C} \setminus \sigma(A)$ the operator-valued funtion

$$P(\lambda) = C(\lambda - A)^{-1}B + D$$

is called the *transfer function* of the system (1.1).

Almost all of the aspects of control theory that are studied for linear systems of the form (1.1) (as well as some additional ones) can be equivalently studied in the *frequency domain* by considering only the transfer functions of the plant. In many cases the analysis of the transfer function of the system leads to simpler and more natural analysis and control techniques.

In the case of a finite-dimensional control system transfer function $P(\cdot)$ is a matrix-valued function whose components are rational functions. The definition of $P(\cdot)$ can be extended analytically to all points λ that are not eigenvalues of A. Conversely, if we are given a matrix-valued function $P(\cdot)$ consisting of rational functions, then the control system corresponding to this transfer function can be written as a linear system of the form (1.1) (this is called the *realization* of the transfer function). This means that finite-dimensional linear systems have a good correspondence with the matrix-valued functions consisting of rational functions.

Also most classes of infinite-dimensional systems, especially those described by partial differential equations, can be studied using their transfer functions [14, Ch. 12]. However, in many cases the connection between the original time-domain system and its transfer function is considerably weaker than in the case of finite-dimensional systems. As is the case for the time-domain theory of infinite-dimensional system, also their frequency domain theory and realization theory are under active research.

1.3 Finite-Dimensional Examples

1.3.1 A Damped Harmonic Oscillator

The motion of a simple damped harmonic oscillator (see Figure 1.3) is described by the equations [14, Ex. 1.1.3]

$$m\ddot{q}(t) + r\dot{q}(t) + kq(t) = F(t)$$

where $m, k > 0$ and $r \geq 0$. The situation $r = 0$ corresponds to the undamped oscillator. In this example we consider external force $F(t)$ as our control input, i.e., $u(t) = F(t)$, and we measure the position $q(t)$ of the oscillator, i.e., $y(t) = q(t)$.

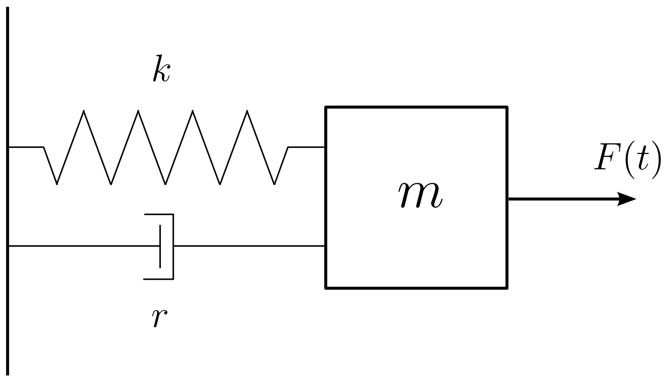

Figure 1.3: A damped harmonic oscillator.

By choosing the state space as $X = \mathbb{R}^2$ and the state of the system as $x(t) = (q(t), \dot{q}(t))^T$, we can see that our system is described by the equations

$$\dot{x}(t) = \begin{pmatrix} \dot{q}(t) \\ \ddot{q}(t) \end{pmatrix} = \begin{pmatrix} \dot{q}(t) \\ -\frac{r}{m}\dot{q}(t) - \frac{k}{m}q(t) + \frac{1}{m}F(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{r}{m} \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} u(t)$$
$$y(t) = q(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} x(t).$$

This system is of the form (1.1) with matrices

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{r}{m} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \qquad D = 0 \in \mathbb{R}.$$

The characteristic polynomial of A is

$$p(\lambda) = \det(\lambda - A) = \lambda^2 + \frac{r}{m}\lambda + \frac{k}{m},$$

and thus the eigenvalues of A are given by

$$\sigma(A) = \left\{ \frac{-r \pm \sqrt{r^2 - 4km}}{2m} \right\}.$$

Since $k, m > 0$, the real parts of the eigenvalues of A are negative whenever $r > 0$, and equal to zero if $r = 0$. By Theorem 1.2.3 we thus have that the oscillator system is exponentially stable whenever $r > 0$, and that it is not asymptotically stable if $r = 0$.

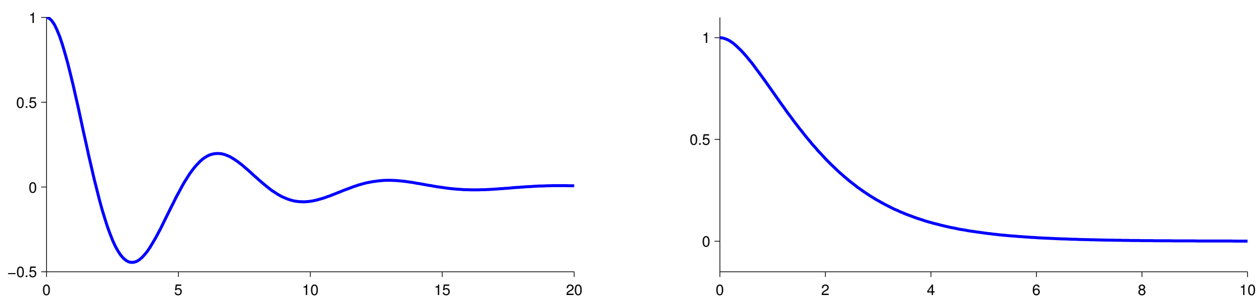

Figure 1.4: The damped harmonic oscillator with $r = 0.5$ (left) and $r = 2$ (right)

1.3.2 Moving Robots

A very simple linearized model for a small moving robot can be given by

$$\dot{x}(t) = u(t), \qquad x(0) \in \mathbb{C}$$

where $x(\cdot)$ and $u(\cdot)$ are both complex-valued functions. The solution $x(t)$ of the above differential equation describes the motion of the robot in the xy-plane once we identify the real axis of $\mathbb{C}$ with the x-axis and the imaginary axis with the y-axis.

The system consisting of $n \in \mathbb{N}$ identical robots $x_k(t)$ is then described by the equations

$$\begin{aligned} \dot{x}_1(t) &= u_1(t), & x_1(0) &\in \mathbb{C} \\ \dot{x}_2(t) &= u_2(t), & x_2(0) &\in \mathbb{C} \\ &\vdots \\ \dot{x}_n(t) &= u_n(t), & x_n(0) &\in \mathbb{C}. \end{aligned}$$

If we measure the positions of the robots in the xy-coordinates, this leads to measurements $y_k(t) = x_k(t)$ for $k \in \{1, \dots, n\}$. If we choose the state space of the full system as $X = \mathbb{C}^n$ and the state of the system as $x(t) = (x_1(t), \dots, x_n(t))^T \in \mathbb{C}^n$, with $u(t) = (u_1(t), \dots, u_n(t))^T \in \mathbb{C}^n$, and $y(t) = (y_1(t), \dots, y_n(t))^T \in \mathbb{C}^n$, then the behaviour of the group of n robots is described by the equations

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} = \begin{pmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{pmatrix}, \qquad \begin{pmatrix} x_1(0) \\ \vdots \\ x_n(0) \end{pmatrix} \in X$$
$$\begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix} = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

which is of the form (1.1) with matrices

$$A = 0_{n\times n}, \qquad B = I_{n\times n}, \qquad C = I_{n\times n}, \qquad D = 0_{n\times n}.$$

Since $A = 0 \in \mathbb{C}^{n\times n}$, its eigenvalues are given by $\sigma(A) = \{0\}$. By Theorem 1.2.3 the system of robots is therefore not asymptotically stable. We can, however, make the system stable using state feedback. Indeed, we can implement a control law which steers each of the robots to the direction of the origin if there is no other input present. This can be done by commanding each robot to move into the direction $-x_k(t)$, which is exactly the direction of the origin. We can therefore choose a control law $u_k(t) = -\alpha x_k(t) + \tilde{u}_k(t)$, where $\alpha > 0$ is a constant parameter that expresses how fast we want the robots to move, and where $\tilde{u}_k(t)$ is the new input. Since

$$u(t) = -\alpha x(t) + \tilde{u}(t),$$

where $\tilde{u}(t) = (\tilde{u}_1(t), \dots, \tilde{u}_n(t))$, the feedback operator $K : X \to U$ is given by $K = -\alpha I_{n\times n}$. With this state feedback the system of robots becomes

$$\begin{aligned} \dot{x}(t) &= -\alpha x(t) + \tilde{u}(t), \qquad x(0) = x_0 \in X \\ y(t) &= x(t), \end{aligned}$$

which is exponentially stable by Theorem 1.2.3 since $\sigma(A + BK) = \sigma(-\alpha I) = \{-\alpha\} \subset \mathbb{C}_-$. Figure 1.5 depicts the behaviour of the stabilized system of robots for two different initial configurations.

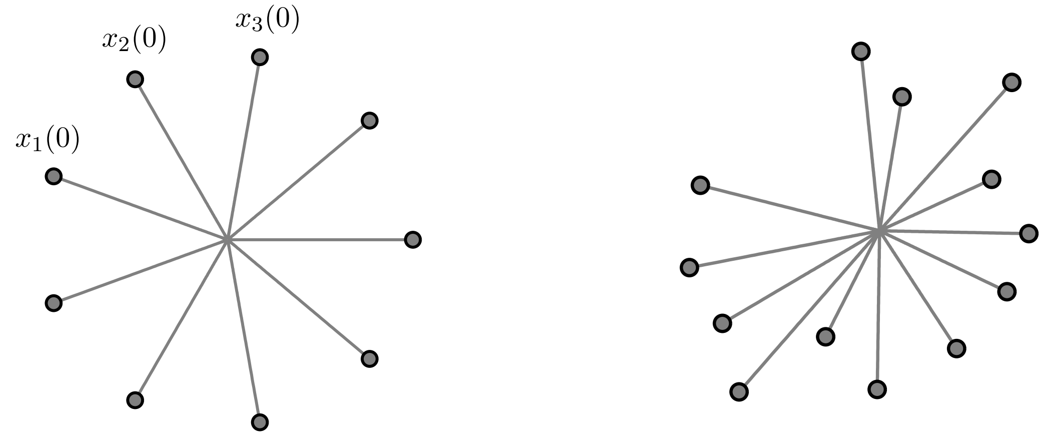

Figure 1.5: Stabilized system of robots.

1.4 Infinite-Dimensional Examples

In this section we present some examples of control systems modeled by linear partial differential examples. On this course we concentrate on simple examples such as the one-dimensional heat and wave equations. However, the approach that we use can also be used in dealing with more complicated equations.

1.4.1 A One-Dimensional Heat Equation

The distribution of heat in a uniform metal rod of a unit length can be modeled by a partial differential equation of the form

$$\frac{\partial v}{\partial t}(\xi,t) = \alpha \frac{\partial^2 v}{\partial \xi^2}(\xi,t) + b(\xi)u(t), \qquad \xi \in (0,1) \tag{1.2a}$$

$$v(0) = 0, \quad v(1) = 0, \tag{1.2b}$$

$$v(\xi,0) = v_0(\xi), \tag{1.2c}$$

where $\alpha > 0$ describes the thermal conductivity of the material. The *boundary conditions* $v(0) = 0$ and $v(1) = 0$ indicate that the two ends of the metal rod are kept at constant temperatures (zero degrees), and $v(\xi,0) = v_0(\xi)$ is an *initial condition* that describes the solution of the system at time $t = 0$.

The control input of the system acts effectively adds or removes heat from certain parts of the rod that are determined by the function $b \in L^2(0,1;\mathbb{R})$. Different choices of the function b describe different types of control systems. For example, if the rod can be heated or cooled down from the part corresponding to the interval $[0,1/2]$, we can choose the function b defined using an indicator function

$$b(\xi) = 2 \cdot \chi_{[0,1/2]}(\xi) = \begin{cases} 2 & \xi \in [0,1/2] \\ 0 & \xi \in (1/2,1] \end{cases}$$

In this example we assume that the state of the heat system can be observed by measuring a weighted average of the temperature on certain parts of the rod. Such a measurement can be written in the form

$$y(t) = \int_0^1 v(\xi,t)c(\xi)d\xi$$

with a given function $c \in L^2(0,1;\mathbb{R})$. For example, if we measure the average temperature on the interval $[1/2,1]$, we can choose an appropriate function $c(\cdot) = 2\cdot\chi_{[1/2,1]}(\cdot)$, and the output of the heat system is given by

$$y(t) = \int_0^1 v(\xi,t)\cdot 2\cdot \chi_{[1/2,1]}(\xi)d\xi = 2\int_{1/2}^1 v(\xi,t)d\xi.$$

The controlled heat equation can be written in the form (1.1) on an infinite-dimensional Hilbert space $X = L^2(0,1;\mathbb{C})$ (the solutions of the origianl heat equation are real-valued, but on this course we consider complex Hilbert spaces for the sake of being uniform). We choose the state $x(t) \in X$ to be the solution of the equation (1.2) at time $t \geq 0$, i.e., $x(t) = v(\cdot,t) \in L^2(0,1)$. The system operator A is an unbounded *second order differential operator* with respect to the spatial variable $\xi \in (0,1)$

$$Af = \alpha f''(\cdot)$$

for a function $f \in X$ that belongs to the *domain of definition* of the operator A that includes the boundary conditions of the original heat equation,

$$\mathcal{D}(A) = \left\{ f \in L^2(0,1) \;\middle|\; f, f' \text{ are absolutely continuous } f'' \in L^2(0,1), \text{ and } f(0) = f(1) = 0 \right\}.$$

The absolute continuity of f and f' for the elements $f \in \mathcal{D}(A)$ guarantee that the two derivatives can be computed in a suitable sense, and that the resulting function Af belongs to the original space $X = L^2(0,1)$.

The inputs and outputs of the heat system are scalar-valued functions, and we therefore have $U = \mathbb{C}$ and $Y = \mathbb{C}$. The operators $B : \mathbb{C} \to X$ and $C : X \to \mathbb{C}$ are bounded linear operators defined by

$$\begin{aligned} Bu &= b(\cdot)u \in X, \qquad \forall u \in \mathbb{C} \\ Cx &= \int_0^1 x(\xi)c(\xi)d\xi, \qquad \forall x \in X. \end{aligned}$$

1.4.2 A One-Dimensional Wave Equation

The vibrations in a uniform undamped string that is fixed at constant positions at both ends are described by the partial differential equation

$$\frac{\partial^2 w}{\partial t^2}(\xi,t) + \alpha\frac{\partial^2 w}{\partial \xi^2}(\xi,t) = b(\xi)u(t), \qquad \xi \in (0,1) \tag{1.3a}$$

$$w(0) = 0, \quad w(1) = 0, \tag{1.3b}$$

$$w(\xi,0) = w_0(\xi), \quad \frac{dw}{dt}(\xi,0) = w_1(\xi). \tag{1.3c}$$

The solution $w(\xi, t)$ of the equation determines the displacement of the string at the position $\xi \in (0,1)$ and at the time instant $t \geq 0$. Here $w(0) = 0$ and $w(1) = 0$ are again the boundary conditions of the equation, and the second order time derivative in the equation requires that the states of the equation are given at time $t = 0$ for both the solution of the system and its first order time derivative. If the measured output of the system is a weighted average of the displacement of the string, then

$$y(t) = \int_0^1 w(\xi, t) c(\xi) d\xi, \qquad t \geq 0$$

for some function $c \in L^2(0,1;\mathbb{R})$.

In order to write the controlled wave system (1.3) in the form (1.1) we in particular need reduce the order of differentiations with respect to the variable t. This can formally be done with a similar approach that is used to reduce higher order ordinary differential equations to systems of first order equations. Namely, we choose the state $x(t)$ to include both the solution $w(\cdot, t)$ of the equation as its time derivative (for brevity, we denote the time derivative as $w_t(\cdot, t) = \frac{dw}{dt}(\cdot, t)$) so that $x(t) = (w(\cdot, t), w_t(\cdot, t))^T$. Now a direct computation shows that

$$\begin{aligned} \dot{x}(t) &= \frac{d}{dt}\begin{pmatrix} w(\cdot, t) \\ w_t(\cdot, t) \end{pmatrix} = \begin{pmatrix} w_t(\cdot, t) \\ w_{tt}(\cdot, t) \end{pmatrix} = \begin{pmatrix} w_t(\cdot, t) \\ -\alpha w_{\xi\xi}(\cdot, t) + b(\cdot) u(t) \end{pmatrix} \\ &= \begin{pmatrix} 0 & I \\ -A_0 & 0 \end{pmatrix} \begin{pmatrix} w(\cdot, t) \\ w_t(\cdot, t) \end{pmatrix} + \begin{pmatrix} 0 \\ b(\cdot) \end{pmatrix} u(t), \end{aligned}$$

where the operator A_0 is the second order differentiation with respect to the spatial variable $\xi \in (0,1)$, which is exactly the same as the operator A in the example concerning the heat equation in the previous section. Since the measured output can be written in the form

$$y(t) = \int_0^1 w(\xi, t) c(\xi) d\xi = \int_0^1 \left(c(\xi),\ 0\right) \begin{pmatrix} w(\xi, t) \\ w_t(\xi, t) \end{pmatrix} d\xi$$

we could choose the operators A, B, and C in the system (1.1) as

$$A = \begin{pmatrix} 0 & I \\ -A_0 & 0 \end{pmatrix}, \qquad Bu = \begin{pmatrix} 0 \\ b(\cdot) \end{pmatrix} u, \qquad C \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \int_0^1 \left(c(\xi),\ 0\right) \begin{pmatrix} f_1(\xi) \\ f_2(\xi) \end{pmatrix} d\xi.$$

However, it turns out that the state space X must be chosen with care, and in particular the most obvious choice $X = L^2(0,1;\mathbb{C}) \times L^2(0,1;\mathbb{C})$ does not lead to a useful infinite-dimensional system (1.1). The choice of the space X is discussed in detail later in Chapter 3.

1.5 Numerical Simulation with Matlab

In this section we develop techniques to simulate the behaviour of the system and its output using Matlab. We begin by considering finite-dimensional linear systems. Simulating systems modeled by partial differential equations require more involved numerical approximations, and these techniques will be considered separately in the later chapters.

Matlab has its own powerful tools for simulation and control of linear systems. These include Simulink, Robust Control Toolbox, Control System Toolbox, Model Predictive Control Toolbox (see Matlab documentation for more infomation). On this course we aim to understand how the simulation and the control algorithms work, and for this reason we write our own simple codes. However, you are also encouraged to get to know and experiment with the built-in Matlab methods related to linear systems and control. There the best place to start is the Control System Toolbox which concentrates on the analysis and control of finite-dimensional linear systems of the form (1.1).

In the following sections we start writing some helpful functions for simulation, analysis, and tweaking of a finite-dimensional control system of the form (1.1).

1.5.1 LinSysSim — Simulation of the State of the System

We begin by writing a Matlab function `LinSysSim` that simulates the state of the system (1.1) with given matrices A and B and given initial state x_0 and a control function $u(\cdot)$ over a specified time-interval. This data is given in the following variables

`A`, `B`	Matrices A and B of the system
`x0`	The initial state x_0
`ufun`	The control function $u(\cdot)$ (Matlab function handle)
`tspan`	The start and end times of the simulation (a vector with two elements)

The differential equation

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0 \tag{1.4}$$

can be solve numerically using one of the available solvers in Matlab. There are many variations of the solver, e.g., `ode23`, `ode45` and `ode15s`. We choose to use the solver `ode15s`, because it can handle some difficulties that arise later in the simulation of approximations of partial differential equations. As the output from the function `LinSysSim` we return the solution structure `sol` that we obtain from the differential equation solver `ode15s`. The structure `sol` contains the instances t of time at which the numerical solution was computed in the variable "`sol.x`" and the corresponding values in the variable "`sol.y`". We will see that the structure is very convenient way of storing the information about the state $x(t)$ of the system.

The code for the function is presented in the following. The first lines of comment are documentation for the function and they can be shown by typing "`help LinSysSim`" in the Matlab command line.

```
function sol = LinSysSim(A,B,x0,ufun,tspan)
% function sol = LinSysSim(A,B,x0,ufun,tspan)
%
% Simulate the state of the differential equation x'(t)=Ax(t)+Bu(t)
% with initial state x(0)=x0, and u(t) = ufun(t) ('ufun' is a function
% handle) over the time interval 'tspan'. The returned variable 'sol' is
% the output of the Matlab's differential equation solver 'ode15s'.

odefun = @(t,x) A*x + B*ufun(t);

sol = ode15s(odefun,tspan,x0);
```

The first line of the code defines how the derivative $\dot{x}(t)$ in the equation (1.4) depends on the variable t and the function $x(t)$. Here we compute the value of the input function $u(t)$ using the function handle `ufun` provided as the parameter in the function `LinSysSim`.

On the second line we ask the solver `ode15s` to solve the differential equation (1.4) on the time-interval determined by the input variable `tspan`.

1.5.2 LinSysOutputPlot — Plotting the Output of the System

The second function that we write uses the output of our first function `LinSysSim` to plot the output of the system (1.1). The input parameters we provide are the variable "`sol`" containing the solution of the differential equation (1.4), matrices C and D, the input function $u(\cdot)$ and a parameter N specifying how many points we want to use in the plotting. We also give a possibility to provide two optional parameters `axlim` and `LineW` that can be used to customize the style of the plot (feel free to add your additional customization parameters if you like!)

`sol`	The output of the function `LinSysSim`
`C, D`	Matrices C and D of the system
`ufun`	The control function $u(\cdot)$ (Matlab function handle)
`N`	Number of points used in the plotting
`axlim`	Custom limits for the axes of the plot, set to "`[]`" for default limits
`LineW`	Line width in the plots, default is equal to 1.

In addition to the plotting the function returns the vector `tt` of points where the output was plotted and a vector `yy` of corresponding values of the function $y(t)$.

```
function [tt,yy] = LinSysOutputPlot(sol,C,D,ufun,N,axlim,LineW)
% function [tt,yy] = LinSysOutputPlot(sol,C,D,ufun,N,axlim,LineW)
%
% Plots the measured output of a linear system when 'sol' is the solution
% variable obtained from the ODE solver, C and D are parameters of the
% system and 'ufun' is the function handle for the input function. Uses a
% uniform grid with N points.
% 'axlim' are the limits for the axes (input '[]' for default) and 'LineW'
% is the line width.

tt = linspace(sol.x(1),sol.x(end),N);
yy = C*deval(sol,tt)+D*ufun(tt);

if nargin <= 6
    LineW = 1;
end

plot(tt,yy,'Linewidth',LineW);

if nargin >5 && ~isempty(axlim)
    axis(axlim)
end
```

The first line of the code initializes an evenly spaced grid of `N` points on the interval where the state $x(t)$ of the system was solved. The second line uses the Matlab function `deval` to evaluate the numerical solution $x(t)$ at these points (the command `deval(sol,tt)`) and computes the output $y(t)$ at these points. Finally, the output is plotted with the command `plot`.

1.5.3 LinSysStatePlot — Plotting the State of the System

There are situations where we might want to plot the state $x(t)$ of the system as well. For this purpose, we can modify the function `LinSysOutputPlot` in the following way. The input variables are the same as in the case of the function `LinSysOutputPlot`, and the output variable `xx` contains the values of $x(t)$ evaluated at the points `tt` of the grid.

```
function [tt,xx] = LinSysStatePlot(sol,N,axlim,LineW)
% function [tt,xx] = LinSysStatePlot(sol,N,axlim,LineW)
%
% Plots the state variables of a linear system when 'sol' is the solution
% variable obtained from the ODE solver. Uses a uniform grid with N points.
% 'axlim' are the limits for the axes (input '[]' for default) and 'LineW'
% is the line width.

tt = linspace(sol.x(1),sol.x(end),N);
xx = deval(sol,tt);

if nargin <= 3
    LineW = 1;
end

plot(tt,xx,'Linewidth',LineW);

if nargin >2 & ~isempty(axlim)
    axis(axlim)
end
```

1.5.4 Example: Simulating the Damped Harmonic Oscillator

We can use our new functions for simulating the behaviour of the damped harmonic oscillator in Section 1.3. The following code defines the matrices (A, B, C, D) of the system, and calls the functions `LinSysSim` and `LinSysOutputPlot` to simulate the behaviour output of the plant with a chosen input function $u(\cdot)$.

```
r = 1;  k = 1;  m = 2;

A = [0 1;-k/m -r/m];
B = [0;1/m];
C = [1 0];
D = 0;

x0 = [1;0];
tspan = [0 15];
```

```
ufun = @(t) zeros(size(t));
%ufun = @(t) sin(t).*cos(t);
%ufun = @(t) sin(t).^2;
%ufun = @(t) sqrt(t);
%ufun = @(t) rem(t,2)<=1;

sol = LinSysSim(A,B,x0,ufun,tspan);
LinSysOutputPlot(sol,C,D,ufun,200,[],2);
```

1.6 References and Further Reading

- Finite-dimensional linear systems [15, 13]
- Semigroup Theory [2, 10, 11, 9]
- Infinite-dimensional linear systems and control [14, 9, 3, 20]
- Free books (at TUT)! [10, 11, 14]

2. Finite-Dimensional Control Theory

In this chapter we concentrate on investigating the controllability and stability of finite-dimensional linear systems. Although the considered results do not directly generalize to infinite-dimensional systems, considering these questions for finite-dimensional systems illustrates the common methodology in the field of control theory. In particular, investigating the degree to which the behaviour of the system of can be influenced using its inputs is a fundamental question that is equally relevant for all classes and types of control systems.

2.1 Controllability of Finite-Dimensional Systems

In this section we will study the controllability of a finite-dimensional linear system. For this we will use the following concepts.

Definition 2.1.1. Let $X = \mathbb{C}^n$ and $u(\cdot) : [0, \infty) \to U = \mathbb{C}^m$. The *controllability matrix* associated to the system (1.1) is defined as

$$\begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix} \in \mathbb{C}^{n \times nm}.$$

For $t > 0$ its *controllability Gramian* is

$$W_t = \int_0^t e^{As} BB^* e^{A^* s} ds \in \mathbb{C}^{n \times n}.$$

For any $t > 0$ the controllability Gramian has the properties that

$$(W_t)^* = \int_0^t \left(e^{As} BB^* e^{A^* s} \right)^* ds = \int_0^t e^{As} BB^* e^{A^* s} ds = W_t$$
$$\langle W_t x, x \rangle = \int_0^t \langle e^{As} BB^* e^{A^* s} x, x \rangle ds = \int_0^t \| B^* e^{A^* s} x \| ds \geq 0 \qquad \forall x \in \mathbb{C}^n.$$

This means that for all $t > 0$ the matrix W_t is *symmetric* (or *Hermitian*) and *positive semi-definite*. The controllability matrix and the controllability Gramian are related in the following way. Here $\mathcal{R}(Q)$ denotes the *range space* of a matrix $Q \in \mathbb{C}^{m \times n}$, i.e., $\mathcal{R}(Q) = \{ y \in \mathbb{C}^m \mid y = Qx \text{ for some } x \in \mathbb{C}^n \}$.

Lemma 2.1.2. *For every $t > 0$ we have*

$$\mathcal{R}\left[\begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix} \right] = \mathcal{R}\left(W_t \right)$$

and $W(t)$ is nonsingular if and only if $\operatorname{rank} \begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix} = n$.

Proof. See [14, Prop. 3.1.5]. □

The following theorem shows that the controllability of a finite-dimensional system can be tested simply by computing the number of linearly independent columns in the controllability matrix.

Theorem 2.1.3. *Let $X = \mathbb{C}^n$. The following are equivalent for every $\tau > 0$.*

(a) *The system* (1.1) *is controllable, i.e., for every initial state $x_0 \in X$ and for every target state $x_1 \in X$ there exists a control input $u(\cdot) \in L^1_{\text{loc}}(0,\tau;U)$ such that at time $\tau > 0$ the state of the system is $x(\tau) = x_1$.*

(b) *The controllability matrix satisfies* $\operatorname{rank}\begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix} = n$.

Proof. We begin by showing that (b) implies (a). Assume $\operatorname{rank}\begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix} = n$ and let $x_0, x_1 \in X$ be arbitrary. To achieve $x(\tau) = x_1$ we need to find an input $u(\cdot)$ such that

$$x_1 = x(\tau) = e^{A\tau}x_0 + \int_0^\tau e^{A(\tau-s)}Bu(s)ds.$$

By Lemma 2.1.2 the controllability Gramian W_τ is invertible. Our aim is to use this property in finding a suitable input. In particular, if we choose a function of the form $u(s) = B^*e^{A^*(\tau-s)}y$ for some $y \in X$ and for all $s \geq 0$, then $u(\cdot) \in L^1_{\text{loc}}(0,\tau;U)$ and

$$x_1 - e^{A\tau}x_0 = \int_0^\tau e^{A(\tau-s)}Bu(s)ds = \int_0^\tau e^{A(\tau-s)}BB^*e^{A^*(\tau-s)}yds = \int_0^\tau e^{Ar}BB^*e^{A^*r}ydr = W_\tau y.$$

This implies that if we choose $y = W_\tau^{-1}(x_1 - e^{A\tau}x_0)$ in the control, then $x(\tau) = x_1$.

For the proof of the implication from (a) to (b), see [14, Thm. 3.1.6]. □

The proof of Theorem 2.1.3 shows that the controllability of a finite-dimensional system for *some* time $\tau > 0$ implies the controllability of the same system for *any* time $\tau > 0$. In particular, we can steer a controllable system to any target state in any arbitrarily small time $\tau > 0$. However, a faster control necessarily requires a control input with a large norm, which in applications is subject to physical constraints. This is also visible in the chosen control input

$$u(t) = B^*e^{A^*(\tau-t)}W_\tau^{-1}(x_1 - e^{A\tau}x_0).$$

Indeed, if $\tau > 0$ becomes small, then also the norm W_τ will be small, which in turn implies that W_τ^{-1} will have large norm due to

$$\|W_\tau^{-1}\| \geq \frac{1}{\|W_\tau\|}.$$

Example 2.1.4. Consider the damped harmonic oscillator in Section 1.3.1. The matrices of the linear system were given by

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{r}{m} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \qquad D = 0 \in \mathbb{R}$$

with $m, k > 0$ and $r \geq 0$ ($r = 0$ corresponds to the situation with no damping). A direct computation shows that the controllability matrix is (now $n = 2$)

$$\begin{pmatrix} B & AB \end{pmatrix} = \frac{1}{m}\begin{pmatrix} 0 & 1 \\ 1 & -r/m \end{pmatrix}$$

which has rank equal to 2 for all $r \geq 0$ and $m > 0$. Thus the system is controllable. ◇

2.2 Stability of a System

In Chapter 1 we learned that the asymptotic and exponential stability of a finite-dimensional linear system can be determined based on the locations of the eigenvalues of the matrix A. We already applied the result in the examples presented in Section 1.3. We will now prove the result using the properties of the matrix exponential function e^{At}.

Theorem 2.2.1. *If $X = \mathbb{C}^n$, then the following are equivalent.*

(i) *The system* (1.1) *is asymptotically stable.*

(ii) *The system* (1.1) *is exponentially stable.*

(iii) $\operatorname{Re}\lambda < 0$ *for every* $\lambda \in \sigma(A)$.

Proof. Clearly (ii) implies (i). We will begin by showing that (i) implies (iii). To this end, assume the system is asymptotically stable. With the constant input $u(t) \equiv 0$ the state $x(t)$ of the system is given by $x(t) = e^{At}x_0$. The asymptotic stability of the system (1.1) therefore means that $\|e^{At}x\| \to 0$ as $t \to \infty$ for all $x \in X$. Let $\lambda \in \sigma(A)$ and let $x \neq 0$ be such that $Ax = \lambda x$. Then also $A^k x = \lambda^k x$ and

$$e^{At}x = \sum_{k=0}^{\infty} \frac{t^k A^k x}{k!} = \sum_{k=0}^{\infty} \frac{t^k \lambda^k}{k!} x = e^{\lambda t}x$$

(both infinite series converge absolutely and uniformly for t on compact intervals of $\mathbb{R}$). The assumption $\|e^{At}x\| \to 0$ as $t \to \infty$ now implies that

$$0 \leftarrow \|e^{At}x\| = \|e^{\lambda t}x\| = |e^{\lambda t}|\|x\| = e^{\operatorname{Re}\lambda t}\|x\|$$

as $t \to \infty$. Since $\|x\| \neq 0$, this is only possible if $\operatorname{Re}\lambda < 0$. Since $\lambda \in \sigma(A)$ was arbitrary, we have that (iii) holds.

Finally, assume that (iii) holds. Let $A = SJS^{-1}$ be the Jordan canonical form of A where $J = \operatorname{diag}(J_1, \dots, J_q)$. We have (see Section A.3)

$$\|e^{At}\| = \|Se^{Jt}S^{-1}\| \leq \|S\|\|S^{-1}\|\|e^{Jt}\| \leq \|S\|\|S^{-1}\| \cdot \max\left\{\|e^{J_1 t}\|, \dots, \|e^{J_q t}\|\right\}$$

For every $k \in \{1, \dots, q\}$ the matrix-valued function $e^{J_k t}$ is of the form $e^{J_k t} = e^{\lambda_k t}Q(t)$ where λ_k is the eigenvalue of the Jordan block and $\|Q(t)\| \leq \tilde{M}_k \max\{1, t^{n_k - 1}\}$ where $n_k = \dim J_k$ (see Theorem A.3.1). If we choose any $0 > \omega_k > \operatorname{Re}\lambda_k$, then there exists $M_k \geq 0$ such that $\|e^{J_k t}\| \leq M_k e^{\omega_k t}$ for all $t \geq 0$. Since this holds for all $k \in \{1, \dots, q\}$, we can estimate

$$\|e^{At}\| \leq \|S\|\|S^{-1}\| \max\left\{\|e^{J_1 t}\|, \dots, \|e^{J_q t}\|\right\} \leq \|S\|\|S^{-1}\| \max\left\{M_1 e^{\omega_1 t}, \dots, M_q e^{\omega_q t}\right\} \leq Me^{\omega t}$$

if we choose $M = \|S\|\|S^{-1}\| \max\{M_1, \dots, M_q\}$ and $\omega = \max\{\omega_1, \dots, \omega_q\} < 0$. This immediately implies that the system is exponentially stable, and thus (ii) holds. □

2.3 Stabilizability of a System

In this section we consider a weaker notion of *stabilizability* of the system. As the following definition shows, this concept means that the system can be made stable with state feedback $u(t) = Kx(t) + \tilde{u}(t)$. We remark that stabilizability is defined in a more general way in [14, Def. 4.1.3], but it is shown in [14, Sec. 4.2] that the two properties coincide.

Definition 2.3.1. Let $X = \mathbb{C}^n$ and $U = \mathbb{C}^m$. The system (1.1) is *stabilizable* if there exists $K \in \mathbb{C}^{m\times n}$ such that $\sigma(A + BK) \subset \mathbb{C}^-$.

It is shown in [14, Cor. 4.2.6] that if the system (1.1) is controllable, then it is also stabilizable. However, controllability actually implies a stronger property which allows us to place the eigenvalues of the matrix $A + BK$ arbitrarily in the complex plane with an appropriate choice of a matrix $K \in \mathbb{C}^{m\times n}$. If the system has this latter property, then it is said that the *pole placement problem* is solvable (the "poles" being the eigenvalues of the matrix $A + BK$). This is a strictly stronger property than stabilizability, because stabilizability does not require us to be able to move the eigenvalues of A that are already in the "*stable half-plane*" $\mathbb{C}^-$. This is illustrated in the following example.

Example 2.3.2. Consider a system with

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \text{and} \qquad B = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

(the matrices C and D do not play a role in controllability and stabilizability). Now $n = 3$ and $m = 1$, and the matrices $K \in \mathbb{C}^{m\times n}$ are of the form $K = (k_1, k_2, k_3)$ with $k_l \in \mathbb{C}$. We have

$$A + BK = \begin{pmatrix} 1 & 1 & 0 \\ k_1 & 1 + k_2 & k_3 \\ 0 & 0 & -1 \end{pmatrix}.$$

A direct computation shows that the characteristic polynomial of $A + BK$ is

$$\det(\lambda - A - BK) = (\lambda + 1)(\lambda^2 + (-k_2 - 2)\lambda - k_1 + k_2 + 1).$$

If we choose $k_1 = 12$ and $k_2 = -7$, and $k_3 \in \mathbb{C}$, then the roots of $\det(\lambda - A - BK)$ are $\sigma(A + BK) = \{-1, -2, -3\}$. Thus the system is stabilizable.

However, the controllability matrix of the system is given by

$$\begin{pmatrix} B & AB & A^2B \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

which has rank equal to $2 < n = 3$. Because of this, the system is not controllable. We can also observe that for all choices of K the matrix $A+BK$ will still have one eigenvalue equal to -1. Because of this, the full pole placement problem is not solvable. $\diamond$

The stabilizability of the system can be tested using the eigenvalues and eigenvectors of A^* in the following way. Since $\sigma(A^*) = \overline{\sigma(A)}$, the corresponding eigenvalues of A and A^* have the same real parts.

Theorem 2.3.3. *Let* $X = \mathbb{C}^n$*. The system* (1.1) *is stabilizable if and only if* A *and* B *have the following property.*

If $\lambda \in \sigma(A^*)$ *is such that* $\operatorname{Re}\lambda \geq 0$ *and* $A^*x = \lambda x$ *with* $x \neq 0$*, then* $x^*B \neq 0$.

Proof. See [14, Thm. 4.3.1]. □

3. Infinite-Dimensional Differential Equations

The purpose of this chapter is to study the behaviour and properties of infinite-dimensional differential equations of the form

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0 \in X \tag{3.1}$$

and

$$\dot{x}(t) = Ax(t) + f(t), \qquad x(0) = x_0 \in X \tag{3.2}$$

when X is an infinite-dimensional vector space. We use terminology and properties of vector spaces defined in Appendix B.

3.1 Strongly Continuous Semigroups

In this section we begin studying the extension of the matrix exponential function e^{At} to infinite-dimensional spaces X and linear operators A. This leads to the theory of strongly continuous semigroups of operators.

3.1.1 Characteristic Properties of the Matrix Exponential Function

The main property that of the matrix exponential function that we are interested in is that if $X = \mathbb{C}^n$, we can then express the solution of the initial value problem (3.1) as $x(t) = e^{At}x_0$ for any $x_0 \in X$. In view of the generalization to operators, the matrix exponential function e^{At} has the following four fundamental properties.:

(1) $e^{A0} = I$ (i.e., for $t = 0$ we have $e^{At} = I$).

(2) $e^{A(t+s)} = e^{At}e^{As}$ if $s, t \in \mathbb{R}$.

(3) The function $t \mapsto e^{At}$ is continuous.

(4) We have $\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$.

The properties (1)–(3) are related to the time-evolution of the differential equation (3.1). In particular, if we consider the solution $x(t) = e^{At}x_0$, then the property (2) tells us that if we let the system evolve for $t + s$ time units we end up in the same state as where we would be if we first let the system evolve for s time units and then for another t time units. The property (1) tells us that if no time passes, the state of the system does not change, and finally, property (3) tells us that the changes in the state happen continuously.

The property (4) can be seen to provide a connection between the matrix exponential function e^{At} and the matrix A. Indeed, we do use A in defining e^{At} either through the series expansion or using the Jordan canonical form of A. However, if we were given a matrix exponential function e^{At}, we could use property (4) to recover the matrix A. This can be done by simply differentiating e^{At} with respect to t and by evaluating the result at $t = 0$,

$$\left[\frac{d}{dt}e^{tA}\right]_{t=0} = \left[Ae^{At}\right]_{t=0} = Ae^{A0} = A.$$

Here we have also used the property (1).

Finally, a small remark on terminology. Besides thinking about e^{At} as a matrix-valued *function* of the variable $t \geq 0$, we can see $\left(e^{At}\right)_{t\geq 0}$ as a *family of matrices* that is parametrized by the variable $t \geq 0$. In particular, for every $t \geq 0$ we have that e^{At} is a matrix that maps the initial state x_0 of (3.1) to the solution $x(t)$ of (3.1) at time t, i.e., $x_0 \mapsto e^{At}x_0$.

3.1.2 Strongly Continuous Semigroups

Motivated by the characteristic features of e^{At}, we define a family $(T(t))_{t\geq 0}$ of bounded linear operators on X with analogous properties. The family $(T(t))_{t\geq 0}$ is parametrized by the variable $t \geq 0$, and $T(t) \in \mathcal{L}(X)$ (the space of bounded linear operators on X) for each $t \geq 0$. Our main objective is to carry out the axiomatic construction of the semigroup $(T(t))_{t\geq 0}$ in such a way that

> The solution of the differential equation (3.1) can be written in the form $x(t) = T(t)x_0$ for all x_0.

Definition 3.1.1. A family $(T(t))_{t\geq 0}$ of bounded linear operators on X is a *strongly continuous semigroup* if it has the following properties.

(1) $T(0) = I$.

(2) $T(t + s) = T(t)T(s)$ if $t, s \geq 0$.

(3) The function $t \mapsto T(t)$ satisfies $\|T(t)x - x\| \to 0$ as $t \to 0+$ for all $x \in X$.

It is clear that the properties (1) and (2) correspond directly to the first two properties of the matrix exponential function. Part (2) is called the *semigroup property*. On the other hand, it turns out that requiring the mapping $t \mapsto T(t)$ to be continuous would be too restrictive. Because of this, we instead require property (3) which together with (2) implies that the function $t \mapsto T(t)x$ (which we aim to be the solution of our differential equation) is continuous for every $x \in X$. The property that $t \mapsto T(t)x$ is continuous for all $x \in X$ is called *strong continuity* of $t \mapsto T(t)$, as opposed to *uniform continuity* $t \to T(t)$ where we require that $\|T(t) - T(s)\| \to 0$ as $t \to s$ for all $t, s \geq 0$.

Lemma 3.1.2. *If $(T(t))_{t\geq 0}$ is a strongly continuous semigroup, then $t \mapsto T(t)x$ is a continuous function for all $x \in X$, i.e., if $x \in X$, then*

$$\|T(t)x - T(s)x\| \to 0 \qquad \text{as } t \to s, \quad t, s \geq 0.$$

Proof. Let $x \in X$ and $s \geq 0$ be arbitrary. If $s = 0$ the continuity at s follows directly from the property (3). On the other hand, if $s > 0$ and $t \geq s$, then we can denote $y = T(s)x \in X$ and use the semigroup property (2) to deduce

$$\|T(t)x - T(s)x\| = \|T(t-s)T(s)x - T(s)x\| = \|T(t-s)y - y\| \to 0$$

as $t \to s+$ due to property (3). Moreover, if $t < s$, then

$$\|T(t)x - T(s)x\| = \|T(t)x - T(s-t)T(t)x\| \leq \|T(t)\|\|x - T(s-t)x\| \to 0$$

as $t \to s$ provided that $s \mapsto T(t)$ is uniformly bounded on compact intervals. To prove this, we will first show that there exists $\varepsilon > 0$ such that $\|T(t)\|$ is bounded on $[0,\varepsilon]$. If this is not true, then there exists $(t_k)_{k\in\mathbb{N}}$ such that $t_k \to 0+$ and $\|T(t_k)\| \to \infty$ as $k \to \infty$. By the uniform boundedness principle [11, Prop. A.2] there also exists $x \in X$ so that $\|T(t_k)x\| \to \infty$ as $k \to \infty$, but this contradicts property (3). Thus there exist $\varepsilon > 0$ and $M \geq 1$ such that $\sup_{0\leq t\leq\varepsilon}\|T(t)\| \leq M < \infty$. On the other hand, if $t_0 > 0$ is arbitrary, then $t_0 < n\varepsilon$ for some $n \in \mathbb{N}$ and if $t \in [0, t_0]$ is such that $t \in [m\varepsilon, (m+1)\varepsilon)$, then

$$\begin{aligned}\|T(t)\| &= \|T(m\varepsilon + t - m\varepsilon)\| = \|T(\varepsilon)\cdots T(\varepsilon)T(t - m\varepsilon)\| \\ &\leq \|T(\varepsilon)\|^m\|T(t - m\varepsilon)\| \leq M^{m+1} \leq M^{n+1}\end{aligned}$$

and thus $\|T(t)\| \leq M^{n+1}$ for all $t \in [0, t_0]$. This shows that the function $t \mapsto T(t)$ is uniformly bounded on compact invervals of $[0,\infty)$ and further implies the continuity of $t \mapsto T(t)x$. □

Example 3.1.3. In this example we consider a *diagonal semigroup*. In particular, let $(\lambda_k)_{k=1}^\infty \subset \mathbb{C}$ be an ordered sequence of complex numbers and assume there exists $\omega \in \mathbb{R}$ such that $\operatorname{Re}\lambda_k \leq \omega$ for all $k \in \mathbb{N}$. We will define the diagonal semigroup on the space $X = \ell^2(\mathbb{C}) = \{\, (x_1, x_2, \ldots) \mid x_k \in \mathbb{C},\quad \sum_{k=1}^\infty |x_k|^2 < \infty \,\}$ of square summable infinite sequences (see also Example B.1.3 in Appendix B).

For every $t \geq 0$ define the operator $T(t) : X \to X$ such that for $x = (x_1, x_2, \ldots)$ we have

$$T(t)x = \left(e^{\lambda_1 t}x_1, e^{\lambda_2 t}x_2, \ldots\right).$$

This is of the same form as the matrix exponential function e^{At} of a diagonal matrix $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, since in that case $e^{At}x = (e^{\lambda_1 t}x_1, \ldots, e^{\lambda_n t}x_n)^T$ for all $x = (x_1, \ldots, x_n)^T$. Likewise, for every $t \geq 0$ the operator $T(t)$ defined above has a representation as an infinite diagonal matrix, and we can denote $T(t) = \operatorname{diag}(e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots) \in \mathcal{L}(X)$.

We need to verify that $T(t)$ is a bounded operator for all $t \geq 0$ and that the properties (1)-(3) of the semigroup are satisfied. For all $t, s \geq 0$ and for every $x = (x_1, x_2, \ldots) \in X$

$$\begin{aligned}T(0)x &= \left(e^{\lambda_1 0}x_1, e^{\lambda_2 0}x_2, \ldots\right) = (x_1, x_2, \ldots) = x \\ T(t+s)x &= \left(e^{\lambda_1(t+s)}x_1, e^{\lambda_2(t+s)}x_2, \ldots\right) = \left(e^{\lambda_1 t}e^{\lambda_1 s}x_1, e^{\lambda_2 t}e^{\lambda_2 s}x_2, \ldots\right) \\ &= T(t)\left(e^{\lambda_1 s}x_1, e^{\lambda_2 s}x_2, \ldots\right) = T(t)T(s)\left(x_1, x_2, \ldots\right) = T(t)T(s)x\end{aligned}$$

and thus $(T(t))_{t\geq0}$ satisfies the properties (1) and (2). The proof of property (3) is left as an exercise. ◇

3.2 The Generator of a Semigroup

At this point we have been able to define a family $(T(t))_{t\geq 0}$ of operators with properties that are suitable for $x(t) = T(t)x_0$ for $x_0 \in X$ to be a solution of *some* differential equation. However, we have not yet linked the semigroup to any particular operator A. We will accomplish this by defining the suitable operator A using the properties of the semigroup $(T(t))_{t\geq 0}$. In particular, since we want the operator A and $(T(t))_{t\geq 0}$ to have a relationship that is similar to the connection between a matrix A and e^{At}, we require that for every "suitable" $x \in X$ the derivative of $T(t)x$ evaluated at $t = 0$ is equal to Ax. We will shortly see that this approach does indeed define a linear operator. Moreover, for suitable initial states $x_0 \in X$ this the function $x(t) = T(t)x_0$ the derivative is equal to

$$\dot{x}(t) = \frac{d}{dt}T(t)x_0 = AT(t)x_0 = Ax(t), \qquad \text{and} \qquad x(0) = T(0)x_0 = Ix_0 = x_0,$$

which means that $x(t)$ is the solution of the differential equation (3.1).

Definition 3.2.1. Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X. We define the *infinitesimal generator* $A : \mathcal{D}(A) \subset X \to X$ in such a way that

$$Ax = \left[\frac{d}{dt}T(t)x\right]_{t=0} = \lim_{t\to 0+} \frac{T(t)x - x}{t}$$

and the domain $\mathcal{D}(A)$ of A is defined to consist of precisely those $x \in X$ for which the above limit exists.

As can be seen from the definition, the relationship between A and $T(t)$ resembles the relationship between a matrix A and e^{At}, but in general it has a much more complicated nature. In particular, we may no longer be able to *compute* $T(t)$ even if we know our operator A well.

The domain $\mathcal{D}(A) \subset X$ of A is the set of elements x for which Ax is defined or "makes sense". In our situation it in particular consits of elements $x \in X$ for which the function $t \mapsto T(t)x$ is differentiable at $t = 0$. Similarly as with continuity, we will see that the semigroup property $T(t + s) = T(t)T(s)$ then implies that the function $t \mapsto T(t)x$ will be (continuously) differentiable at all points $t \geq 0$.

We begin by showing that A is a linear operator. To this end, let $\alpha, \beta \in \mathbb{C}$ and $x, y \in \mathcal{D}(A)$. To show that $\alpha x + \beta y$ we must verify that the limit in Definition 3.2.1 exists. For $t > 0$ we have

$$\frac{1}{t}\big(T(t)(\alpha x + \beta y) - (\alpha x + \beta y)\big) = \alpha\frac{1}{t}(T(t)x - x) + \beta\frac{1}{t}(T(t)y - y) \to \alpha Ax + \beta Ay \in X$$

as $t \to 0+$ because $\frac{1}{t}(T(t)x - x) \to Ax$ and $\frac{1}{t}(T(t)y - y) \to Ay$. Since the limit exists in X, we have $\alpha x + \beta y \in \mathcal{D}(A)$, and by definition the limit is equal to $A(\alpha x + \beta y)$, which further shows that

$$A(\alpha x + \beta y) = \alpha Ax + \beta Ay.$$

Thus A is a linear operator. Although there may not be an expression for the semigroup generated by a given operator $A : \mathcal{D}(A) \subset X \to X$, there is still a direct correspondence between the semigroup and its generator. Indeed, it is shown in [14, Thm. 5.2.3] that an operator $A : \mathcal{D}(A) \subset X \to X$ may generate at most one semigroup. More precisely, if A_1 and A_2 generate semigroup $(T_1(t))_{t\geq 0}$ and $(T_2(t))_{t\geq 0}$, respectively, and if $A_1 = A_2$, then also $T_1(t) = T_2(t)$ for all $t \geq 0$.

Example 3.2.2. Consider the diagonal semigroup $(T(t))_{t\geq 0}$ on $X = \ell^2(\mathbb{C})$ in Example 3.1.3. We have $T(t) = \operatorname{diag}(e^{\lambda_1 t}, e^{\lambda_2 t}, \dots) \in \mathcal{L}(X)$ where $(\lambda_k)_{k\in\mathbb{N}} \subset \mathbb{C}$ are such that $\operatorname{Re}\lambda_k \leq \omega$ for some constant $\omega \in \mathbb{R}$.

In this example we will show that the generator A of the diagonal semigroup $(T(t))_{t\geq 0}$ is an operator

$$A = \operatorname{diag}(\lambda_1, \lambda_2, \dots), \qquad \mathcal{D}(A) = \left\{ x = (x_1, x_2, \dots) \in X \;\middle|\; \sum_{k=1}^{\infty} |\lambda_k|^2 |x_k|^2 < \infty \right\}.$$

It should be noted that if there exists $R > 0$ such that $|\lambda_k| \leq R$ for all $k \in \mathbb{N}$ (that is, all λ_k are contained in some disk centered at 0 and with radius R in the complex plane), then the operator A will be bounded since

$$\|Ax\|_{\ell^2}^2 = \|(\lambda_1 x_1, \lambda_2 x_2, \dots)\|_{\ell^2}^2 = \sum_{k=1}^{\infty} |\lambda_k|^2 |x_k|^2 \leq R^2 \sum_{k=1}^{\infty} |x_k|^2 = R^2 \|x\|_{\ell^2}^2.$$

However, if no such $R > 0$ exists, then the operator A is *unbounded* and $\mathcal{D}(A) \neq X$.

Since $(T(t))_{t\geq 0}$ is a strongly continuous semigroup, it has an infinitesimal generator that we can denote with $A_1 : \mathcal{D}(A_1) \subset X \to X$. In order to show that this generator is actually our operator A, we need to show that $\mathcal{D}(A_1) = \mathcal{D}(A)$ and $A_1 x = Ax$ for all $x \in \mathcal{D}(A)$.

We begin by showing that $\mathcal{D}(A_1) \subset \mathcal{D}(A)$ and $A_1 x = Ax$ for all $x \in \mathcal{D}(A_1)$. To this end, let $x \in \mathcal{D}(A_1)$ be arbitrary. This means that

$$A_1 x = \lim_{t\to 0+} \frac{T(t)x - x}{t}.$$

Denote by $e_k \in X$ a vector whose kth element is 1 and whose other elements are 0 (these vectors actually form a *basis* of the space $X = \ell^2(\mathbb{C})$). Since the inner product $\langle \cdot, \cdot \rangle$ is a continuous function, for every $k \in \mathbb{N}$ we can compute

$$\langle A_1 x, e_k \rangle_{\ell^2} = \left\langle \lim_{t\to 0+} \frac{T(t)x - x}{t}, e_k \right\rangle_{\ell^2} = \lim_{t\to 0+} \frac{\langle T(t)x - x, e_k \rangle_{\ell^2}}{t} = \lim_{t\to 0+} \frac{e^{\lambda_k t} x_k - x_k}{t} = \lambda_k x_k$$

since $\langle x, e_k \rangle = \sum_{l=1}^{\infty} \delta_{lk} x_l = x_k$ for all $x \in X$ (here δ_{kl} is the *Kronecker delta* for which $\delta_{kk} = 1$ and $\delta_{kl} = 0$ for $k \neq l$). Thus $A_1 x = (\lambda_1 x_1, \lambda_2 x_2, \dots) = Ax$. Because we know that $y = A_1 x \in X$, we must have

$$\infty > \sum_{k=1}^{\infty} |y_k|^2 = \sum_{k=1}^{\infty} |\lambda_k x_k|^2 = \sum_{k=1}^{\infty} |\lambda_k|^2 |x_k|^2$$

and thus $x \in \mathcal{D}(A)$ by definition. Since $x \in \mathcal{D}(A_1)$ was arbitrary, we have that $\mathcal{D}(A_1) \subset \mathcal{D}(A)$ and $A_1 x = Ax$ for all $x \in \mathcal{D}(A_1)$.

It remains to show $\mathcal{D}(A) \subset \mathcal{D}(A_1)$. This is a bit trickier thing to do. Let $x \in \mathcal{D}(A)$, i.e., $\sum_{k=1}^{\infty} |\lambda_k|^2 |x_k|^2 < \infty$. Our aim is to show that the limit $\lim_{t\to 0+}(T(t)x - x)/t$ exists. We already know that the limit should be equal to $Ax = (\lambda_1 x_1, \lambda_2 x_2, \ldots)$. For all $0 < t \leq 1$ we have

$$\left\| \frac{T(t)x - x}{t} - Ax \right\|_{\ell^2}^2 = \sum_{k=1}^{\infty} \left| \frac{e^{\lambda_k t} x_k - x_k}{t} - \lambda_k x_k \right|^2 = \sum_{k=1}^{\infty} \left| \frac{e^{\lambda_k t} - 1}{t} - \lambda_k \right|^2 |x_k|^2.$$

Let $\varepsilon > 0$ be arbitrary. We aim to show that we can choose $t_0 \leq 1$ such that the above norm is smaller than ε for all $0 < t \leq t_0$. For all $k \in \mathbb{N}$ and for all $0 < t \leq 1$ we can estimate

$$\begin{aligned} \left| \frac{e^{\lambda_k t} - 1}{t} - \lambda_k \right| &\leq \left| \frac{e^{\lambda_k t} - e^{\lambda_k \cdot 0}}{t} \right| + |\lambda_k| = \frac{1}{t} \left| \int_0^t \lambda_k e^{\lambda_k s} ds \right| + |\lambda_k| \leq \lambda_k \max_{0 \leq t \leq 1} |e^{\lambda_k t}| + |\lambda_k| \\ &\leq |\lambda_k| \left(\max_{0 \leq t \leq 1} e^{\operatorname{Re} \lambda_k t} + 1 \right) \leq |\lambda_k| \left(\max_{0 \leq t \leq 1} e^{\omega t} + 1 \right) \leq |\lambda_k| \left(e^{\omega} + 1 \right) \end{aligned}$$

since $\operatorname{Re} \lambda_k \leq \omega$ and $t \leq 1$ by assumption. Choose $N \in \mathbb{N}$ in such a way that $\sum_{k=N+1}^{\infty} |\lambda_k|^2 |x_k|^2 < \frac{\varepsilon^2}{2(e^{\omega}+1)}$ and choose $t_0 \leq 1$ such that

$$\max_{1 \leq k \leq N} \left| \frac{e^{\lambda_k t} - 1}{t} - \lambda_k \right|^2 < \frac{\varepsilon^2}{2\|x\|^2}$$

for all $0 < t \leq t_0$. Then for every $0 < t \leq t_0$ we also have

$$\begin{aligned} \left\| \frac{T(t)x - x}{t} - Ax \right\|_{\ell^2}^2 &= \sum_{k=1}^{N} \left| \frac{e^{\lambda_k t} - 1}{t} - \lambda_k \right|^2 |x_k|^2 + \sum_{k=N+1}^{\infty} \left| \frac{e^{\lambda_k t} - 1}{t} - \lambda_k \right|^2 |x_k|^2 \\ &\leq \max_{1 \leq k \leq N} \left| \frac{e^{\lambda_k t} - 1}{t} - \lambda_k \right|^2 \sum_{k=1}^{N} |x_k|^2 + (e^{\omega} + 1) \sum_{k=N+1}^{\infty} |\lambda_k|^2 |x_k|^2 \\ &\leq \frac{\varepsilon^2}{2\|x\|^2} \|x\|^2 + (e^{\omega} + 1) \frac{\varepsilon^2}{2(e^{\omega} + 1)} = \varepsilon^2. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we have now shown that $\lim_{t \to 0+} (T(t)x - x)/t = Ax$. This finally implies that $x \in \mathcal{D}(A_1)$ and $A_1 x = Ax$. Since $x \in \mathcal{D}(A)$ was arbitrary, we have $\mathcal{D}(A) \subset \mathcal{D}(A_1)$, and thus the proof of $A = A_1$ is complete. ◇

Remark 3.2.3. Note that instead of indexing the diagonal elements with $\mathbb{N}$ in Examples 3.1.3 and 3.2.2 we could have also chosen to index them with $k \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Moreover, the same results hold for *doubly infinite* matrices, in which case we would have

$$T(t)x = (\ldots, e^{\lambda_{-1} t} x_{-1}, e^{\lambda_0 t} x_0, e^{\lambda_1 t} x_1, \ldots)$$

for all $x = (\ldots, x_{-1}, x_0, x_1, \ldots) \in \ell^2(\mathbb{Z}; \mathbb{C})$.

The following theorem shows that a strongly continuous semigroup indeed gives us the solution of the initial value problem (3.1).

Theorem 3.2.4. *Let $(T(t))_{t \geq 0}$ be a strongly continuous semigroup with an infinitesimal generator $A : \mathcal{D}(A) \subset X \to X$. If $x_0 \in \mathcal{D}(A)$, then the function $t \mapsto T(t)x_0$ is continuously differentiable on $[0, \infty)$ and*

$$\frac{d}{dt} T(t)x_0 = AT(t)x_0 = T(t)Ax_0, \qquad t \geq 0. \tag{3.3}$$

Moreover, if $x_0 \in \mathcal{D}(A)$, then the function $t \mapsto x(t) = T(t)x_0$ is the unique solution of (3.1).

Proof. Let $x_0 \in \mathcal{D}(A)$. The definition of A and $\mathcal{D}(A)$ imply that the function $t \mapsto x(t) = T(t)x_0$ is (right) differentiable at $t = 0$. Using the semigroup property $T(t+s) = T(t)T(s)$ we can then show that it is also differentiable for all $t > 0$. Indeed, if $t > 0$ and $h > 0$, then denoting $y = T(t)x$

$$\frac{T(t+h)x_0 - T(t)x_0}{h} = T(t)\frac{T(h)x_0 - x_0}{h} = \frac{T(h)y - y}{h}. \tag{3.4}$$

Since $x_0 \in \mathcal{D}(A)$ and $T(t) \in \mathcal{L}(X)$, the expression in the middle converges as $h \to 0+$ and its limit is equal to $T(t)Ax_0$ by Definition 3.2.1. Because of this, also the right and left limits exist as $h \to 0+$. The existence of the rightmost limit implies that $y = T(t)x \in \mathcal{D}(A)$ and the limit is equal to $Ay = AT(t)x_0$, and thus we conclude $T(t)Ax_0 = AT(t)x_0$.

On the other hand, if $h < 0$, then

$$\frac{T(t+h)x_0 - T(t)x_0}{h} = T(t+h)\frac{T(-h)x_0 - x_0}{-h} \tag{3.5}$$

and the limit the right-hand side as $h \to 0-$ exists and equals $T(t)Ax_0$ since $x_0 \in \mathcal{D}(A)$ and $T(\cdot)$ is strongly continuous. The limits in (3.4) and (3.5) as $h \to 0$ show that $t \mapsto T(t)x_0$ is differentiable at t and (3.3) holds. Furthermore, the derivative is continuous, since $t \mapsto T(t)Ax_0$ is a continuous function due to strong continuity of $t \mapsto T(t)$.

If we let $x(t) = T(t)x_0$ with $x_0 \in \mathcal{D}(A)$, then $x(0) = T(0)x_0 = x_0$, and $\dot{x}(t) = Ax(t)$ for all $t \geq 0$ by (3.3). Since $x(\cdot)$ is continuously differentiable, it is a solution of the differential equation (3.1).

For the proof of the uniqueness of the solution, see [14, Thm. 5.3.2]. □

By the above theorem, the function $x(t) = T(t)x_0$ is a solution of the differential equation (3.1) whenever $x_0 \in \mathcal{D}(A)$. The requirement that the initial state x_0 belongs to the domain of the generator A guarantees that the solution $x(t)$ of the equation is continuously differentiable. Such solutions of (3.1) are called *classical solutions* of the equation. Moreover, it turns out that the function $x(t) = T(t)x_0$ is differentiable *only when* $x_0 \in \mathcal{D}(A)$ [11, Lem. 1.1]. However, we can define the function $x(t) = T(t)x_0$ even when $x_0 \notin \mathcal{D}(A)$. These more general functions are called *mild solutions* of the equation (3.1).

Definition 3.2.5. For every $x_0 \in X$ the function $t \mapsto x(t) = T(t)x_0$ is called the *mild solution* of (3.1).

The mild solution $x(t) = T(t)x_0$ does not have a derivative with respect to t if $x_0 \notin \mathcal{D}(A)$, but it does satisfy an “integrated version” of the differential equation (3.1),

$$x(t) = x(0) + A\int_0^t x(s)ds, \qquad t \geq 0,$$

see [14, Def. 5.3.3] for details. In the treatment of linear partial differential equations the mild solutions in particular correspond to solutions that originating from initial states that are not differentiable or initial states that do not satisfy the boundary conditions of the original equation.

The property that the operator A in the differential equation (3.1) is a generator of a strongly continuous semigroup guarantees that the differential equation is *well-posed* in the sense that (i) for every suitable x_0 the equation has a solution, (ii) this solution is unique and (iii) the solution depends continuously on the initial state x_0 [11, Sec. II.6]. In particular we have the following property (which depends on the precise definition of "well-posedness", see [11, SEc. II.6] for details).

Theorem 3.2.6. *The differential equation* (3.1) *is well-posed if and only if the operator* $A : \mathcal{D}(A) \subset X \to X$ *is a generator of a strongly continuous semigroup on* X.

Proof. See [11, Cor. II.6.8]. □

It is also reasonable to ask if the differential equation (3.1) can really have classical solutions, and how many classical solutions exist. In other words, we would like to know whether or not $\mathcal{D}(A)$ is nonempty and to know large it is. The properties of the semigroup actually guarantee that the if A is a generator of a semigroup, then $\mathcal{D}(A)$ is always nonempty and quite large. In particular, $\mathcal{D}(A)$ is *dense* in X [11, Thm. II.1.4], which means that for every $x \in X$ and $\varepsilon > 0$ there exists $y \in \mathcal{D}(A)$ such that $\|x - y\| < \varepsilon$. This guarantees that the equation (3.1) always has a large set of classical solutions.

3.3 When Does an Operator Generate a Semigroup?

We have now learned that every semigroup has a generator, but usually when we are studying a particular equation, we are more interested in whether or not a given operator A generates a strongly continuous semigroup on X.

There are many results that answer this imporant question. On this course we concentrate on studying differential equations where the operator A is *similar* to a diagonal operator, and we can then deduce the semigroup generation property using the properties of the diagonal operators in Examples 3.1.3 and 3.2.2. This approach is restricted to situations where we can analyze the eigenvalues and eigenfunctions of the operator A in detail. However, even with this limitation, the class of systems where A is similar to a diagonal operator cover many interesting examples, such as linear heat, wave and beam equations on

- one-dimensional spatial intervals.
- two-dimensional rectangular and certain triangular spatial domains, and on disks.
- n-dimensional rectangular and spherical spatial domains.

In more general situations, the property that a closed and densely defined operator A is a generator of a strongly continuous semigroup can be characterized by the so-called *Hille-Yosida generation theorems* [11, Thm. II.3.8]. I recommend studying Chapter 3 of [11] for a very good overview of the relationship between a semigroup $(T(t))_{t\geq 0}$ and its generator $A : \mathcal{D}(A) \subset X \to X$. In a situation where we want to show that an operator A is the generator of a semigroup that is *contractive*, i.e., $\|T(t)\| \leq 1$ for all $t \geq 0$, we can also use the so-called Lumer–Phillips Theorem [14, Thm. 6.1.7]. Here $\mathcal{R}(A)$ denotes the *range space* of the operator $A : \mathcal{D}(A) \subset X \to X$, i.e. $\mathcal{R}(A) = \{\, y \in X \mid y = Ax \text{ for some } x \in \mathcal{D}(A) \,\}$.

Theorem 3.3.1 (Lumer–Phillips Theorem). *Let X be a Hilbert space. A closed and densely defined operator is a generator of a contraction semigroup if and only if*

$$\mathrm{Re}\langle Ax, x\rangle \leq 0$$

for all $x \in \mathcal{D}(A)$ and $\mathcal{R}(I - A) = X$.

Proof. See [14, Thm. 6.1.7]. □

3.3.1 Operators That Are Similar to Diagonal Ones

In this section we consider operators that are *similar* to diagonal operators of the form

$$A = \mathrm{diag}(\lambda_1, \lambda_2, \ldots), \qquad \mathcal{D}(A) = \Big\{ x = (x_1, x_2, \ldots) \in X \;\Big|\; \sum_{k=1}^{\infty} |\lambda_k|^2 |x_k|^2 < \infty \Big\}.$$

It should be noted that all the results also remain valid with the appropriate modifications for doubly infinite diagonal operators of the form

$$A = \mathrm{diag}(\ldots, \lambda_{-1}, \lambda_0, \lambda_1, \ldots),$$
$$\mathcal{D}(A) = \Big\{ x = (\ldots, x_{-1}, x_0, x_1, \ldots) \in X \;\Big|\; \sum_{k=-\infty}^{\infty} |\lambda_k|^2 |x_k|^2 < \infty \Big\}.$$

Theorem 3.3.2. *A diagonal operator A on $X = \ell^2(\mathbb{C})$ generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ if (and only if) there exists $\omega \in \mathbb{R}$ such that $\mathrm{Re}\,\lambda_k \leq \omega$ for all k. The semigroup satisfies $T(t) = \mathrm{diag}(e^{\lambda_k})_k$ for all $t \geq 0$.*

Proof. We saw in Examples 3.1.3 and 3.2.2 that if there exists $\omega \in \mathbb{R}$ such that $\mathrm{Re}\,\lambda_k \leq \omega$ for all k, then the diagonal semigroup $(T(t))_{t\geq 0}$ satisfying $T(t) = \mathrm{diag}(\lambda_k)_k$ is a strongly continuous semigroup and its generator is exactly the diagonal operator A.

We omit the proof for the property that if no such $\omega \in \mathbb{R}$ exists, then A does not generate a semigroup on X. □

Definition 3.3.3. Let X_1 and X_2 be Banach spaces. Two operators $A_1 : \mathcal{D}(A_1) \subset X_1 \to X_1$ and $A_2 : \mathcal{D}(A_2) \subset X_2 \to X_2$ are *(boundedly) similar* if there exists a bounded operator $S \in \mathcal{L}(X_1, X_2)$ with a bounded inverse $S^{-1} \in \mathcal{L}(X_2, X_1)$ (i.e., $SS^{-1} = I_{X_2}$ and $S^{-1}S = I_{X_1}$) such that

$$\mathcal{D}(A_2) = \big\{ x \in X \;\big|\; S^{-1}x \in \mathcal{D}(A_1) \big\}$$

and $A_2 x = SA_1S^{-1}x$ for all $x \in \mathcal{D}(A_2)$.

Similarity is a symmetric relation, and if A_1 and A_2 are similar, then $\mathcal{D}(A_1) = \big\{ x \in X \;\big|\; Q^{-1}x \in \mathcal{D}(A_2) \big\}$ and $A_1 x = QA_2Q^{-1}x$ for all $x \in \mathcal{D}(A_1)$ for the boundedly invertible operator $Q = S^{-1}$.

Lemma 3.3.4. *Assume X_1 and X_2 are linear vector spaces assume $S \in \mathcal{L}(X_1, X_2)$ is boundedly invertible.*

(a) If $(T_1(t))_{t\geq 0}$ is a strongly continuous semigroup on X_1 and $T_2(t) = ST_1(t)S^{-1}$ for all $t \geq 0$, then $(T_2(t))_{t\geq 0}$ is a strongly continuous semigroup on X_2.

(b) If $A_1 : \mathcal{D}(A_1) \subset X_1 \to X_1$ is the generator of $(T_1(t))_{t\geq 0}$, then the generator of $(T_2(t))_{t\geq 0}$ is $A_2 = SA_1S^{-1} : \mathcal{D}(A_2)$ with domain $\mathcal{D}(A_2) = \{ x \in X_2 \mid S^{-1}x \in \mathcal{D}(A_1) \}$.

Proof. The proof of part (a) is left as an exercise, see Exercise 5.4 in Jacob & Zwart.

To prove part (b), denote the generator of $(T_2(t))_{t\geq 0}$ by $\tilde{A} : \mathcal{D}(\tilde{A}) \subset X_2 \to X_2$. Our aim is to show that $\mathcal{D}(\tilde{A}) = \mathcal{D}(A_2)$ and $\tilde{A}x = A_2x$ for all $x \in \mathcal{D}(A_2)$. Let $x \in \mathcal{D}(A_2)$ be artbitrary. Then $S^{-1}x \in \mathcal{D}(A_1)$ and using $T_2(t) = ST_1(t)S^{-1}$ we have

$$\frac{T_2(t)x - x}{t} = S\frac{T_1(t)S^{-1}x - S^{-1}x}{t} \to SA_1S^{-1}x$$

as $t \to 0+$ since $S^{-1}x \in \mathcal{D}(A_1)$ and S is a bounded operator. Since $x \in \mathcal{D}(A_2)$ was arbitrary, we have by definition that $\mathcal{D}(A_2) \subset \mathcal{D}(\tilde{A})$ and $\tilde{A}x = A_2x$ for all $x \in \mathcal{D}(A_2)$. On the other hand, if $x \in \mathcal{D}(\tilde{A})$, then

$$\frac{T_1(t)S^{-1}x - S^{-1}x}{t} = S^{-1}\frac{T_2(t)x - x}{t} \to S^{-1}\tilde{A}x$$

as $t \to 0+$. Thus $S^{-1}x \in \mathcal{D}(A_1)$ which implies $x \in \mathcal{D}(A_2)$. Since $x \in \mathcal{D}(\tilde{A})$ was arbitrary, we conclude that $\mathcal{D}(\tilde{A}) = \mathcal{D}(A_2)$. □

Corollary 3.3.5. *Let X be a Hilbert space and assume $A : \mathcal{D}(A) \subset X \to X$ is similar to a diagonal operator $D = \operatorname{diag}(\lambda_k)_k$ on $\ell^2(\mathbb{C})$. If there exists $\omega \in \mathbb{R}$ such that $\operatorname{Re}\lambda_k \leq \omega$ for all k. Then A generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ on X.*

If $S \in \mathcal{L}(\ell^2(\mathbb{C}), X)$ is boundedly invertible and $A = SDS^{-1}$, and $(T_D(t))_{t\geq 0}$ denotes the semigroup generated by D on $\ell^2(\mathbb{C})$, then $T(t) = ST_D(t)S^{-1}$ for all $t \geq 0$.

3.3.2 The Diagonal Semigroup for the Heat Equation

Consider the uncontrolled distribution of heat in a uniform metal rod of a unit length modeled by a heat equation of the form (see Section 1.4.1)

$$\frac{\partial v}{\partial t}(\xi, t) = \alpha\frac{\partial^2 v}{\partial \xi^2}(\xi, t), \qquad \xi \in (0,1) \tag{3.6a}$$

$$v(0) = 0, \quad v(1) = 0, \tag{3.6b}$$

$$v(\xi, 0) = v_0(\xi), \tag{3.6c}$$

where $\alpha > 0$ describes the thermal conductivity of the material. As discussed in Section 1.4.1, the heat equation (3.6) can be written as an abstract differential equation of the form (3.1) on the space $X = L^2(0,1)$ by choosing $x(t) = v(\cdot, t) \in X$ for all $t \geq 0$ and defining the operator A such that

$$Af = \alpha f''(\cdot),$$

with domain

$$\mathcal{D}(A) = \left\{ f \in X \;\middle|\; f, f' \text{ abs. cont. } f'' \in L^2(0,1), \text{ and } f(0) = f(1) = 0 \right\}.$$

Note again that the domain $\mathcal{D}(A)$ of A contains the boundary conditions of the original partial differential equation. The absolute continuity of f and f' for the elements $f \in \mathcal{D}(A)$ guarantee that the two derivatives can be computed in a suitable sense, and that the resulting function Af belongs to the original space $X = L^2(0,1)$.

Our aim is to show that A is similar to a diagonal operator on $\ell^2(\mathbb{C})$. This requires studying the *eigenvalues* and *spectrum* of the operator A.

Definition 3.3.6. Let X be a Banach space and let $A : \mathcal{D}(A) \subset X \to X$.

(a) A scalar $\lambda \in \mathbb{C}$ is called *an eigenvalue* of A if there exists $x \in \mathcal{D}(A)$ such that $x \neq 0$ and $Ax = \lambda x$. Then x is called *an eigenvector* (or *eigenfunction*) corresponding to the eigenvalue λ. The set of eigenvalues of A is denoted by $\sigma_p(A)$.

(b) The set of scalars $\lambda \in \mathbb{C}$ for which the operator $\lambda - A$ is boundedly invertible, i.e., $(\lambda - A)^{-1}\mathcal{L}(X)$, is called *the resolvent set* of A and it is denoted by $\rho(A)$.

We begin by finding the eigenvalues and eigenvectors of the operator $A = \alpha\frac{d^2}{d\xi^2}$. If $k \in \mathbb{N}$, define $\phi_k(\cdot) = \sqrt{2}\sin(k\pi\cdot) \in X$. Since $\phi_k(0) = \phi_k(1) = 0$, and $\phi_k \in \mathcal{D}(A)$ for all $k \in \mathbb{N}$. A direct computation shows that

$$A\phi_k = \sqrt{2}\alpha\frac{d^2}{d\xi^2}\sin(k\pi\cdot) = -\alpha k^2\pi^2\sqrt{2}\sin(k\pi\cdot) = -\alpha k^2\pi^2\phi_k.$$

Since $\phi_k \neq 0$, we have that $\lambda_k = -\alpha k^2\pi^2$ are eigenvalues of A for all $k \in \mathbb{N}$, and the corresponding eigenvectors are given by $\phi_k = \sqrt{2}\sin(k\pi\cdot)$. Since for all $k, l \in \mathbb{N}$ with $k \neq l$ we have

$$\langle \phi_k, \phi_l \rangle = \int_0^1 \phi_k(\xi)\overline{\phi_l(\xi)}d\xi = 2\int_0^1 \sin(k\pi\xi)\sin(l\pi\xi)d\xi = 0$$
$$\langle \phi_k, \phi_k \rangle = \|\phi_k\|^2 = \int_0^1 |\phi_k(\xi)|^2 d\xi = 2\int_0^1 \sin^2(k\pi\xi)\,d\xi = 1,$$

the set $\{\phi_k\}_{k\in\mathbb{N}}$ of eigenvectors of A is *orthonormal*. We also know from Fourier analysis that any function $f \in L^2(0,1)$ can be expressed as a limit of functions $f_n(\cdot)$ such that

$$f_n = \sum_{k=1}^n \beta_k\phi_k(\cdot), \qquad \lim_{n\to\infty}\|f - f_n\|_{L^2} = 0.$$

This property implies that the set $\{\phi_k\}_k \subset X$ is an orthonormal *Schauder basis* of the Hilbert space X. In particular, if $x \in X$, then there exist constants $(\beta_k)_{k\in\mathbb{N}}$ such that $x = \sum_{k=1}^\infty \beta_k\phi_k$, and since the set $\{\phi_k\}_k$ is orthonormal, the norm of x satisfies

$$\|x\|^2 = \langle x, x\rangle = \lim_{n,m\to\infty}\left\langle \sum_{k=1}^n \beta_k\phi_k, \sum_{l=1}^m \beta_l\phi_l \right\rangle = \lim_{n,m\to\infty}\sum_{k=1}^n\sum_{l=1}^m \overline{\beta_k}\beta_l \underbrace{\langle \phi_k, \phi_l\rangle}_{=\delta_{kl}} = \sum_{k=1}^\infty |\beta_k|^2.$$

Finally, the operator A has a very nice (and comparatively rare!) property that its effect on a vector $x \in X$ can be expressed in terms of its eigenvalues and eigenvectors. Due to this property, the operator A can be compared to a *diagonalizable matrix* that has a full set of linearly independent eigenvectors. However, for infinite-dimensional operators the existence of such an *eigenfunction expansion* is a much stronger assumption than the diagonalizability of a matrix. It can be shown that

$$Ax = \sum_{k=1}^{\infty} \lambda_k \beta_k \phi_k, \qquad x = \sum_{k=1}^{\infty} \beta_k \phi_k \in \mathcal{D}(A) = \left\{ \sum_{k=1}^{\infty} \beta_k \phi_k \in X \;\middle|\; \sum_{k=1}^{\infty} |\lambda_k|^2 |\beta_k|^2 < \infty \right\}.$$

In particular, the domain $\mathcal{D}(A)$ can be characterized using the summability of the coefficients $(\beta_k)_k$ of the elements x, and the application of A to an elemenent $x \in \mathcal{D}(A)$ produces an element where each component $\beta_k \phi_k$ of x is multiplied with the corresponding eigenvalue λ_k of A. The precise proof of the eigenfunction expansion uses the property that the generator A of a semigroup is a *closed* operator [11, Def. A.5].

We will now define an operator $S : \ell^2(\mathbb{C}) \to X$ that is the similarity transformation between A and the corresponding infinite diagonal matrix $A_D : \mathcal{D}(A_D) \subset \ell^2(\mathbb{C}) \to \ell^2(\mathbb{C})$. We do this by defining an operator S that maps the kth natural basis vector of $\ell^2(\mathbb{C})$ to the function $\phi_k \in X$. More precisely, if $\{e_k\}_{k\in\mathbb{N}} \in \ell^2(\mathbb{C})$ denote the natural basis vectors of $\ell^2(\mathbb{C})$ (the kth element of the vector e_k is equal to 1 and others are zero), then we define

$$Se_k = \phi_k \in X \qquad \text{for all} \quad k \in \mathbb{N}.$$

As an exercise we will show that this defines a bounded and boundedly invertible linear operator $S \in \mathcal{L}(\ell^2(\mathbb{C}), X)$. In particular, $\|Sy\| = \|y\|$ for all $y \in \ell^2(\mathbb{C})$. If $y = (y_1, y_2, \dots) \in \ell^2(\mathbb{C})$, then

$$Sy = S \sum_{k=1}^{\infty} x_k e_k = \sum_{k=1}^{\infty} x_k S e_k = \sum_{k=1}^{\infty} x_k \phi_k \in X.$$

Our aim is to show that $A = SA_D S^{-1}$ where $A_D = \operatorname{diag}(\lambda_1, \lambda_2, \dots)$ with domain $\mathcal{D}(A_D) = \{ (y_1, y_2, \dots) \mid \sum_{k=1}^{\infty} |\lambda_k|^2 |y_k|^2 < \infty \}$. However, since $S^{-1}\phi_k = e_k$ and

$$S^{-1}x = \sum_{k=1}^{\infty} \beta_k S^{-1} \phi_k = \sum_{k=1}^{\infty} \beta_k e_k = y$$

for $y = (\beta_1, \beta_2, \dots) \in \ell^2(\mathbb{C})$, it is immediate that $\mathcal{D}(A) = \{ x \in X \mid S^{-1}x \in \mathcal{D}(A_D) \}$. Furthermore, if $x = \sum_{k=1}^{\infty} \beta_k \phi_k \in \mathcal{D}(A)$, then using $A_D(\beta_1, \beta_2, \dots) = (\lambda_1 \beta_1, \lambda_2 \beta_2, \dots)$ we get

$$SA_D S^{-1} x = SA_D \sum_{k=1}^{\infty} \beta_k e_k = S \sum_{k=1}^{\infty} \lambda_k \beta_k e_k = \sum_{k=1}^{\infty} \lambda_k \beta_k S e_k = \sum_{k=1}^{\infty} \lambda_k \beta_k \phi_k = Ax.$$

This shows that A is similar to the infinite diagonal operator A_D. Since we saw in Example 3.2.2 that the semigroup generated by A_D on $\ell^2(\mathbb{C})$ is given by $T_D(t) = \operatorname{diag}(e^{\lambda_k t})_{k=1}^{\infty}$, Corollary 3.3.5 immediately gives us an expression for the semigroup generated by the operator A on X. In particular, if $x_0 = \sum_{k=1}^{\infty} \beta_k \phi_k \in X$, then

$$T(t)x_0 = ST_D(t)S^{-1}x_0 = ST_D(t) \sum_{k=1}^{\infty} \beta_k e_k = \sum_{k=1}^{\infty} e^{\lambda_k t} \beta_k \phi_k = \sqrt{2} \sum_{k=1}^{\infty} e^{-\alpha k^2 \pi^2 t} \beta_k \sin(k\pi \cdot).$$

The solution of the original partial differential equation is now given by $v(\cdot,t) = x(t) = T(t)x_0$, and the infinite series representation above can be used to easily approximate the solution numerically. Indeed this can be done by simply *truncating* the series representation, i.e., for any $N \in \mathbb{N}$ we can approximate

$$v(\cdot,t) = T(t)x_0 \approx \sqrt{2}\sum_{k=1}^{N} e^{-\alpha k^2\pi^2 t}\beta_k \sin(k\pi\cdot).$$

Since $x_0 = \sum_{k=1}^{\infty} \beta_k\phi_k$, the values $\beta_k \in \mathbb{C}$ are the *coordinates* of x_0 in the Fourier basis $\{\phi_k\}_k$. Since the basis is orthonormal, we can further derive an expression for β_k by observing that for any $l \in \mathbb{N}$ we have

$$\langle x_0, \phi_l\rangle = \sum_{k=1}^{\infty} \beta_k\langle\phi_k,\phi_l\rangle = \sum_{k=1}^{\infty}\beta_k\delta_{kl} = \beta_l.$$

Since $x_0 = v_0(\cdot) \in X$, we have that $\beta_k = \langle x_0, \phi_k\rangle_{L^2} = \int_0^1 v_0(\xi)\phi_k(\xi)d\xi$ for all $k \in \mathbb{N}$.

If we denote the truncated series with N elements by $v_N(\cdot,t)$, then

$$\|v(\cdot,t) - v_N(\cdot,t)\|_{L^2}^2 = \left\|\sum_{k=N+1}^{\infty} e^{-\alpha k^2\pi^2 t}\beta_k\phi(\cdot)\right\|_{L^2}^2 = \sum_{k=N+1}^{\infty} e^{-2\alpha k^2\pi^2 t}|\beta_k|^2 \le \sum_{k=N+1}^{\infty}|\beta_k|^2 \to 0$$

as $N \to \infty$. This means that for every initial condition $v_0(\cdot)$ the numerical approximation converges to the actual solution of the equation uniformly in $t \ge 0$ and in the sense of the L^2-norm in the spatial variable $\xi \in [0,1]$. This would suggest we get better and better numerical approximations for higher values of $N \in \mathbb{N}$. In practice, however, large values of N also lead to an increased amount of numerical errors.

Figure 3.1 shows the approximate solution of the heat equation with parameters $\alpha = 1/10$, $N = 30$ and with an initial condition $v_0(\xi) = 10\xi^3(1-\xi)$ for $\xi \in [0,1]$. The initial condition in particular satisfies $x_0 \in \mathcal{D}(A)$.

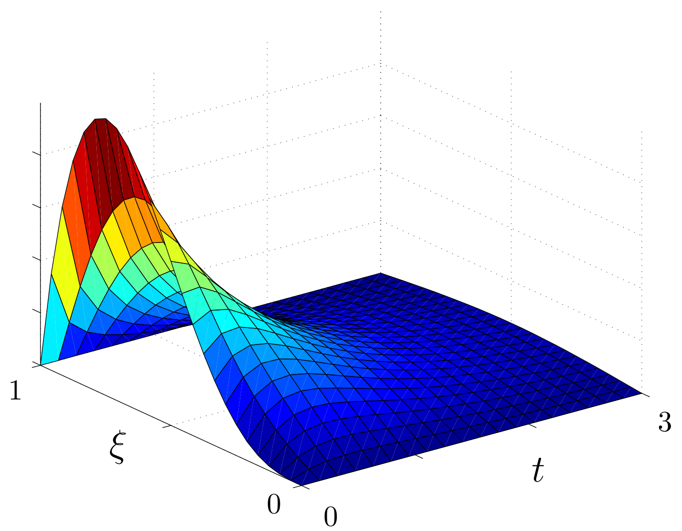

Figure 3.1: Numerical approximation of the solution with $\alpha = 1/10$.

In the above analysis we defined the operator S using the eigenfunctions of the operator A. This connection to the spectrum of A is not a coincidence, as the following lemma illustrates.

Lemma 3.3.7. *Assume $A : \mathcal{D}(A) \subset X \to X$ is similar to a diagonal operator $A_D = \operatorname{diag}(\lambda_k)_{k\in\mathbb{N}}$ in such a way that $A = SA_DS^{-1}$. Then every λ_k is an eigenvalue of A with a corresponding eigenvector $Se_k \in X$, where $e_k = (\delta_{kl})_l \in \ell^2(\mathbb{C})$.*

Proof. Let $k \in \mathbb{N}$. Since clearly e_k is an eigenvector of A_D corresponding to the eigenvalue λ_k, we have

$$A_De_k = \lambda_ke_k \qquad \Leftrightarrow \qquad S^{-1}ASe_k = \lambda_ke_k \qquad \Leftrightarrow \qquad ASe_k = \lambda_kSe_k.$$

Since $Se_k \neq 0$ due to invertibility of S, we have by definition that λ_k is an eigenvalue of A with a corresponding eigenvector $Se_k \in X$. □

3.4 Nonhomogeneous Differential Equations

Semigroups can also be used to study the solutions of *nonhomogeneous* differential equations

$$\dot{x}(t) = Ax(t) + f(t), \qquad x(0) = x_0 \in X \tag{3.7}$$

where A generates a strongly continuous semigroup $T(t)$ on X and $f : [0,\infty) \to X$. In particular, the solution of (3.7) has exactly the same "variation of parameters form" as the solution of a finite-dimensional matrix differential equation of the form (3.7). However, we again need to be more careful in defining what we mean by a "solution" of (3.7). Here we again call $x(\cdot)$ the *classical solution* of (3.7) on $[0,\tau]$ for some $\tau > 0$ if $x(\cdot) \in C^1([0,\tau];X)$, $x(t) \in \mathcal{D}(A)$ for all $t \geq 0$ and (3.7) is satisfied for all $t \in [0,\tau]$. Moreover, the function is a classical solution of (3.7) if it is its classical solution on $[0,\tau]$ for all $\tau > 0$.

Theorem 3.4.1. *Assume A generates a strongly continuous semigroup $T(t)$ on X. If $f \in C^1([0,\tau];X)$ and $x_0 \in \mathcal{D}(A)$, then (3.7) has a unique classical solution given by*

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s)ds, \qquad t \geq 0. \tag{3.8}$$

Proof. See [14, Thm. 10.1.3]. □

Similarly as in the case for homogeneous abstract differential equations, it is often useful to be able to consider weaker forms of solutions of the differential equation (3.7). The *mild solution* of the equation is again defined using the form of the classical solution.

Definition 3.4.2. Assume A generates a strongly continuous semigroup $T(t)$ on X. If $f \in L^1_{\text{loc}}(0,\infty;X)$ and $x_0 \in X$, then the function defined in (3.8) is called the *mild solution* of (3.7).

It is shown in [14, Lem. 10.1.6] that the mild solution $x(\cdot) : [0,\infty) \to X$ is a continuous function.

4. Infinite-Dimensional Linear Control Systems

In this chapter we define the basic properties of an infinite-dimensional control system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0 \in X \tag{4.1a}$$
$$y(t) = Cx(t) + Du(t). \tag{4.1b}$$

on a Banach or a Hilbert space X. To guarantee that the differential equation (4.1a) has a well-defined solution for a suitable class of inputs, we make a standing assumption that the operator A generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ on X.

4.1 Inputs and Outputs

We assume the *input space* U is a finite-dimensional linear vector space, that is, $U = \mathbb{C}^m$ for some $m \in \mathbb{N}$. The control input $u(\cdot)$ is again a function $u(\cdot) : [0, \infty) \to U$. The *output space* is assumed to be $Y = \mathbb{C}^p$ for some $p \in \mathbb{N}$, and $y(\cdot) : [0, \infty) \to Y$, which means that we take p independent measurements from the state of the system.

Definition 4.1.1. If A generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space X, and if $B \in \mathcal{L}(U, X)$, $C \in \mathcal{L}(X, Y)$, and $D \in \mathcal{L}(U, Y)$ for some Hilbert spaces U and Y, then we call (4.1) an *infinite-dimensional linear system*.

Usually in applications we either have $U = Y = \mathbb{C}$, or more generally $U = \mathbb{C}^m$ and $Y = \mathbb{C}^p$ for some $m, p \in \mathbb{N}$. This corresponds to the system having m independent inputs and p independent outputs.

The use of the term "an infinite-dimensional linear system" varies in the literature. On this course we mainly use the above definition to collect our standing assumptions on the operators A, B, C, and D. We assume the operators B, and C are bounded, but sometimes these operators need to be allowed to be unbounded instead. This is the case especially when we would like to consider the control of partial differential equations where the control input and the measurement act through the boundary of the spatial domain. However, the theory for systems with unbounded input and output operators requires certain advanced techniques, and because of this, we concentrate on bounded operators B and C. For more information on more general classes of infinite-dimensional linear systems, see [14, Ch. 11] and [19, 20].

Theorem 4.1.2. *The infinite-dimensional control system* (4.1) *has well-defined mild state* $x(t)$ *and output* $y(t)$ *for every initial state* $x_0 \in X$ *and every input* $u(\cdot) \in L^1_{loc}(0, \infty; U)$.

Proof. If $u(\cdot) \in L^1_{\mathrm{loc}}(0,\infty;U)$ the boundedness of B implies that $Bu(\cdot) \in L^1_{\mathrm{loc}}(0,\infty;U)$, and thus by Definition 3.4.2 the mild solution of the differential equation (4.1a) is given by

$$x(t) = T(t)x_0 + \int_0^t T(t-s)Bu(s)ds, \qquad t \geq 0.$$

Using the formula (4.1b) shows that the output $y(t)$ is given by

$$y(t) = Cx(t) + Du(t) = CT(t)x_0 + \int_0^t CT(t-s)Bu(s)ds + Du(t), \qquad t \geq 0.$$

The state $x(\cdot)$ and the output $y(\cdot)$ of (4.1) are both continuous functions. □

Example 4.1.3. We can now consider adding inputs and outputs to the heat equation considered in Section 3.3.2. If we consider a situation where we have one input and one output (this situation is called *single-input single-output*, or SISO), our partial differential equation is of the form

$$\frac{\partial v}{\partial t}(\xi,t) = \alpha\frac{\partial^2 v}{\partial \xi^2}(\xi,t) + b(\xi)u(t), \qquad \xi \in (0,1) \tag{4.2a}$$

$$v(0,t) = 0, \quad v(1,t) = 0, \quad v(\xi,0) = v_0(\xi), \tag{4.2b}$$

$$y(t) = \int_0^1 v(\xi,t)c(\xi)d\xi + du(t) \tag{4.2c}$$

where the function $b(\cdot) \in L^2(0,1;\mathbb{C})$ describes the way the control input affects the heat distribution of the partial differential equation and $c(\cdot) \in L^2(0,1;\mathbb{C})$ describes how the measurement is taken from the state of the system. In particular, if $c(\cdot) \geq 0$, the integral in the formula for the output is a weighted average of the heat over a part of the domain $[0,1]$. Finally, $d \in \mathbb{C}$ is the feedthrough, i.e., the direct effect of the control input to the output. The value of d depends on the physical situation, and in the absence of feedthrough we simply have $d = 0$.

If we choose $x(t) = v(\cdot,t)$, $X = L^2(0,1)$ and $Af = \alpha f''$ with domain $\mathcal{D}(A) = \{\, f \in X \mid f, f' \text{ abs. cont. } f'' \in X,\ f(0) = f(1) = 0 \,\}$, then the heat equation can be written in the form (4.1) if the operators $B \in \mathcal{L}(\mathbb{C},X)$, $C \in \mathcal{L}(X,\mathbb{C})$ and $D \in \mathcal{L}(\mathbb{C},\mathbb{C})$ are chosen in such a way that

$$\begin{aligned} Bu &= b(\cdot)u, \qquad u \in \mathbb{C} \\ Cx &= \int_0^1 v(\xi)c(\xi)d\xi = \langle v, \overline{c}\rangle_{L^2}, \qquad u \in \mathbb{C} \\ Du &= du, \qquad u \in \mathbb{C}. \end{aligned}$$

We saw in Section 3.3.2 that the operator A is diagonalizable in such a way that $A = SA_DS^{-1}$ for a boundedly invertible $S \in \mathcal{L}(\ell^2(\mathbb{C}), X)$. Writing $z(t) = S^{-1}x(t)$ for all $t \geq 0$, we can write (4.1) in the form

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t), \qquad x(0) = x_0 \in X \\ y(t) &= Cx(t) + Du(t) \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} \dot{x}(t) &= SA_D S^{-1}x(t) + Bu(t), \qquad S^{-1}x(0) = S^{-1}x_0 \in \ell^2(\mathbb{C}) \\ y(t) &= CSS^{-1}x(t) + Du(t) \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} \dot{z}(t) &= A_D z(t) + B_D u(t), \qquad z(0) = S^{-1}x_0 \in Z \\ y(t) &= C_D z(t) + Du(t) \end{cases}$$

This is a new infinite-dimensional linear system on the space $Z = \ell^2(\mathbb{C})$ with state $z(t) \in Z$ and new operators (A_D, B_D, C_D, D), where $A_D = S^{-1}AS = \operatorname{diag}(\lambda_k)_k$ is an infinite diagonal matrix, $B_D = S^{-1}B$, and $C_D = CS$. Let us find out what the new input and output operators look like. Since $b(\cdot) \in X$ and the eigenvectors $\phi_k(\cdot) = \sqrt{2}\sin(k\pi\cdot)$ form a basis of X, we can write $b(\cdot)$ in the form

$$b(\cdot) = \sum_{k=1}^{\infty} b_k \phi_k(\cdot)$$

where $(b_k)_{k\in\mathbb{N}} \subset \ell^2(\mathbb{C})$ are the Fourier coefficients of the function $b(\cdot) \in L^2(0,1)$ in the basis $(\phi_k)_{k\in\mathbb{N}}$. Recalling that we defined the operator S in such a way that $Se_k = \phi_k$ and $S^{-1}\phi_k = e_k$, we can see that for every $u \in \mathbb{C}$

$$S^{-1}Bu = S^{-1}b(\cdot)u = S^{-1}\left(\sum_{k=1}^{\infty} b_k\phi_k\right)u = u\sum_{k=1}^{\infty} b_k S^{-1}\phi_k = u\sum_{k=1}^{\infty} b_k e_k = u \cdot (b_k)_{k\in\mathbb{N}} \in \ell^2(\mathbb{C})$$

Where $(b_k)_{k\in\mathbb{N}}$ is an infinite vector whose elements are the Fourier coefficients of the function $b(\cdot)$. Similarly, we have

$$\overline{c(\cdot)} = \sum_{k=1}^{\infty} c_k \phi_k(\cdot)$$

where $(c_k)_{k\in\mathbb{N}} \in \ell^2(\mathbb{C})$ are the Fourier coefficients of $\overline{c(\cdot)}$, and for any $z = (z_k)_{k\in\mathbb{N}} \in \ell^2(\mathbb{C})$ we have

$$C_D z = CS\sum_{k=1}^{\infty} z_k e_k = \sum_{k=1}^{\infty} z_k CSe_k = \sum_{k=1}^{\infty} z_k C\phi_k = \sum_{k=1}^{\infty} z_k\langle \phi_k, \overline{c}\rangle_{L^2} = \sum_{k=1}^{\infty} \overline{c_k} z_k = \langle z, (c_k)_{k\in\mathbb{N}}\rangle_{\ell^2}$$

since

$$\langle \phi_k, \overline{c}\rangle_{L^2} = \left\langle \phi_k, \sum_{l=1}^{\infty} c_l\phi_l \right\rangle = \sum_{l=1}^{\infty} \overline{c_l}\langle \phi_k, \phi_l\rangle = \overline{c_k}$$

for all $k \in \mathbb{N}$. Thus for all $z \in \ell^2(\mathbb{C})$ we have $C_D z = \langle z, (c_k)_{k\in\mathbb{N}}\rangle$, where $(c_k)_{k\in\mathbb{N}}$ is an infinite vector consisting of the Fourier coefficients of the function $\overline{c(\cdot)}$ in the basis $(\phi_k)_{k\in\mathbb{N}}$. Finally, for the initial state z_0 of the diagonalized system we likewise have that if the initial state $x_0 = v_0(\cdot)$ of the original system satisfies

$$x_0 = v_0(\cdot) = \sum_{k=1}^{\infty} v_{0k}\phi_k(\cdot),$$

then

$$z_0 = S^{-1}x_0 = \sum_{k=1}^{\infty} v_{0k}S^{-1}\phi_k = \sum_{k=1}^{\infty} v_{0k}e_k = (v_{0k})_{k\in\mathbb{N}}.$$

Thus $z_0 = (v_{0k})_{k\in\mathbb{N}} \in Z$ consists of the Fourier coefficients of the initial state $v_0(\cdot)$.

The diagonal representation can again be used to numerically approximate the original infinite-dimensional system (with certain constraints). Indeed, since

$$\left\| b(\cdot) - \sum_{k=1}^{N} b_k\phi_k \right\|_{L^2} \to 0, \qquad \left\| c(\cdot) - \sum_{k=1}^{N} c_k\phi_k \right\|_{L^2} \to 0, \qquad \left\| x_0(\cdot) - \sum_{k=1}^{N} v_{0k}\phi_k \right\|_{L^2} \to 0$$

as $N \to \infty$, we can again *truncate* the infinite vectors $z(t) = (z_k(t))_{k\in\mathbb{N}}$, $(b_k)_{k\in\mathbb{N}}$, $(c_k)_{k\in\mathbb{N}}$ and $z_0 = (v_{0k})_{k\in\mathbb{N}}$ to finite vectors of length $N \in \mathbb{N}$. This way we obtain a finite-dimensional system

$$\begin{aligned} \dot{z}_N(t) &= A_D^N z_N(t) + B_D^N u(t), \qquad z_N(0) = z_{0N} \in \mathbb{C}^N \\ y_N(t) &= C_D^N z(t) + Du(t) \end{aligned}$$

on a finite-dimensional space $Z_N = \mathbb{C}^N$. Here

$$\begin{aligned} &A_D^N = \operatorname{diag}(\lambda_k)_{k=1}^N \in \mathbb{C}^{N\times N} \qquad B_D^N = (b_k)_{k=1}^N \in \mathbb{C}^N \qquad C_D^N = (c_1, \dots, c_N) \in \mathbb{C}^{1\times N} \\ &z_N(t) = (z_k(t))_{k=1}^N \in \mathbb{C}^N \qquad z_{0N} = (v_{0k})_{k=1}^N \in \mathbb{C}^N \end{aligned}$$

Note that $y_N(\cdot)$ is not a truncation, but instead it is still a scalar valued function. However, we denote it this way because it is different from the original input $y(\cdot)$, and it depends on N. The numerical approximation using the truncated series works best when $b(\cdot)$ and $c(\cdot)$ are continuous functions, and when they satisfy the boundary conditions of the original partial differential equation, i.e., when $b(0) = b(1) = 0$ and $c(0) = c(1) = 0$. If the functions are not continuous or the boundary conditions are not satisfied, the numerical approximation is still guaranteed to converge in the L^2-*sense*, meaning that for all $t \geq 0$

$$\|z_N(t) - z(t)\|_{L^2} \to 0, \qquad \text{as } N \to \infty.$$

However, the truncation results in the well-known *Gibbs phenomenon* if the functions $b(\cdot)$, $c(\cdot)$ and $v_0(\cdot)$ are not continuous or if they do not satisfy the boundary conditions. These situations are illustrated in Figure 4.1 which presents a numerical approximation for three different initial states $v_0(\cdot) \in L^2(0,1)$.

Figure 4.2 plots the state and the output of the control system with $\alpha = 1/10$, with input and output operators $b(\cdot) = \chi_{[1/2,1]}(\cdot)$ and $c(\cdot) = \chi_{[0,3/4]}(\cdot)$ and with $d = 0$.

◇

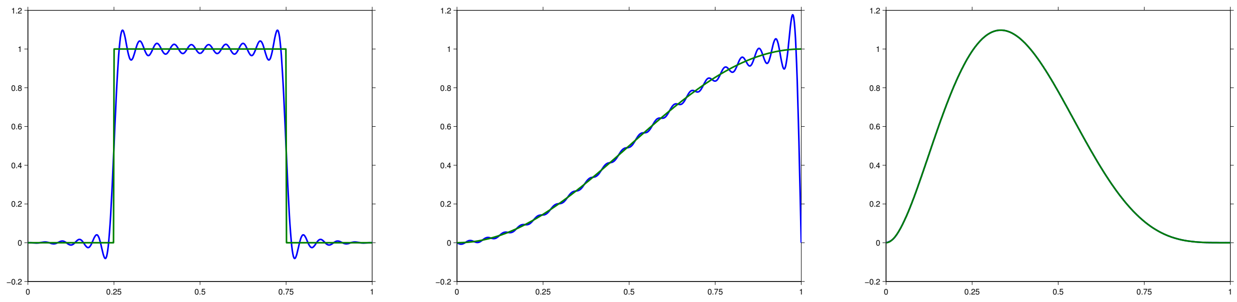

Figure 4.1: Numerical approximations (blue) for three initial states (green) with $N = 40$.

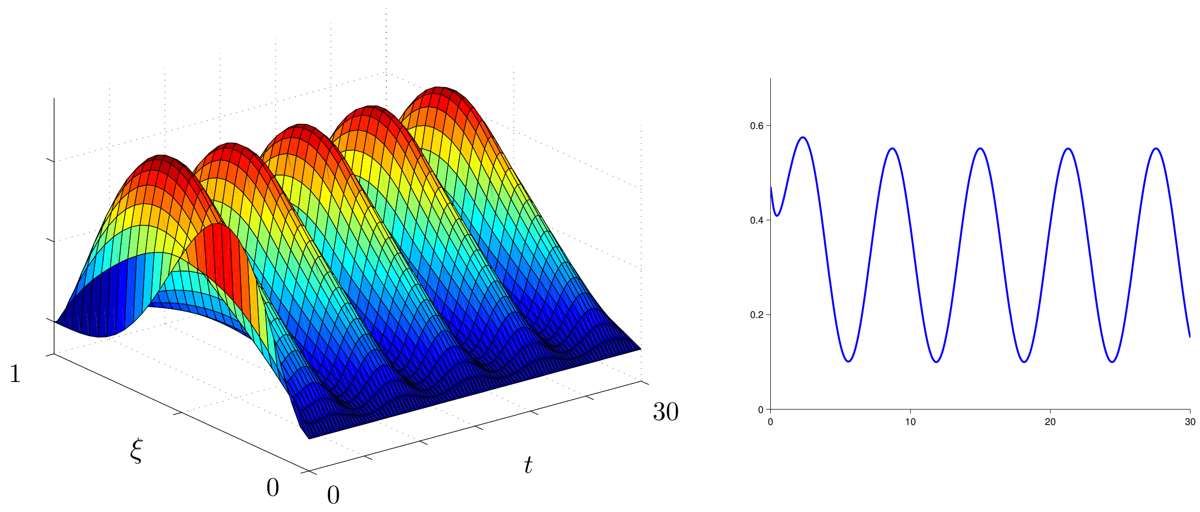

Figure 4.2: Numerical approximation of the solution with $N = 40$.

4.2 Stability of Infinite-Dimensional Systems

In this section we consider some fundamental properties and results on the stability of infinite-dimensional linear systems. In particular, we concentrate on "internal" stability types, i.e., types that are only related to the properties of the semigroup $T(t)$ generated by A. The following three definitions are the main concepts that we study.

Definition 4.2.1. The semigroup $(T(t))_{t\geq 0}$ is called *uniformly bounded* if there exists $M \geq 1$ such that $\|T(t)\| \leq M$ for all $t \geq 0$.

Definition 4.2.2. The system (4.1) is called *strongly stable* (or *asymptotically stable*), if in the case of the constant zero input $u(t) \equiv 0$ the state of the system (4.1) satisfies $x(t) \to 0$ as $t \to \infty$ for all $x_0 \in X$.

Definition 4.2.3. The system (4.1) is called *exponentially stable*, if there exist $\omega > 0$ and $M \geq 1$ such that in the case of the constant zero input $u(t) \equiv 0$ the state of the system (4.1) satisfies

$$\|x(t)\| \leq Me^{-\omega t}\|x_0\|, \qquad \forall t \geq 0, \quad x_0 \in X.$$

Since in Definition 4.2.3 the state $x(t)$ of the system with input $u(t) \equiv 0$ is given by $x(t) = T(t)x_0$, the condition for exponential stability is equivalent to the property that the semigroup satisfies $\|T(t)\| \leq Me^{-\omega t}$. Because of this, it is also common to say that the *semigroup* is exponentially stable, if such $M \geq 1$ and $\omega > 0$ exist. Similarly, the semigroup $(T(t))_{t\geq 0}$ is called *strongly stable* if $T(t)x \to 0$ as $t \to \infty$ for all $x \in X$.

Of all the stability types of semigroups and systems (there are others as well!) exponential stability is the most commonly used and the one that is understood most profoundly. On the other hand, the properties and characterizations for strongly stable semigroups are under active research, see for instance [5, 6, 4].

The following theorem presents some properties of the different stability types. In particular, Gearhart–Greiner–Prüss Theorem in part (d) shows that on a Hilbert space X the exponential stability of a semigroup can be characterized using the *resolvent operator* $R(\lambda, A) = (\lambda - A)^{-1}$ defined for all

$$\lambda \in \rho(A) = \{\, \lambda \in \mathbb{C} \mid (\lambda - A)^{-1} \text{ exists and is bounded} \,\}.$$

In part (d) the notation $i\mathbb{R}$ denotes the imaginary axis, i.e., $i\mathbb{R} = \{\, is \mid s \in \mathbb{R} \,\}$.

Theorem 4.2.4. *Assume A generates a semigroup $T(t)$ on a Banach space X.*

(a) *If $(T(t))_{t\geq 0}$ is uniformly bounded, then $\operatorname{Re}\lambda \leq 0$ for all $\lambda \in \sigma_p(A)$.*

(b) *If $(T(t))_{t\geq 0}$ is strongly stable, then it is uniformly bounded.*

(c) *If $(T(t))_{t\geq 0}$ is strongly stable, then $\operatorname{Re}\lambda < 0$ for all $\lambda \in \sigma_p(A)$.*

(d) *If X is Hilbert and $(T(t))_{t\geq 0}$ is uniformly bounded, then $(T(t))_{t\geq 0}$ is exponentially stable if and only if $i\mathbb{R} \subset \rho(A)$ and*

$$\sup_{s\in\mathbb{R}} \|R(is, A)\| < \infty.$$

(e) *If $(T(t))_{t\geq 0}$ is exponentially stable in such a way that $\|T(t)\| \leq Me^{-\omega t}$ for some $M \geq 1$ and $\omega > 0$, then $\operatorname{Re}\lambda \leq -\omega < 0$ for all $\lambda \in \sigma_p(A)$.*

Proof. If $\lambda \in \sigma_p(A)$ and $\phi \in X$ is such that $A\phi = \lambda\phi$ and $\phi \neq 0$, then it also follows that

$$T(t)\phi = e^{\lambda t}\phi, \qquad \forall t \geq 0.$$

Indeed, if for any $t > 0$ we consider the function $s \mapsto f(s) = e^{\lambda s}T(t-s)\phi$ on $[0, t]$, then

$$\frac{d}{ds}f(s) = \lambda e^{\lambda s}T(t-s)\phi - e^{\lambda s}T(t-s)A\phi = \lambda e^{\lambda s}T(t-s)\phi - e^{\lambda s}T(t-s)\lambda\phi = 0.$$

Thus $f(\cdot)$ is a constant function on $[0,t]$, and in particular $T(t)\phi = f(0) = f(t) = e^{\lambda t}\phi$. Now

$$\|T(t)\phi\| = \|e^{\lambda t}\phi\| = |e^{\lambda t}|\|\phi\| = e^{\operatorname{Re}\lambda t}\|\phi\|, \qquad t \geq 0.$$

Since $\|\phi\| \neq 0$, we immediately have that (i) $\|T(t)\phi\|$ stays bounded for all $t \geq 0$ only if $\operatorname{Re}\lambda \leq 0$, (ii) $\|T(t)\phi\| \to 0$ only if $\operatorname{Re}\lambda < 0$, and finally (iii) $\|T(t)\phi\| \leq Me^{-\omega t}\|\phi\|$ only if $\operatorname{Re}\lambda \leq -\omega$. Thus we get that if $T(t)$ is uniformly bounded, $\operatorname{Re}\lambda \leq 0$ (part (a)), if it is strongly stable, then $\operatorname{Re}\lambda < 0$ (part (c)), and if it is exponentially stable, then $\operatorname{Re}\lambda \leq -\omega < 0$ (part (d)).

Part (b) follows from the *uniform boundedness principle* (also known as the *Banach–Steinhauss Theorem*), which states if $\sup_{t\geq 0}\|T(t)x\| < \infty$ for all $x \in X$, then we also have $\sup_{t\geq 0}\|T(t)\| < \infty$. Here the property $\sup_{t\geq 0}\|T(t)x\| < \infty$ for all $x \in X$ follows from the continuity of $t \to T(t)x$ and the fact that $\|T(t)x\| \to 0$ as $t \to \infty$.

Part (d) is the Gearhart–Greiner–Prüss Theorem [11, Thm. V.3.8], [14, Thm. 8.1.4]. The "only if" part of this theorem remains valid also if X is a Banach space. □

Example 4.2.5. We can now study the stability of the heat equation in Example (4.1.3) (and in Section 3.3.2). The operator A has a representation

$$Ax = \sum_{k=1}^{\infty} \lambda_k \langle x, \phi_k\rangle \phi_k, \qquad x \in \mathcal{D}(A) = \left\{ x \in X \;\middle|\; \sum_{k=1}^{\infty} |\lambda_k|^2 |\langle x, \phi_k\rangle|^2 < \infty \right\}.$$

where $\lambda_k = -\alpha\pi^2 k^2$, $\alpha > 0$, and $\phi_k = \sqrt{2}\sin(\pi k \cdot) \in X$. The semigroup $T(t)$ generated by A is likewise given by the formula

$$T(t)x = \sum_{k=1}^{\infty} e^{\lambda_k t}\langle x, \phi_k\rangle \phi_k, \qquad x \in X, \quad t \geq 0.$$

Our aim is to show that the heat equation is exponentially stable. We already know that the set $\sigma_p(A) = \{\lambda_k\}_{k=1}^{\infty}$ of eigenvalues of A belongs to the "stable" half-plane $\mathbb{C}^-$. This is a necessary condition for exponential stability, but not yet sufficient. However, since the set $(\phi_k)_{k=1}^{\infty}$ is orthonormal, we can estimate $\|T(t)x\|$ by

$$\begin{aligned}
\|T(t)x\|^2 = \langle T(t)x, T(t)x\rangle &= \left\langle \sum_{k=1}^{\infty} e^{\lambda_k t}\langle x, \phi_k\rangle \phi_k, \sum_{l=1}^{\infty} e^{\lambda_l t}\langle x, \phi_l\rangle \phi_l \right\rangle \\
&= \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} e^{\lambda_k t}\langle x, \phi_k\rangle \overline{e^{\lambda_l t}\langle x, \phi_l\rangle}\, \langle \phi_k, \phi_l\rangle = \sum_{k=1}^{\infty} |e^{\lambda_k t}|^2 |\langle x, \phi_k\rangle|^2 \\
&\leq \sup_{k\in\mathbb{N}}\; e^{2\operatorname{Re}\lambda_k t} \sum_{k=1}^{\infty} |\langle x, \phi_k\rangle|^2 \leq e^{-2\alpha\pi^2 t}\|x\|^2.
\end{aligned}$$

The final estimate follows from the fact that $\operatorname{Re}\lambda_k = -\alpha\pi^2 k^2 \leq -\alpha\pi^2$ for all $k \in \mathbb{N}$. The above estimate implies that

$$\|T(t)\| \leq e^{-\alpha\pi^2 t}, \qquad \forall t \geq 0,$$

and thus the condition for exponential stability is satisfied if we choose $\omega = -\alpha\pi^2$ and $M = 1$. ◇

In the above example we saw that the exponential stability and the exponential rate of decay of the heat equation was determined completely by the eigenvalue with the largest real part. This result does also hold more generally for diagonalizable systems. The following theorem shows that if A is diagonalizable, then the exponential stability of the semigroup generated by A can be completely determined by the locations of its eigenvalues. As was already remarked earlier, this is not true for general semigroups.

Theorem 4.2.6. *Assume* $A = SA_DS^{-1}$ *where* $A_D = \operatorname{diag}(\lambda_k)_{k=1}^\infty$ *and* $S \in \mathcal{L}(\ell^2(\mathbb{C}), X)$ *is boundedly invertible. Then the following hold.*

(a) *A generates an exponentially stable semigroup $T(t)$ if and only if there exists $\omega > 0$ such that* $\operatorname{Re}\lambda_k \leq -\omega < 0$ *for all* $k \in \mathbb{N}$.

(b) *A generates a uniformly bounded semigroup $T(t)$ if and only* $\operatorname{Re}\lambda_k \leq 0$ *for all* $k \in \mathbb{N}$.

(c) *A generates a strongly stable semigroup $T(t)$ if and only* $\operatorname{Re}\lambda_k < 0$ *for all* $k \in \mathbb{N}$.

Proof. The necessity of each of the conditions follow from Theorem 4.2.4, and thus it remains to prove that these conditions are also sufficient.

Part (a): If such $\omega > 0$ exists, then A_D generates a semigroup $T_D(t)$ on $\ell^2(\mathbb{C})$. For every $y = (y_k)_{k=1}^\infty \in \ell^2(\mathbb{C})$ we have

$$\|T_D(t)y\|_{\ell^2}^2 = \left\|\sum_{k=1}^\infty e^{\lambda_k t} y_k e_k\right\|^2 = \sum_{k=1}^\infty |e^{\lambda_k t}|^2 |y_k|^2 \leq \sup_{k\in\mathbb{N}} \, e^{2\operatorname{Re}\lambda_k t} \sum_{k=1}^\infty |y_k|^2 \leq e^{-2\omega t}\|y\|_{\ell^2}^2$$

since $|e^{\lambda_k t}|^2 = e^{2\operatorname{Re}\lambda_k t}$ and $\operatorname{Re}\lambda_k \leq -\omega < 0$ for all $t \geq 0$. This implies that $\|T_D(t)\| \leq e^{-\omega t}$ for all $t \geq 0$, and we further have

$$\|T(t)\| = \|ST_D(t)S^{-1}\| \leq \|S\|\|T_D(t)\|\|S^{-1}\| \leq \|S\|\|S^{-1}\|e^{-\omega t}, \qquad \forall t \geq 0.$$

Thus $T_D(t)$ the condition for exponential stability of $(T(t))_{t\geq 0}$ is satisfied if we choose $M = \|S\|\|S^{-1}\|$.

Part (b): If $\operatorname{Re}\lambda_k \leq 0$ for all $k \in \mathbb{N}$, we have similarly as above that for all $t \geq 0$

$$\|T(t)\| \leq \|S\|\|S^{-1}\|\|T_D(t)\| \leq \|S\|\|S^{-1}\| \sup_{k\in\mathbb{N}} e^{\operatorname{Re}\lambda_k t} \leq \|S\|\|S^{-1}\|$$

since $e^{\operatorname{Re}\lambda_k t} \leq e^0 = 1$. Thus $(T(t))_{t\geq 0}$ is uniformly bounded.

Part (c): Assume $\operatorname{Re}\lambda_k < 0$ for all $k \in \mathbb{N}$. We will first show that the diagonal semigroup $(T_D(t))_{t\geq 0}$ is strongly stable. Let $y = (y_k)_{k=1}^\infty \in \ell^2(\mathbb{C})$ be fixed and let $\varepsilon > 0$ be artbitrary. Our aim is to show that there exists $t_0 \geq 0$ such that $\|T_D(t)x\| < \varepsilon$ for all $t \geq t_0$. Since $y \in \ell^2(\mathbb{C})$, there exists $N \in \mathbb{N}$ such that $\sum_{k=N+1}^\infty |y_k|^2 < \varepsilon^2/2$. Moreover, since $e^{-2\operatorname{Re}\lambda_k t} \to 0$ as $t \to \infty$, there exists $t_0 \geq 0$ such that

$$\max_{1\leq k\leq N} e^{-2\operatorname{Re}\lambda_k t} < \frac{\varepsilon^2}{2\|y\|_{\ell^2}^2}, \qquad \forall t \geq t_0.$$

For all $t \geq t_0$ we then have

$$\begin{aligned}\|T_D(t)y\|^2 &= \sum_{k=1}^N e^{-2\operatorname{Re}\lambda_k t}|y_k|^2 + \sum_{k=N+1}^\infty e^{-2\operatorname{Re}\lambda_k t}|y_k|^2 \leq \max_{1\leq k\leq N} e^{-2\operatorname{Re}\lambda_k t} \sum_{k=1}^N |y_k|^2 + \sum_{k=N+1}^\infty |y_k|^2 \\ &< \frac{\varepsilon^2}{2\|y\|^2}\|y\|^2 + \frac{\varepsilon^2}{2} = \varepsilon^2.\end{aligned}$$

Thus $\|T_D(t)y\| < \varepsilon$ for all $t \geq t_0$. This means that $\|T_D(t)y\| \to 0$ as $t \to \infty$, and thus $(T_D(t))_{t\geq 0}$ is strongly stable. Now if $x \in X$, then

$$\|T(t)x\| = \|ST_D(t)S^{-1}x\| \leq \|S\|\|T_D(t)S^{-1}x\| \to 0$$

as $t \to \infty$ since $S^{-1}x \in \ell^2(\mathbb{C})$, and thus $(T(t))_{t\geq 0}$ is strongly stable as well. □

The above proof also implies that if the operator A is itself a diagonal operator, or more generally if $\|S\| = \|S^{-1}\| = 1$, then we can choose $M = 1$ in the condition for exponential stability. The similarity transform S for the heat equation in Examples 4.1.3 and 4.2.5 in particular has the property $\|S\| = \|S^{-1}\| = 1$. This happens because the operator S is an *isometry* between the spaces ℓ^2 and $X = L^2(0,1)$, i.e., $\|Sy\|_X = \|y\|_{\ell^2}$ for all $y \in \ell^2(\mathbb{C})$.

4.3 Controllability and Observability of Infinite-Dimensional Systems

We consider the following controllability concepts.

Definition 4.3.1. Let X be a Banach space and $u(\cdot) : [0,\infty) \to U = \mathbb{C}^m$. For $t > 0$ the *controllability map* $\Phi_t \in \mathcal{L}(L^2(0,t;U),X)$ associated to the system (4.1) is defined as

$$\Phi_t u = \int_0^t T(t-s)Bu(s)ds, \qquad u \in L^2(0,t;U).$$

(a) The system (4.1) is *exactly controllable* (in time $\tau > 0$) if the controllability map satisfies $\mathcal{R}(\Phi_\tau) = X$.

(b) The system (4.1) is *approximately controllable* **in time** $\tau > 0$ if $\mathcal{R}(\Phi_\tau)$ is *dense* in X.

(c) The system (4.1) is *approximately controllable* if $\bigcup_{\tau>0} \mathcal{R}(\Phi_\tau)$ is dense in X.

Here $\mathcal{R}(\Phi_\tau) = \{ x \in X \mid x = \Phi_t u \text{ for some } u \in L^2(0,\tau;U) \}$ is the *range space* of the operator $\Phi_\tau \in \mathcal{L}(L^2(0,\tau;U),X)$. In the condition for approximate controllability, the property that $\mathcal{R}(\Phi_\tau)$ is dense in X means that for every $x \in X$ and for every $\varepsilon > 0$ there exists $y \in \mathcal{R}(\Phi_\tau)$ such that $\|x-y\|_X < \varepsilon$.

The concept of exact controllability corresponds to the controllability for finite-dimensional linear systems. Indeed, if $\mathcal{R}(\Phi_\tau) = X$ for some $\tau > 0$, then for any $x_0 \in X$ and $x_1 \in X$ there exists $u \in L^2(0,\tau;U)$ such that $\Phi_\tau u = x_1 - T(\tau)x_0$. This means that with this input the state $x(\cdot)$ of the system (4.1) at time τ satisfies

$$x(\tau) = T(\tau)x_0 + \int_0^\tau T(\tau-s)Bu(s)ds = T(\tau)x_0 + \Phi_\tau u = x_1.$$

This means that for any initial state $x_0 \in X$ and every target state $x_1 \in X$ we can find an input $u(\cdot)$ that steers the state of the system from x_0 to x_1 in time τ. However, exact controllability is not a common property in infinite-dimensional control theory. In fact, it is shown in [9, Thm. 4.1.5] that if X is infinite-dimensional and the values of the control function $u(t)$ are finite-dimensional vectors, i.e., $u : [0,\infty) \to \mathbb{C}^m$ for some $m \in N$, then the system (4.1) is not exactly controllable. However, it should be mentioned that exact controllability for infinite-dimensional systems does appear naturally in connection with control from acting on the boundaries of partial differential equations.

Approximate controllability, on the other hand, means that we can steer from any initial state (either in some specific time $\tau > 0$ or without such restrictions) to *arbitrarily close* to any given target state. These properties of inifinite-dimensional systems are much more common than exact controllability.

The observability of a system means that the output of the system completely determines the initial state of the system (4.1).

Definition 4.3.2. Let X be a Banach space. For $t > 0$ the *observability map* $\Psi_t \in \mathcal{L}(X, L^2(0,t;Y))$ associated to the system (4.1) is so that

$$\Psi_t x_0 = CT(\cdot)x_0 \in L^2(0,t;Y), \qquad x_0 \in X.$$

(a) The system (4.1) is *exactly observable* (in time $\tau > 0$) if there exists $c > 0$ such that $\|\Psi_\tau x_0\| \geq c\|x_0\|$ for all $x_0 \in X$.

(b) The system (4.1) is *approximately observable* **in time** $\tau > 0$ if $\Psi_\tau x_0 = 0$ implies $x_0 = 0$.

(c) The system (4.1) is *approximately observable* if $\Psi_\tau x_0 = 0$ for all $\tau > 0$ implies $x_0 = 0$.

Definition 4.3.2 indeed implies that the output of the system uniquely determines the initial state $x_0 \in X$ of the system. Indeed, if for some control input $u \in L^1_{\text{loc}}(0,\infty;U)$ and two initial states $x_0^1 \in X$ and $x_0^2 \in X$ the system (4.1) produces the outputs $y_1(\cdot)$ and $y_2(\cdot)$ such that $y_1(t) = y_2(t)$ for all $t \geq 0$, then

$$\begin{aligned} 0 = y_1(t) - y_2(t) &= CT(t)x_0^1 + \int_0^t T(t-s)Bu(s)ds - \left(CT(t)x_0^2 + \int_0^t T(t-s)Bu(s)ds\right) \\ &= CT(t)(x_0^1 - x_0^2) \end{aligned}$$

for all $t \geq 0$, and approximate controllability of the system implies that necessarily $x_0^1 = x_0^2$.

The following theorem shows that controllability and observability for a system on a Hilbert space X are *dual* concepts in the sense that the controllability of a system (A,B,C,D) is equivalent to the observability of the *dual system* (A^*,C^*,B^*,D^*). As is shown in [9, Sec. 2.2], on a Hilbert space X the operator A^* generates a strongly continuous semigroup $(T(t)^*)_{t\geq 0}$. We only prove the duality result for approximate controllability. For the corresponding results for exact controllability and approximate controllability in time $\tau > 0$ see [9, Lem. 4.1.13].

Theorem 4.3.3. *Assume A generates a semigroup $T(t)$ on a Hilbert space X and $B \in \mathcal{L}(U,X)$. The system (A,B,C,D) is approximately controllable if and only if (A^*,C^*,B^*,D^*) is approximately observable.*

Proof. The property that a set $Y \subset X$ is dense in a Hilbert space X is equivalent to the property that if $\langle x,y\rangle = 0$ for some $x \in X$ and for every $y \in Y$, then necessarily $x = 0$.

The approximate controllability of (A,B,C,D), i.e., the property that $\bigcup_{\tau>0}\mathcal{R}(\Phi_\tau)$ is dense in X, is therefore equivalent to the property that

$$\text{If} \quad \langle \Phi_\tau u, x\rangle_X = 0 \quad \text{for all} \quad \tau > 0 \quad \text{and} \quad u \in L^2(0,\tau;U), \qquad \text{then} \quad x = 0.$$

We want to show that this is equivalent to the approximate observability of (A^*, C^*, B^*, D^*), which means that for every $x \in X$

$$B^*T(t)^*x = 0 \quad \forall t \geq 0 \qquad \text{only if} \quad x = 0.$$

For all $\tau > 0$, $u \in L^2(0, \tau; U)$ and $x \in X$ we have

$$\begin{aligned}\langle \Phi_\tau u, x\rangle_X &= \left\langle \int_0^\tau T(\tau - s)Bu(s)ds, x \right\rangle_X = \int_0^\tau \langle T(\tau - s)Bu(s), x\rangle_X ds \\ &= \int_0^\tau \langle u(s), B^*T(\tau - s)^*x\rangle_U ds = \int_0^\tau \langle v(\tau - s), B^*T(\tau - s)^*x\rangle_U ds = \langle v, B^*T(\cdot)^*x\rangle_{L^2}\end{aligned}$$

where we have denoted $v \in L^2(0, \tau; U)$ such that $v(\cdot) = u(\tau - \cdot)$. Since the function $t \to B^*T(t)^*x$ is continuous and since $L^2(0, \tau; U)$ is a Hilbert space, we have that the property

$$\langle v, B^*T(\cdot)^*x\rangle_{L^2} = 0 \quad \text{for all} \quad \tau > 0 \quad \text{and} \quad v \in L^2(0, \tau; U)$$

is equivalent to $B^*T(t)^*x = 0$ for all $t \geq 0$. Indeed, the necessity of this condition can be seen conveniently seen by choosing $v = B^*T(\cdot)^*x \in \mathcal{L}(0, \tau; U)$, in which case we have

$$0 = \langle v, B^*T(\cdot)^*x\rangle_{L^2} = \langle B^*T(\cdot)^*x, B^*T(\cdot)^*x\rangle_{L^2} = \int_0^\tau \|B^*T(s)^*x\|^2 ds,$$

which implies $B^*T(t)^*x = 0$ for all $t \geq 0$ since the integrand is continuous. Combining the above properties shows that the claim of the theorem holds. □

The following result shows that approximate controllability of a system where A diagonalizable can be tested quite easily.

Theorem 4.3.4. *Assume A generates a semigroup $T(t)$ and is diagonalizable in such a way that $A = SA_DS^{-1}$ where $A_D = \operatorname{diag}(\lambda_k)_{k\in\mathbb{N}}$ are such that $\lambda_k \neq \lambda_l$ for all $k \neq l$. Let $B = (b_1, \ldots, b_m) \in \mathcal{L}(\mathbb{C}^m, X)$. The system (A, B, C, D) is approximately controllable if and only if*

$$(\langle b_1, \psi_k\rangle, \ldots, \langle b_m, \psi_k\rangle) \neq 0 \qquad \forall k \in \mathbb{N}, \tag{4.3}$$

where $\psi_k = (S^{-1})^*e_k$ are eigenvectors of A^* associated to its eigenvalues $\overline{\lambda_k}$.

The result also has a more general version where the eigenvalues λ_k of A are allowed to have *finite multiplicity*, i.e., for every λ_k there exists at most finite indices $l \in \mathbb{N}$ for which $\lambda_l = \lambda_k$, see [9, Thm. 4.2.1].

Proof of Theorem 4.3.4. We only prove the result in the case where λ_k are real. For the more general situation, see [9, Thm. 4.2.3]. By possibly rearranging the indexing we can assume that $\lambda_1 > \lambda_2 > \cdots$. Note that since $B \in \mathcal{L}(\mathbb{C}^m, X)$ by assumption and for any $u = (u_1, \ldots, u_m)^T \in \mathbb{C}^m$ and $x \in X$ we have $Bu = \sum_{k=1}^m b_k u_k \in X$ and

$$\langle Bu, x\rangle_X = \left\langle \sum_{k=1}^m b_k u_k, x \right\rangle_X = \sum_{k=1}^m u_k \langle b_k, x\rangle_X = \sum_{k=1}^m u_k \overline{\langle x, b_k\rangle_X} = \langle u, (\langle x, b_1\rangle, \ldots, \langle x, b_m\rangle)^T\rangle_U,$$

we have by definition that the adjoint B^* of B is an operator $B^* \in \mathcal{L}(X, \mathbb{C}^m)$ given by

$$B^*x = \begin{pmatrix} \langle x, b_1 \rangle_X \\ \vdots \\ \langle x, b_m \rangle_X \end{pmatrix} \in \mathbb{C}^m.$$

As shown in the exercises, the semigroup $T(t)$ has the form

$$T(t)x = \sum_{k=1}^{\infty} e^{\lambda_k t} \langle x, \psi_k \rangle \phi_k, \qquad \forall x \in X, \ t \geq 0,$$

where $\psi_k = (S^{-1})^* e_k$ are the eigenvectors of A^*. Since λ_k were assumed to be real, a direct computation shows that $T(t)^*$ are given by

$$T(t)^*x = \sum_{k=1}^{\infty} e^{\lambda_k t} \langle x, \phi_k \rangle \psi_k, \qquad \forall x \in X, \ t \geq 0,$$

If $x \in X$ is such that $B^*T(t)^*x = 0$ for all $t \geq 0$, then also $e^{-\lambda_1 t} B^*T(t)^*x = 0$ for all $t \geq 0$, and

$$\begin{aligned} 0 = e^{-\lambda_1 t} B^*T(t)^*x = e^{-\lambda_1 t} \sum_{k=1}^{\infty} e^{\lambda_k t} \langle x, \phi_k \rangle B^*\psi_k = B^*\psi_1 \langle x, \phi_1 \rangle + \sum_{k=2}^{\infty} e^{(\lambda_k - \lambda_1)t} \langle x, \phi_k \rangle B^*\psi_k \\ \to B^*\psi_1 \langle x, \phi_1 \rangle \end{aligned}$$

as $t \to \infty$. This is due to the fact that the assumption $\lambda_1 > \lambda_2 > \cdots$ implies $\lambda_k - \lambda_1 \leq \lambda_2 - \lambda_1 < 0$ for all $k \geq 2$, and thus

$$\begin{aligned} \left\| \sum_{k=2}^{\infty} e^{(\lambda_k - \lambda_1)t} \langle x, \phi_k \rangle B^*\psi_k \right\| &\leq e^{(\lambda_2 - \lambda_1)t} \sum_{k=2}^{\infty} \| \langle x, \phi_k \rangle B^*\psi_k \| \\ &\leq e^{(\lambda_2 - \lambda_1)t} \left(\sum_{k=2}^{\infty} |\langle x, \phi_k \rangle|^2 \right)^{\frac{1}{2}} \left(\sum_{k=2}^{\infty} \|B^*\psi_k\|^2 \right)^{\frac{1}{2}} \to 0 \end{aligned}$$

as $t \to \infty$ since the two infinite series are convergent and independent of t (for details, see [9]). We have thus shown that if $B^*T(\cdot)^*x \equiv 0$, then necessarily $B^*\psi_1 \langle x, \phi_1 \rangle = 0$, and similarly we have that

$$0 = e^{\lambda_2 t} B^*T(t)^*x = B^*\psi_2 \langle x, \phi_2 \rangle + \sum_{k=3}^{\infty} e^{(\lambda_k - \lambda_2)t} \langle x, \phi_k \rangle B^*\psi_k \quad \longrightarrow \quad B^*\psi_2 \langle x, \phi_2 \rangle$$

which in turn implies $B^*\psi_2 \langle x, \phi_2 \rangle = 0$. Continuing this procedure we can show that $B^*T(\cdot)^*x \equiv 0$ if and only if

$$B^*\psi_k \langle x, \phi_k \rangle = 0, \qquad k \in \mathbb{N}.$$

The pair (A, B) is controllable if and only if this condition implies that $x = 0$. However, this is true precisely if $B^*\psi_k \neq 0$ for all $k \in \mathbb{N}$, since

$$x = 0 \qquad \Leftrightarrow \qquad S^{-1}x = 0 \qquad \Leftrightarrow \qquad \langle S^{-1}x, e_k \rangle = 0 \quad \forall k \qquad \Leftrightarrow \qquad \langle x, \psi_k \rangle = 0 \quad \forall k$$

since $\psi_k = (S^{-1})^* e_k$. The proof is concluded by noting that $B^*\psi_k \neq 0$ for all $k \in \mathbb{N}$ is equivalent to (4.3) since

$$B^*\psi_k = \begin{pmatrix} \langle \psi_k, b_1 \rangle_X \\ \vdots \\ \langle \psi_k, b_m \rangle_X \end{pmatrix}$$

for all $k \in \mathbb{N}$. □

The duality between controllability and observability allows us to use the condition in Theorem 4.3.4 to also test for the approximate observability of the system, which is equivalent to the approximate controllability of the dual system (A^*, B^*, C^*, D^*). If $C \in \mathcal{L}(X, \mathbb{C}^p)$ is such that

$$Cx = \begin{pmatrix} \langle x, c_1 \rangle \\ \vdots \\ \langle x, c_p \rangle \end{pmatrix} \in \mathbb{C}^p, \qquad x \in X,$$

we can see similarly as in the case of the operator B in the beginning of the proof of Theorem 4.3.4 that

$$C^* = \big(c_1, \ldots, c_p\big) \in \mathcal{L}(\mathbb{C}^p, X),$$

Since $A^* = (S^{-1})^* A_D^* S^*$ (requires some more careful analysis, but it does indeed hold) the test for observability of diagonalizable systems has the following form.

Corollary 4.3.5. *Assume A generates a semigroup $T(t)$ and is diagonalizable in such a way that $A = SA_D S^{-1}$ where $A_D = \operatorname{diag}(\lambda_k)_{k\in\mathbb{N}}$ are such that $\lambda_k \neq \lambda_l$ for all $k \neq l$. Let $C \in \mathcal{L}(X, \mathbb{C}^p)$ such that $Cx = (\langle x, c_1 \rangle, \ldots, \langle x, c_p \rangle)^T \in \mathbb{C}^p$ for all $x \in X$. The system (A, B, C, D) is approximately observable if and only if*

$$(\langle c_1, \phi_k \rangle, \ldots, \langle c_p, \phi_k \rangle) \neq 0 \qquad \forall k \in \mathbb{N}, \tag{4.4}$$

where $\phi_k = Se_k$ are eigenvectors of A associated to its eigenvalues λ_k.

Example 4.3.6. We want to study the controllability and observability of the controlled heat equation studied in Example 4.1.3

$$\begin{aligned} \frac{\partial v}{\partial t}(\xi, t) &= \alpha \frac{\partial^2 v}{\partial \xi^2}(\xi, t) + b(\xi)u(t), \qquad \xi \in (0,1) \\ v(0) &= 0, \quad v(1) = 0, \quad v(\xi, 0) = v_0(\xi), \\ y(t) &= \int_0^1 v(\xi, t)c(\xi)d\xi + du(t) \end{aligned}$$

The operators $B \in \mathcal{L}(\mathbb{C}, X)$ and $C \in \mathcal{L}(X, \mathbb{C})$ are of the form

$$\begin{aligned} Bu &= b(\cdot)u, \qquad u \in \mathbb{C} \\ Cx &= \int_0^1 v(\xi)c(\xi)d\xi = \langle v, \overline{c} \rangle_{L^2}, \qquad u \in \mathbb{C} \end{aligned}$$

with given $b, c \in X$. Since the set $(\phi_k)_{k\in\mathbb{N}} = (\sqrt{2}\sin(k\pi\cdot))_{k\in\mathbb{N}}$ of eigenfunctions of A is orthonormal and its eigenvalues $\lambda_k = -\alpha\pi^2k^2$ are all real, the biorthonormal seqence $(\psi_k)_{k\in\mathbb{N}}$ of $(\phi_k)_{k\in\mathbb{N}}$ is the sequence itself, i.e., $\psi_k = \phi_k$. We have from Theorem 4.3.4 and Corollary 4.3.5 that the controlled heat equation is approximately controllable if and only if

$$\langle b, \phi_k\rangle_{L^2} \neq 0 \qquad \forall k \in \mathbb{N}$$

and it is approximately observable if and only if

$$\langle c, \phi_k\rangle_{L^2} \neq 0 \qquad \forall k \in \mathbb{N}.$$

In the following we will only consider approximate controllability, as approximate observability can be studied analogously.

Consider first the situation where $b(\cdot) \equiv 1$, i.e., $b(\xi) = 1$ for all $\xi \in [0, 1]$. A direct computation yields

$$\langle b, \phi_k\rangle_{L^2} = \sqrt{2}\int_0^1 b(\xi)\sin(k\pi\xi)d\xi = \sqrt{2}\int_0^1 \sin(k\pi\xi)d\xi = \sqrt{2}\frac{1-\cos(k\pi)}{k\pi} = \begin{cases} 0 & k \text{ is even} \\ \frac{2\sqrt{2}}{k\pi} & k \text{ is odd} \end{cases}$$

Thus $\langle b, \phi_k\rangle = 0$ for every even index k, and by Theorem 4.3.4 the heat equation is not approximately controllable.

Consider now the function $b(\cdot) = 2\xi_{[0,1/2]}(\cdot)$, i.e., $b(\xi) = 2$ for $\xi \in [0, 1/2]$ and zero otherwise. We have

$$\langle b, \phi_k\rangle_{L^2} = \sqrt{2}\int_0^1 b(\xi)\sin(k\pi\xi)d\xi = 2\sqrt{2}\int_0^{1/2} \sin(k\pi\xi)d\xi = 2\sqrt{2}\frac{1-\cos(k\pi/2)}{k\pi}$$

which is zero whenever $\cos(k\pi/2) = 1$, or equivalently $k = 4\pi$. This means that the heat equation is again not approximately controllable. More generally, if we let $b(\cdot) = \frac{1}{2\varepsilon}\xi_{[\xi_0-\varepsilon,\xi_0+\varepsilon]}(\cdot)$ (the constant $1/(2\varepsilon)$ is chosen to guarantee $\|b\|_{L^2} = 1$), then

$$\langle b, \phi_k\rangle_{L^2} = \frac{\sqrt{2}}{2\varepsilon}\int_{\xi_0-\varepsilon}^{\xi_0+\varepsilon} \sin(k\pi\xi)d\xi = \frac{\sqrt{2}}{k\pi\varepsilon}\sin(k\pi\varepsilon)\sin(k\pi\xi_0).$$

This shows that the heat equation is approximately controllable whenever we choose $\xi_0 \in (0, 1)$ and $\varepsilon > 0$ in such a way that $\sin(k\pi\varepsilon) \neq 0$ and $\sin(k\pi\xi_0) \neq 0$ for all $k \in \mathbb{N}$. This happens whenever both ξ_0 and $\varepsilon > 0$ are irrational numbers. $\diamond$

4.4 The Controlled Wave Equation

In this section we will formulate the one-dimensional wave equation that we already briefly discussed in Section 1.4.2 as an infinite-dimensional linear system. In particular we consider the partial differential equation

$$\frac{\partial^2 w}{\partial t^2}(\xi, t) = \alpha^2\frac{\partial^2 w}{\partial \xi^2}(\xi, t) + b(\xi)u(t), \qquad \xi \in (0, 1) \tag{4.5a}$$

$$w(0) = 0, \quad w(1) = 0, \tag{4.5b}$$

$$w(\xi, 0) = w_0(\xi), \quad \frac{\partial w}{\partial t}(\xi, 0) = w_1(\xi) \tag{4.5c}$$

$$y(t) = \int_0^1 w(\xi, t)c(\xi)d\xi, \qquad t \geq 0 \tag{4.5d}$$

where $\alpha > 0$ is the wave speed. The solution $w(\xi, t)$ of the equation describes the displacement of an undamped string on the interval $\xi \in [0, 1]$ at time $t \geq 0$. Here $w(0) = 0$ and $w(1) = 0$ are the Dirichlet boundary conditions of the equation and $w(\xi, 0) = w_0(\xi)$ and $\frac{dw}{dt}(\xi, 0) = w_1(\xi)$ describe the initial state of the system. It was mentioned in Chapter 1 that the wave equation can not be formulated as a linear system on its "most obvious" choice of a state space, $X = L^2(0,1) \times L^2(0,1)$ (see [9, Exer. 2.25]). Instead, we need to do some tricks to guarantee that our operator A will generate a semigroup.

We begin by defining an operator $A_0 : \mathcal{D}(A) \subset L^2(0,1) \to L^2(0,1)$ in such a way that

$$A_0 f = -\alpha^2 \frac{d^2 f}{d\xi^2}, \qquad f \in \mathcal{D}(A_0) = \{\, f \in X \mid f, f' \text{ abs. cont. } f'' \in X,\ f(0) = f(1) = 0 \,\}.$$

Note that this is the same operator (only multiplied by -1) as the operator A in the heat equation example. Because of this, we also know the eigenvalues and eigenfunctions of the operator A_0. In particular, the operator A_0 has a representation

$$A_0 f = \sum_{k=1}^{\infty} \mu_k \langle f, \phi_k \rangle \phi_k, \qquad f \in \mathcal{D}(A_0) = \Big\{\, f \in L^2(0,1) \;\Big|\; \sum_{k=1}^{\infty} |\mu_k|^2 |\langle f, \phi_k \rangle|^2 < \infty \,\Big\}.$$

where $\mu_k = \alpha^2 \pi^2 k^2$, $\alpha > 0$, and $\phi_k = \sqrt{2}\sin(\pi k \cdot) \in X$.

We choose the state $x(t)$ of the linear infinite-dimensional system as $x(t) = (w(\cdot, t), w_t(\cdot, t))$, where $w(\cdot, t)$ is the profile of the string at time $t \geq 0$ and $w_t(\cdot, t) = \frac{\partial w}{\partial t}(\cdot, t)$. For this state we can use the equation (4.5a) formally write

$$\begin{aligned} \dot{x}(t) &= \frac{d}{dt}\begin{pmatrix} w(\cdot,t) \\ w_t(\cdot,t) \end{pmatrix} = \begin{pmatrix} w_t(\cdot,t) \\ w_{tt}(\cdot,t) \end{pmatrix} = \begin{pmatrix} w_t(\cdot,t) \\ \alpha^2 w_{\xi\xi}(\cdot,t) + b(\cdot)u(t) \end{pmatrix} \\ &= \begin{pmatrix} 0 & I \\ -A_0 & 0 \end{pmatrix}\begin{pmatrix} w(\cdot,t) \\ w_t(\cdot,t) \end{pmatrix} + \begin{pmatrix} 0 \\ b(\cdot) \end{pmatrix} u(t) = \begin{pmatrix} 0 & I \\ -A_0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ b(\cdot) \end{pmatrix} u(t). \end{aligned}$$

From this computation we can see that in our linear system the system operator A and the input operator should be chosen as

$$A = \begin{pmatrix} 0 & I \\ -A_0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ b(\cdot) \end{pmatrix}.$$

However, we still need to fix the state space X of the system and the domain $\mathcal{D}(A)$ of the operator A.

4.4.1 The state space $X = \mathcal{D}(A_0^{1/2}) \times L^2(0,1)$ and The Domain of A

Since the eigenvalues of the operator A_0 satisfy $\mu_k > \mu_1 = \alpha^2\pi^2 > 0$ for all $k \in \mathbb{N}$ we can define the *square root* $A_0^{1/2}$ of A_0 with the formula

$$A_0^{1/2} f = \sum_{k=1}^{\infty} \sqrt{\mu_k} \langle f, \phi_k \rangle \phi_k, \qquad f \in \mathcal{D}(A_0^{1/2}) = \Big\{\, f \in L^2(0,1) \;\Big|\; \sum_{k=1}^{\infty} \mu_k |\langle f, \phi_k \rangle|^2 < \infty \,\Big\}.$$

The operator $A_0^{1/2}$ is indeed a square root of A_0 in the sense that for every $f \in \mathcal{D}(A_0)$ we have $A_0^{1/2} f \in \mathcal{D}(A_0^{1/2})$ and $A_0^{1/2}(A_0^{1/2} f) = A_0 f$ as one would expect. We can define an inner product $\langle \cdot, \cdot \rangle_{\mathcal{D}(A_0^{1/2})}$ on the space $\mathcal{D}(A_0^{1/2})$ by

$$\langle f, g \rangle_{\mathcal{D}(A_0^{1/2})} = \langle A_0^{1/2} f, A_0^{1/2} g \rangle_{L^2}, \qquad f, g \in \mathcal{D}(A_0^{1/2}).$$

This inner product satisfies

$$\begin{aligned}\langle f, g \rangle_{\mathcal{D}(A_0^{1/2})} &= \langle A_0^{1/2} f, A_0^{1/2} g \rangle_{L^2} = \left\langle \sum_{k=1}^{\infty} \sqrt{\mu_k} \langle f, \phi_k \rangle \phi_k, \sum_{l=1}^{\infty} \sqrt{\mu_l} \langle g, \phi_l \rangle \phi_l \right\rangle_{L^2} \\ &= \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \sqrt{\mu_k} \sqrt{\mu_l} \langle f, \phi_k \rangle \overline{\langle g, \phi_l \rangle} \langle \phi_k, \phi_l \rangle_{L^2} = \sum_{k=1}^{\infty} \mu_k \langle f, \phi_k \rangle \langle \phi_k, g \rangle\end{aligned}$$

since $\langle \phi_k, \phi_l \rangle_{L^2} = \delta_{kl}$. The inner product induces a norm $\|\cdot\|_{\mathcal{D}(A_0^{1/2})}$ satisfying

$$\|f\|^2_{\mathcal{D}(A_0^{1/2})} = \langle f, f \rangle_{\mathcal{D}(A_0^{1/2})} = \sum_{k=1}^{\infty} \mu_k |\langle f, \phi_k \rangle|^2,$$

and $\mathcal{D}(A_0^{1/2})$ is a Hilbert space with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{D}(A_0^{1/2})}$.

Definition 4.4.1. We choose the state space of our infinite-dimensional linear system as

$$X = \mathcal{D}(A_0^{1/2}) \times L^2(0,1).$$

The elements $f \in X$ are of the form $f = (f_1, f_2)^T$, where $f_1 \in \mathcal{D}(A_0^{1/2})$ and $f_2 \in L^2(0,1)$. The space X is a Hilbert space with inner product defined by

$$\langle f, g \rangle_X = \langle f_1, g_1 \rangle_{\mathcal{D}(A_0^{1/2})} + \langle f_2, g_2 \rangle_{L^2}$$

for all $f = (f_1, f_2)^T \in X$ and $g = (g_1, g_2)^T \in X$.

We now want to choose the domain $\mathcal{D}(A)$ in such a way that if $f = (f_1, f_2)^T \in \mathcal{D}(A)$, then $Af \in X$. We choose

$$\mathcal{D}(A) = \mathcal{D}(A_0) \times \mathcal{D}(A_0^{1/2}),$$

which consists of elements $f = (f_1, f_2)^T$ where $f_1 \in \mathcal{D}(A_0)$ and $f_2 \in \mathcal{D}(A_0^{1/2})$. Then a direct computation using the form $A = \left(\begin{smallmatrix} 0 & I \\ A_0 & 0 \end{smallmatrix}\right)$ we have that for any $f = (f_1, f_2)^T \in \mathcal{D}(A)$ we have

$$\begin{pmatrix} 0 & I \\ A_0 & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} f_2 \\ A_0 f_1 \end{pmatrix} \in X = \mathcal{D}(A_0^{1/2}) \times L^2(0,1)$$

since $f_2 \in \mathcal{D}(A_0)$ and $Af_1 \in L^2(0,1)$.

4.4.2 The Properties of The Operator A

We will now show that the operator A is diagonalizable. We begin by finding the eigenvalues and eigenvectors of A. If $\varphi = (\varphi^1, \varphi^2) \in \mathcal{D}(A) = \mathcal{D}(A_0) \times \mathcal{D}(A_0^{1/2})$ is an eigenvector corresponding to the eigenvalue $\lambda \in \mathbb{C}$ of A, then

$$A\varphi = \lambda\varphi \qquad \Leftrightarrow \qquad \begin{pmatrix} 0 & I \\ -A_0 & 0 \end{pmatrix} \begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix} = \lambda \begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix} \qquad \Leftrightarrow \qquad \begin{pmatrix} \varphi^2 \\ -A_0\varphi^1 \end{pmatrix} = \begin{pmatrix} \lambda\varphi^1 \\ \lambda\varphi^2 \end{pmatrix}$$

which shows that $-A_0\varphi^1 = \lambda\varphi^2 = \lambda^2\varphi^1$, and thus φ_1 is an eigenfunction of $-A_0$ corresponding to the eigenvalue λ^2 of $-A_0$. Since we know that $-A_0$ has eigenvalues $\mu_k = -\alpha^2 k^2 \pi^2$ with corresponding eigenvectors $\phi_k = \sqrt{2}\sin(k\pi\cdot)$, we see that if $\lambda_k = i\alpha\pi k$ for $k \in \mathbb{Z} \setminus \{0\}$, then $\lambda_k^2 = -\alpha^2\pi^2k^2 = \mu_{\pm k}$ are eigenvalues of $-A_0$ with corresponding eigenfunctions $\phi_k = \sqrt{2}\sin(k\pi\cdot)$. Because of this, the eigenvector of A corresponding to the eigenvalue $\lambda_k = i\alpha\pi k$ should then be

$$\begin{pmatrix} \phi_k \\ \lambda_k\phi_k \end{pmatrix} = \sqrt{2} \begin{pmatrix} \sin(k\pi\cdot) \\ \lambda_k \sin(k\pi\cdot) \end{pmatrix}.$$

Indeed, a direct computation shows that for $k \in \mathbb{Z}$, $\lambda_k = i\alpha k\pi$ the vector $\varphi_k = \left(\begin{smallmatrix} \phi_k \\ \lambda_k\phi_k \end{smallmatrix}\right)$ satisfies

$$\begin{pmatrix} 0 & I \\ -A_0 & 0 \end{pmatrix} \begin{pmatrix} \phi_k \\ \lambda_k\phi_k \end{pmatrix} = \begin{pmatrix} \lambda_k\phi_k \\ -A_0\phi_k \end{pmatrix} = \begin{pmatrix} \lambda_k\phi_k \\ \lambda_k^2\phi_k \end{pmatrix} = \lambda_k \begin{pmatrix} \phi_k \\ \lambda_k\phi_k \end{pmatrix},$$

since $-A_0\phi_k = \lambda_k^2\phi_k$. Finally, we want to normalize the eigenvectors of A. A direct computation shows that

$$\begin{aligned} \left\| \begin{pmatrix} \phi_k \\ \lambda_k\phi_k \end{pmatrix} \right\|_X^2 &= \|\phi_k\|_{\mathcal{D}(A_0^{1/2})}^2 + \|\lambda_k\phi_k\|_{L^2}^2 = \langle A_0^{1/2}\phi_k, A_0^{1/2}\phi_k\rangle_{L^2} + |\lambda_k|^2\|\phi_k\|_{L^2}^2 \\ &= \sum_{l=1}^{\infty} \mu_l |\langle \phi_k, \phi_l\rangle|^2 + |\lambda_k|^2\|\phi_k\|_{L^2}^2 = |\lambda_k|^2 + |\lambda_k|^2 = 2|\lambda_k|^2 = -2\lambda_k^2 \end{aligned}$$

since $\mu_k = \alpha^2k^2\pi^2 = |\lambda_k|^2$. Thus a normalized eigenvector of A corresponding to the eigenvalue $\lambda_k = i\alpha k\pi$ for $k \in \mathbb{Z}$ is given by

$$\varphi_k = \frac{1}{\sqrt{2}\lambda_k} \begin{pmatrix} \phi_k \\ \lambda_k\phi_k \end{pmatrix} = \frac{1}{\sqrt{2}\lambda_k} \begin{pmatrix} \sqrt{2}\sin(k\pi\cdot) \\ \lambda_k\sqrt{2}\sin(k\pi\cdot) \end{pmatrix} = \frac{1}{\lambda_k} \begin{pmatrix} \sin(k\pi\cdot) \\ \lambda_k \sin(k\pi\cdot) \end{pmatrix}.$$

The set $\{\varphi_k\}_{k\neq 0} \subset X$ is orthonormal, since $\|\varphi_k\|_X = 1$ for all $k \in \mathbb{Z} \setminus \{0\}$, and for all $k \neq l$ we have

$$\begin{aligned} \langle \varphi_k, \varphi_l\rangle_X &= \frac{1}{2\lambda_k\lambda_l}\langle \phi_k, \phi_l\rangle_{\mathcal{D}(A_0^{1/2})} + \frac{1}{2\lambda_k\lambda_l}\langle \lambda_k\phi_k, \lambda_l\phi_l\rangle_{L^2} \\ &= \frac{1}{2\lambda_k\lambda_l}\sum_{j=1}^{\infty} \mu_j \langle \phi_k, \phi_j\rangle_{L^2}\langle \phi_j, \phi_l\rangle_{L^2} + \frac{1}{2}\langle \phi_k, \phi_l\rangle_{L^2} = 0 \end{aligned}$$

since the set $\{\phi_j\}_{j\in\mathbb{N}}$ is orthonormal. In addition, the set $\{\varphi_k\}_{k\in\mathbb{Z}\setminus\{0\}}$ with

$$\varphi_k = \frac{1}{\lambda_k} \begin{pmatrix} \sin(k\pi\cdot) \\ \lambda_k \sin(k\pi\cdot) \end{pmatrix}, \qquad \lambda_k = i\alpha k\pi$$

is an orthonormal basis of X as is shown in [9, Ex. 2.3.8]. Every $x \in X$ can be represented in the form

$$x = \sum_{k\in\mathbb{Z}\setminus\{0\}} \langle x, \varphi_k\rangle \varphi_k, \qquad \|x\|_X^2 = \sum_{k\in\mathbb{Z}\setminus\{0\}} |\langle x, \varphi_k\rangle|^2.$$

Theorem 4.4.2. *The operator A is diagonalizable in such a way that $A = SA_DS^{-1}$ with $A_D = \operatorname{diag}(\lambda_k)_{k\in\mathbb{Z}\setminus\{0\}}$ and it generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ on X. The operator A and the semigroup $(T(t))_{t\geq 0}$ have the spectral representations*

$$Ax = \sum_{k\in\mathbb{Z}\setminus\{0\}} \lambda_k \langle x, \varphi_k\rangle \varphi_k, \qquad x \in \mathcal{D}(A) = \Big\{\, x\in X \;\Big|\; \sum_{k\in\mathbb{Z}\setminus\{0\}} |\lambda_k|^2 |\langle x, \varphi_k\rangle|^2 < \infty \,\Big\},$$

$$T(t)x = \sum_{k\in\mathbb{Z}\setminus\{0\}} e^{\lambda_k t}\langle x, \varphi_k\rangle \varphi_k, \qquad x\in X,\ t\geq 0.$$

Proof. The diagonalizability of A follows exactly as in the case of the heat equation when we consider the space $\ell^2(\mathbb{C}) = \{\, (x_k)_{k\in\mathbb{Z}\setminus\{0\}} \mid \sum_{k\in\mathbb{Z}\setminus\{0\}} |x_k|^2 < \infty \,\}$ and define $S \in \mathcal{L}(\ell^2(\mathbb{C}), X)$ such that

$$Se_k = \varphi_k, \qquad \text{for all } k \in \mathbb{Z}\setminus\{0\}.$$

Since the operator A is diagonalizable and the real parts of its eigenvalues are bounded from above, we have from Theorem 3.3.2 that A generates a strongly continuous semigroup on $T(t)$. □

Since the wave equation is diagonalizable and all its eigenvalues are on the imaginary axis, Theorem 4.2.6 tells us that the semigroup $T(t)$ generated by A is not strongly or exponentially stable, but it is uniformly bounded. In fact, the semigroup $T(t)$ is *contractive*, which means that $\|T(t)\| \leq 1$ for all. Indeed, for any $x \in X$ the spectral representation of $T(t)$ in Theorem 4.4.2 and $\lambda_k = i\alpha\pi k$ imply that

$$\|T(t)x\|^2 = \left\| \sum_{k\in\mathbb{Z}\setminus\{0\}} e^{i\alpha\pi kt}\langle x, \varphi_k\rangle \varphi_k \right\|^2 = \sum_{k\in\mathbb{Z}\setminus\{0\}} |e^{i\alpha\pi kt}|^2 |\langle x, \varphi_k\rangle|^2 = \sum_{k\in\mathbb{Z}\setminus\{0\}} |\langle x, \varphi_k\rangle|^2 = \|x\|^2.$$

4.4.3 Control and Observation

As we already saw, the controlled wave equation (4.5) can be written in the form

$$\dot{x}(t) = \begin{pmatrix} 0 & I \\ -A_0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ b(\cdot) \end{pmatrix} u(t),$$

and thus the control operator $B \in \mathcal{L}(\mathbb{C}, X)$ of our linear system is given by

$$Bu = \begin{pmatrix} 0 \\ b(\cdot) \end{pmatrix} u \in X \qquad u \in \mathbb{C}.$$

More generally, if the wave equation has $m \in \mathbb{N}$ inputs, we have

$$\dot{x}(t) = \begin{pmatrix} 0 & I \\ -A_0 & 0 \end{pmatrix} x(t) + \sum_{j=1}^{m} \begin{pmatrix} 0 \\ b_j(\cdot) \end{pmatrix} u_j(t),$$

and the control operator $B \in \mathcal{L}(\mathbb{C}^m, X)$ becomes

$$B \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} = \sum_{j=1}^{m} \begin{pmatrix} 0 \\ b_j(\cdot) \end{pmatrix} u_j = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ b_1(\cdot) & b_2(\cdot) & \cdots & b_m(\cdot) \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}.$$

The measured output of the wave equation can be written using the inner product of X in the form

$$\begin{aligned} y(t) &= \int_0^1 w(\xi,t)c(\xi)d\xi = \langle w(\cdot,t), \overline{c(\cdot)} \rangle_{L^2} = \langle A_0^{1/2} w(\cdot,t), A_0^{1/2} A_0^{-1} \overline{c(\cdot)} \rangle_{L^2} \\ &= \langle w(\cdot,t), A_0^{-1}\overline{c(\cdot)} \rangle_{\mathcal{D}(A_0^{1/2})} = \langle w(\cdot,t), A_0^{-1}\overline{c(\cdot)} \rangle_{\mathcal{D}(A_0^{1/2})} = \left\langle x(t), \begin{pmatrix} \tilde{c} \\ 0 \end{pmatrix} \right\rangle_X \end{aligned}$$

where $\tilde{c} = A_0^{-1}\overline{c(\cdot)} \in \mathcal{D}(A_0)$ and where we have used $x(t) = (w(\cdot,t), w_t(\cdot,t))^T \in X$. Thus the observation operator $C \in \mathcal{L}(X, \mathbb{C})$ is such that

$$Cx = \left\langle x, \begin{pmatrix} \tilde{c} \\ 0 \end{pmatrix} \right\rangle_X, \qquad \text{for all} \quad x \in X,$$

where $\tilde{c} = A_0^{-1}\overline{c(\cdot)}$. More generally, if the wave equation has p independent measured outputs of the form

$$y_j(t) = \int_0^1 w(\xi,t)c_j(\xi)d\xi,$$

then the operator $C \in \mathcal{L}(X, \mathbb{C}^p)$ is defined by

$$Cx = \begin{pmatrix} \left\langle x, \begin{pmatrix} \tilde{c}_1 \\ 0 \end{pmatrix} \right\rangle_X \\ \vdots \\ \left\langle x, \begin{pmatrix} \tilde{c}_p \\ 0 \end{pmatrix} \right\rangle_X \end{pmatrix}, \qquad \text{for all} \quad x \in X,$$

where $\tilde{c}_j = A_0^{-1}\overline{c_j(\cdot)}$ for all $j \in \{1, \ldots, p\}$.

4.4.4 The Diagonalized System and Numerical Approximation

If we denote $z(t) = S^{-1}x(t)$ for all $t \geq 0$, we have as in Example 4.1.3 that

$$\begin{aligned} &\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t), \qquad x(0) = x_0 \in X \\ y(t) &= Cx(t) + Du(t) \end{cases} \\ \Leftrightarrow \quad &\begin{cases} \dot{z}(t) &= A_D z(t) + B_D u(t), \qquad z(0) = S^{-1}x_0 \in Z = \ell^2(\mathbb{C}) \\ y(t) &= C_D z(t) + Du(t) \end{cases} \end{aligned}$$

where $B_D = S^{-1}B$, and $C_D = CS$. Our aim is to again find expressions for the operators B_D and C_D.

Let us find out what the new input and output operators look like. Since $\begin{pmatrix} 0 \\ b(\cdot) \end{pmatrix} \in X$ and the eigenvectors $\varphi_k(\cdot) = \frac{1}{\lambda_k}\begin{pmatrix} \sin(k\pi\cdot) \\ \lambda_k \sin(k\pi\cdot) \end{pmatrix}$ are an orthonormal basis of X, we can express

$$\begin{pmatrix} 0 \\ b(\cdot) \end{pmatrix} = \sum_{k\in\mathbb{Z}\setminus\{0\}} b_k\varphi_k(\cdot)$$

where

$$b_k = \left\langle \begin{pmatrix} 0 \\ b(\cdot) \end{pmatrix}, \varphi_k \right\rangle_X = 0 + \langle b(\cdot), \sin(k\pi\cdot)\rangle_{L^2} = \int_0^1 b(\xi)\sin(k\pi\xi)d\xi, \tag{4.6}$$

and for every $k \in \mathbb{N}$ we have

$$b_{-k} = \int_0^1 b(\xi)\sin(-k\pi\xi)d\xi = -\int_0^1 b(\xi)\sin(k\pi\xi)d\xi = -b_k.$$

Since S satisfies $Se_k = \varphi_k$ and $S^{-1}\varphi_k = e_k$, we can see that for every $u \in \mathbb{C}$

$$\begin{aligned} S^{-1}Bu &= S^{-1}\begin{pmatrix} 0 \\ b(\cdot) \end{pmatrix} u = S^{-1}\left(\sum_{k\in\mathbb{Z}\setminus\{0\}} b_k\varphi_k\right) u = u\sum_{k\in\mathbb{Z}\setminus\{0\}} b_k S^{-1}\varphi_k = u\sum_{k\in\mathbb{Z}\setminus\{0\}} b_k e_k \\ &= u\cdot(b_k)_{k\in\mathbb{Z}\setminus\{0\}} \in \ell^2(\mathbb{C}) \end{aligned}$$

Where $(b_k)_{k\in\mathbb{Z}\setminus\{0\}}$ is an infinite vector with elements b_k defined in (4.6). Similarly, since $\begin{pmatrix} \tilde{c}(\cdot) \\ 0 \end{pmatrix} \in X$ where $\tilde{c} = A_0^{-1}\overline{c}$, we can write

$$\begin{pmatrix} \tilde{c}(\cdot) \\ 0 \end{pmatrix} = \sum_{k=1}^{\infty} c_k\varphi_k(\cdot)$$

where

$$c_k = \left\langle \begin{pmatrix} \tilde{c}(\cdot) \\ 0 \end{pmatrix}, \varphi_k \right\rangle = \langle A_0^{-1}\overline{c(\cdot)}, \frac{1}{\lambda_k}\sin(k\pi\cdot)\rangle_{\mathcal{D}(A_0^{1/2})} = \frac{1}{\lambda_k}\langle A_0^{1/2}A_0^{-1}\overline{c(\cdot)}, A_0^{1/2}\sin(k\pi\cdot)\rangle_{L^2} \tag{4.7a}$$

$$= \frac{1}{\lambda_k}\langle\overline{c(\cdot)}, \sin(k\pi\cdot)\rangle_{L^2} = -\frac{i}{\alpha k\pi}\int_0^1 \overline{c(\xi)}\sin(k\pi\xi)d\xi, \tag{4.7b}$$

and for all $k \in \mathbb{N}$ we have

$$c_{-k} = -\frac{i}{\alpha(-k)\pi}\int_0^1 \overline{c(\xi)}\sin(-k\pi\xi)d\xi = -\frac{(-1)(-1)i}{\alpha k\pi}\int_0^1 \overline{c(\xi)}\sin(k\pi\xi)d\xi = c_k.$$

For all $z = (z_k)_{k\in\mathbb{Z}\setminus\{0\}} \in \ell^2(\mathbb{C})$ we have

$$\begin{aligned} C_D z &= CS\sum_{k\neq 0} z_k e_k = \sum_{k\neq 0} z_k CSe_k = \sum_{k\neq 0} z_k C\varphi_k = \sum_{k\neq 0} z_k \left\langle \varphi_k, \begin{pmatrix} \tilde{c} \\ 0 \end{pmatrix} \right\rangle_X = \sum_{k\neq 0} \overline{c_k} z_k \\ &= \langle z, (c_k)_{k\in\mathbb{N}}\rangle_{\ell^2} \end{aligned}$$

since

$$\left\langle \varphi_k, \begin{pmatrix} \tilde{c} \\ 0 \end{pmatrix} \right\rangle_X = \left\langle \varphi_k, \sum_{k\neq 0} c_l \varphi_l \right\rangle = \sum_{k\neq 0} \overline{c_l} \langle \phi_k, \phi_l \rangle = \overline{c_k}$$

for all $k \in \mathbb{Z} \setminus \{0\}$. Thus for all $z \in \ell^2(\mathbb{C})$ we have $C_D z = \langle z, (c_k)_{k\in\mathbb{N}} \rangle$, where c_k are defined in (4.7). If the initial state x_0 of the original equation is

$$x_0 = \begin{pmatrix} w(\cdot, 0) \\ w_t(\cdot, 0) \end{pmatrix} = \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} = \sum_{k\neq 0} w_k \varphi_k$$

where

$$w_k = \left\langle \begin{pmatrix} w_0 \\ w_1 \end{pmatrix}, \varphi_k \right\rangle_X = \langle w_0, \frac{1}{\lambda_k} \sin(k\pi\cdot) \rangle_{\mathcal{D}(A_0^{1/2})} + \langle w_1, \sin(k\pi\cdot) \rangle_{L^2} \tag{4.8a}$$

$$= -i\alpha k\pi \int_0^1 w_0(\xi) \sin(k\pi\xi) d\xi + \int_0^1 w_1(\xi) \sin(k\pi\xi) d\xi \tag{4.8b}$$

for all $k \in \mathbb{Z} \setminus \{0\}$ since

$$\langle w_0, \frac{1}{\lambda_k} \sin(k\pi\cdot) \rangle_{\mathcal{D}(A_0^{1/2})} = \frac{1}{\lambda_k} \langle A_0^{1/2} w_0, A_0^{1/2} \sin(k\pi\cdot) \rangle_{L^2} = -\frac{i}{\alpha k\pi} \langle w_0, A_0 \sin(k\pi\cdot) \rangle_{L^2}$$
$$= -\frac{i}{\alpha k\pi} \langle w_0, -\alpha^2 \frac{d^2}{d\xi^2} \sin(k\pi\cdot) \rangle_{L^2} = -\frac{i\alpha^2 k^2 \pi^2}{\alpha k\pi} \langle w_0, \sin(k\pi\cdot) \rangle_{L^2} = -i\alpha k\pi \langle w_0, \sin(k\pi\cdot) \rangle_{L^2}.$$

Thus the initial state of the diagonalized system becomes then

$$z_0 = S^{-1} x_0 = \sum_{k=1}^{\infty} w_k S^{-1} \varphi_k = \sum_{k=1}^{\infty} w_k e_k = (w_k)_{k\in\mathbb{Z}\setminus\{0\}}$$

where w_k are defined in (4.8).

We can use truncated series to approximate the behaviour of the controlled wave equation. Now for $N \in \mathbb{N}$ we have

$$\left\| \begin{pmatrix} 0 \\ b(\cdot) \end{pmatrix} - \sum_{\substack{k=-N \\ k\neq 0}}^{N} b_k \varphi_k \right\|_X \to 0, \quad \left\| \begin{pmatrix} \tilde{c} \\ 0 \end{pmatrix} - \sum_{\substack{k=-N \\ k\neq 0}}^{N} c_k \varphi_k \right\|_X \to 0, \quad \left\| x_0(\cdot) - \sum_{\substack{k=-N \\ k\neq 0}}^{N} v_{0k} \varphi_k \right\|_X \to 0$$

as $N \to \infty$. The finite-dimensional approximation for the controlled wave equation is of the form

$$\dot{z}_N(t) = A_D^N z_N(t) + B_D^N u(t), \qquad z_N(0) = z_{0N} \in \mathbb{C}^N$$
$$y_N(t) = C_D^N z(t) + Du(t)$$

on the finite-dimensional space $Z_N = \mathbb{C}^{2N}$. Here

$$A_D^N = \operatorname{diag}(\lambda_k)_{0<|k|\leq N} \in \mathbb{C}^{2N\times 2N} \qquad B_D^N = (b_k)_{0<|k|\leq N} \in \mathbb{C}^{2N} \qquad C_D^N = (c_k)_{0<|k|\leq N} \in \mathbb{C}^{1\times 2N}$$

$$z_N(t) = (z_k(t))_{0<|k|\leq N} \in \mathbb{C}^{2N} \qquad z_{0N} = (w_k)_{0<|k|\leq N} \in \mathbb{C}^{2N}$$

Figures 4.3 and 4.4 illustrate the behaviour of the state and the output of the wave equation with initial state where $w_0(\xi) = \sin(2\pi\xi)$ and $w_1(\xi) \equiv 0$. In the first simulation the system has no input ($u(t) \equiv 0$) and in the second one the input is

$$u(t) = \begin{cases} 1 & 0 \leq t \leq 1 \\ 0 & t > 1 \end{cases}$$

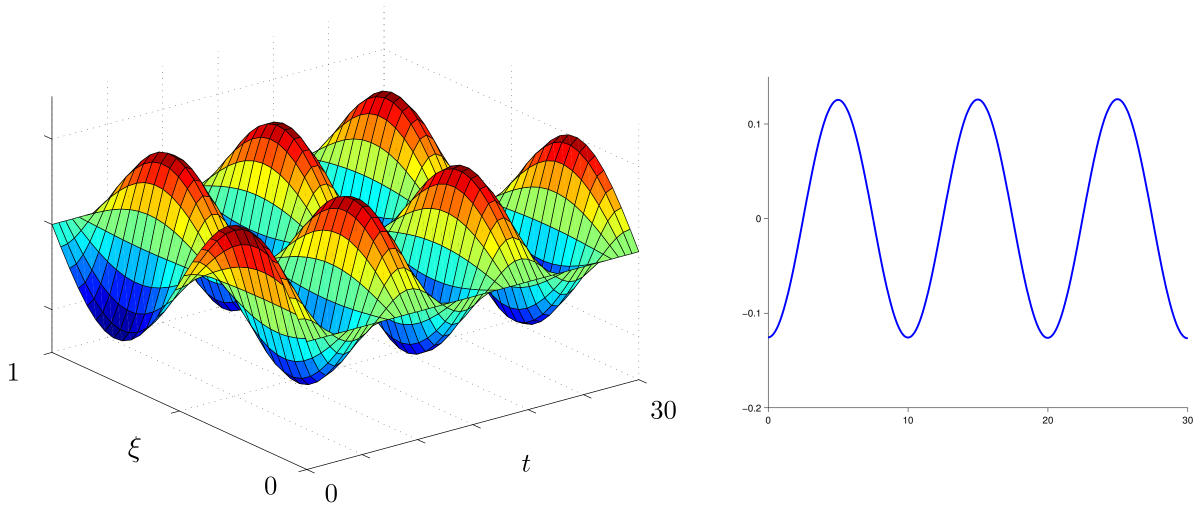

Figure 4.3: Numerical approximation of the solution with $N = 40$, no input.

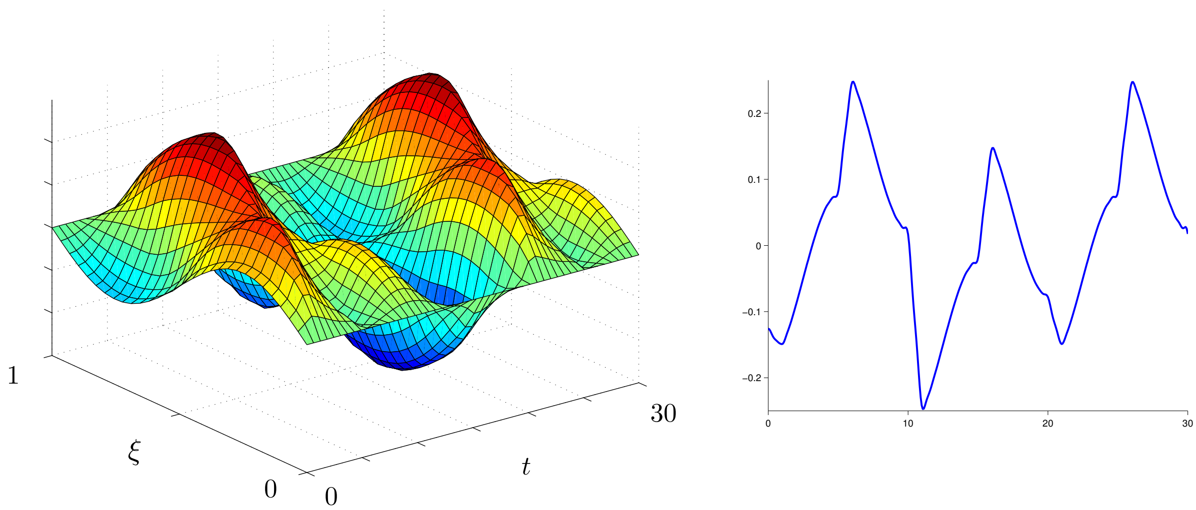

Figure 4.4: Numerical approximation with $N = 40$, input $u(t) = 1$ for $0 \leq t \leq 1$.

4.4.5 The Wave Equation with Neumann Boundary Conditions

We could alternatively consider a controlled wave equation with *Neumann boundary conditions*,

$$\begin{aligned}
\frac{\partial^2 w}{\partial t^2}(\xi,t) &= \alpha^2 \frac{\partial^2 w}{\partial \xi^2}(\xi,t) + b(\xi)u(t), \qquad \xi \in (0,1) \\
\frac{\partial w}{\partial \xi}(0) &= 0, \quad \frac{\partial w}{\partial \xi}(1) = 0, \\
w(\xi,0) &= w_0(\xi), \quad \frac{\partial w}{\partial t}(\xi,0) = w_1(\xi) \\
y(t) &= \int_0^1 w(\xi,t)c(\xi)d\xi, \qquad t \geq 0.
\end{aligned}$$

Also this equation can be formulated as a linear systems on the space $X = \mathcal{D}(A_0^{1/2}) \times L^2(0,1)$ with a system operator

$$A = \begin{pmatrix} 0 & I \\ A_0 & 0 \end{pmatrix}, \qquad \mathcal{D}(A) = \mathcal{D}(A_0) \times \mathcal{D}(A_0^{1/2}),$$

where we now choose

$$A_0 f = -\alpha^2 \frac{d^2 f}{d\xi^2}, \qquad f \in \mathcal{D}(A_0) = \{\, f \in X \mid f, f' \text{ abs. cont. } f'' \in X,\ f'(0) = f'(1) = 0 \,\}.$$

Also the operator A has an orthonormal set $\{\varphi_k\}_{k\in\mathbb{Z}} \subset X$ of eigenfunctions, but the operator A is **not** diagonalizable due to the fact that $\{\varphi_k\}_{k\in\mathbb{Z}}$ is not a basis of the space X. Nevertheless, the operator A generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ on X, but this property needs to be proved using some other method besides diagonalizability. For example, one can use the Lumer–Phillips Theorem presented in Theorem 3.3.1 as is done in [9, Ex. 2.3.9].

5. The Output Tracking Problem

So far we have concentrated on studying the properties of infinite-dimensional linear systems. In this chapter we study an actual control problem, in which our goal is to choose a control input $u(t)$ in such a way that the measured output $y(t)$ of the system converges to a given reference signal $y_{ref}(t)$.

Definition 5.0.1. Let $y_{ref}(\cdot) : \mathbb{R} \to Y = \mathbb{C}^p$ be a τ-periodic function, i.e., $y(t+\tau) = y(t)$ for all $t \in \mathbb{R}$. In *output tracking* the aim is to choose the input $u(t)$ of the system in such a way that

$$\|y(t) - y_{ref}(t)\| \to 0 \qquad \text{as} \quad t \to \infty.$$

This same control problem is also known as the *output regulation problem* [7, 12, 17], and it is also possible and customary to include rejection of external disturbance signals in the control objective.

We begin by considering the output tracking problem in the situation where our infinite-dimensional control system is stable in the sense that the semigroup $(T(t))_{t\geq 0}$ generated by A is exponentially stable. In order to study output tracking for unstable systems, we first need to consider the *stabilizability* of infinite-dimensional systems. This problem consists of finding a suitable state feedback that makes the initially unstable system to become stable. This concept is studied in Section 5.3. Finally, in Section 5.4 we can use the results on stabilizability to achieve output tracking for unstable systems.

For simplicity, throughout this chapter we only consider systems with a finite number of inputs and outputs.

Asumption 5.0.2. Throughout the chapter we assume that the system has $m \in \mathbb{N}$ inputs and $p \in \mathbb{N}$ measured outputs, i.e., $U = \mathbb{C}^m$ and $Y = \mathbb{C}^p$.

5.1 The Reference Signal $y_{ref}(t)$

We consider a reference signal of the form

$$y_{ref}(t) = \sum_{k=-q}^{q} a_k e^{i\omega_0 kt}, \qquad \text{where} \qquad a_k \in \mathbb{C}^p,\ \omega_0 = \frac{2\pi}{\tau}. \tag{5.1}$$

The value $\omega_0 = \frac{2\pi}{\tau} > 0$ is the "base frequency", and y_{ref} is indeed a τ-periodic function since

$$y_{ref}(t+\tau) = \sum_{k=-q}^{q} a_k e^{i\omega_0 k(t+\tau)} = \sum_{k=-q}^{q} a_k e^{i\omega_0 kt} e^{i2\pi k} = \sum_{k=-q}^{q} a_k e^{i\omega_0 kt} = y_{ref}(t).$$

The component functions of $y_{ref}(t)$ are real-valued if (and only if) $a_0 \in \mathbb{R}^p$ and $\overline{a_{-k}} = a_k$ for all $k \in \mathbb{N}$. In this situation for all $t \in \mathbb{R}$ we have

$$\overline{y_{ref}(t)} = \sum_{k=-q}^{q} \overline{a_k} e^{-i\omega_0 kt} = \sum_{k=-q}^{q} a_{-k} e^{i\omega_0(-k)t} = \sum_{n=-q}^{q} a_n e^{i\omega_0 nt} = y_{ref}(t),$$

which implies that $y_{ref}(t) \in \mathbb{R}^p$.

The theory of Fourier Series tells us that functions of the form (5.1) can be used to approximate any τ-periodic function $f(\cdot) \in L^2(0,\tau;\mathbb{C}^p)$ with any given finite accuracy *in the L^2-sense*. This means that $f(\cdot)$ is a τ-periodic function such that $f(\cdot) \in L^2(0,\tau;\mathbb{C}^p)$, then for any $\varepsilon > 0$ there exists $q \in \mathbb{N}$ and $(a_k)_{k=-q}^{q} \subset \mathbb{C}^p$ such that

$$\left\| f(\cdot) - \sum_{k=-q}^{q} a_k e^{i\omega_0 k\cdot} \right\|_{L^2} < \varepsilon.$$

As illustrated in Figure 5.1 the convergence may not happen in the pointwise sense if the function $f(\cdot)$ is not continuous, and in particular the *Gibbs phenomenon* results in overshoots and undershoots that can not be reduced by increasing the number of terms in the approximating function. However, if the τ-periodic function $f(\cdot)$ is continuous on $\mathbb{R}$, then the convergence also happens in the "pointwise sense". In particular, if $f(\cdot)$ is τ-periodic and continuous, then for any $\varepsilon > 0$ there exists $q \in \mathbb{N}$ and $(a_k)_{k=-q}^{q} \subset \mathbb{C}^p$ such that

$$\left\| f(t) - \sum_{k=-q}^{q} a_k e^{i\omega_0 kt} \right\|_{\mathbb{C}^p} < \varepsilon \qquad \text{for all } t \in \mathbb{R}.$$

This property is much stronger than convergence in the L^2-sense, as is illustrated in Figure 5.1 for a periodic "triangle" function that is continuous (but not continuously differentiable).

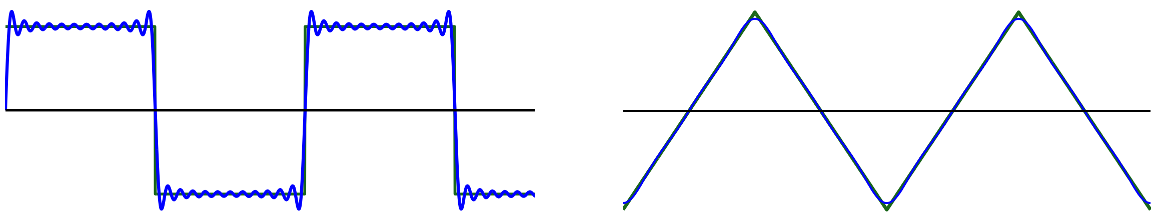

Figure 5.1: Fourier approximations of periodic functions.

Example 5.1.1. If we want to consider a periodic refencence signal

$$y_{ref}(t) = \begin{pmatrix} \sin(2\pi t) + 1 \\ \cos(2\pi t) \end{pmatrix},$$

then we can choose $\tau = 1$ and $\omega_0 = \frac{2\pi}{\tau} = 2\pi$ and using

$$\sin(2\pi t) = \frac{e^{i2\pi t} - e^{-i2\pi t}}{2i}, \qquad \cos(2\pi t) = \frac{e^{i2\pi t} + e^{-i2\pi t}}{2}$$

we can write

$$\begin{aligned} y_{ref}(t) &= \begin{pmatrix} \sin(2\pi t)+1 \\ \cos(2\pi t) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i\omega_0\cdot 0\cdot t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{e^{i\omega_0\cdot 1\cdot t} - e^{i\omega_0\cdot(-1)\cdot t}}{2i} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \frac{e^{i\omega_0\cdot 1\cdot t} + e^{i\omega_0\cdot(-1)\cdot t}}{2} \\ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i\omega_0\cdot 0\cdot t} + \left[\begin{pmatrix} 1/(2i) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}\right] e^{i\omega_0\cdot 1\cdot t} + \left[\begin{pmatrix} -1/(2i) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}\right] e^{i\omega_0\cdot(-1)\cdot t} \\ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i\omega_0\cdot 0\cdot t} + \frac{1}{2}\begin{pmatrix} -i \\ 1 \end{pmatrix} e^{i\omega_0\cdot 1\cdot t} + \frac{1}{2}\begin{pmatrix} i \\ 1 \end{pmatrix} e^{i\omega_0\cdot(-1)\cdot t} \end{aligned}$$

which means that in the form (5.1) we have $q = 1$, $\omega_0 = 2\pi$, and

$$a_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad a_1 = \frac{1}{2}\begin{pmatrix} -i \\ 1 \end{pmatrix}, \qquad a_{-1} = \frac{1}{2}\begin{pmatrix} i \\ 1 \end{pmatrix}.$$

Note that the vectors satisfy $\overline{a_{-1}} = a_1$, since $y_{ref}(t) \in \mathbb{R}^2$ for all $t \in \mathbb{R}$. ◇

5.2 Output Tracking For Stable Systems

In this section we solve the output tracking problem in the situation where the semigroup generated by A is exponentially stable. We look for a control in the form

$$u(t) = \sum_{k=-q}^{q} e^{i\omega_0 kt} u_k, \qquad t \geq 0 \tag{5.2}$$

where $u_k \in \mathbb{C}^m$ are the parameters of the control input that we need to determine. For $\lambda \in \rho(a)$ we denote by $P(\lambda) \in \mathcal{L}(\mathbb{C}^m, \mathbb{C}^p)$ the *transfer function* of our infinite-dimensional system, and it is defined with the formula

$$P(\lambda) = C(\lambda - A)^{-1}B + D.$$

You can find more details on the role of transfer functions in Section 1.2.6, but for the purposes of output tracking it is sufficient to know the above formula for $P(\lambda)$, since it appears naturally in our conditions.

Theorem 5.2.1. *Assume the semigroup $(T(t))_{t\geq 0}$ generated by A is exponentially stable. Let $y_{ref}(t)$ be the reference signal defined by* (5.1). *If we can choose $\{u_k\}_{k=-q}^{q} \subset \mathbb{C}^m$ in such a way that*

$$P(i\omega_0 k)u_k = a_k \qquad k \in \{-q, \dots, q\},$$

then the output of the system satisfies

$$\|y(t) - y_{ref}(t)\|_{\mathbb{C}^p} \to 0, \qquad \textit{as} \quad t \to \infty.$$

The proof uses the following connection between semigroups and the inverses of their generators.

Lemma 5.2.2. *If A generates an exponentially stable semigroup $(T(t))_{t\geq 0}$ on a Banach space X, then for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq 0$ we have that*

$$\int_0^\infty e^{-\lambda t}T(t)xdt = (\lambda - A)^{-1}x, \qquad \textit{for all} \quad x \in X.$$

Proof. See [10, Thm. II.1.10] or [9, Lem. 2.1.11]. □

Proof of Theorem 5.2.1. For a given initial state $x_0 \in X$ and the control input $u(t)$ defined in (5.2) the output $y(t)$ of the system is given by

$$\begin{aligned}
y(t) &= CT(t)x_0 + C\int_0^t T(t-s)Bu(s)ds + Du(t)\\
&= CT(t)x_0 + \sum_{k=-q}^{q}\left[C\int_0^t e^{i\omega_0 ks}T(t-s)Bu_kds + e^{i\omega_0 kt}Du_k\right]\\
&= CT(t)x_0 + \sum_{k=-q}^{q} e^{i\omega_0 kt}\left[C\int_0^t e^{-i\omega_0 k(t-s)}T(t-s)Bu_kds + Du_k\right]\\
&= CT(t)x_0 + \sum_{k=-q}^{q} e^{i\omega_0 kt}\left[C\int_0^t e^{-i\omega_0 ks}T(s)Bu_kds + Du_k\right]\\
&= CT(t)x_0 + \sum_{k=-q}^{q} e^{i\omega_0 kt}\left[-C\int_t^\infty e^{-i\omega_0 ks}T(s)Bu_kds + C\int_0^\infty e^{-i\omega_0 ks}T(s)Bu_kds + Du_k\right]\\
&= CT(t)x_0 - \sum_{k=-q}^{q} e^{i\omega_0 kt}C\int_t^\infty e^{-i\omega_0 ks}T(s)Bu_kds + \sum_{k=-q}^{q} e^{i\omega_0 kt}P(i\omega_0 k)u_k\\
&= y_0(t) + \sum_{k=-q}^{q} e^{i\omega_0 kt}P(i\omega_0 k)u_k.
\end{aligned}$$

where we have denoted

$$y_0(t) = CT(t)x_0 - \sum_{k=-q}^{q} e^{i\omega_0 kt}C\int_t^\infty e^{-i\omega_0 ks}T(s)Bu_kds.$$

If we choose $u_k \in \mathbb{C}^m$ in such a way that $a_k = P(i\omega_0 k)u_k$, then

$$y(t) - y_{ref}(t) = y_0(t) + \sum_{k=-q}^{q} e^{i\omega_0 kt}P(i\omega_0 k)u_k - \sum_{k=-q}^{q} e^{i\omega_0 kt}a_k = y_0(t)$$

for all $t \geq 0$, and thus $\|y(t) - y_{ref}(t)\| \to 0$ as $t \to \infty$ if we can show that $\|y_0(t)\| \to 0$ as $t \to \infty$. Since $(T(t))_{t\geq 0}$ is exponentially stable, there exists $M \geq 1$ and $c > 0$ such that $\|T(t)\| \leq Me^{-ct}$ for all $t \geq 0$. For all $k \in \{-q, \ldots, q\}$

$$\begin{aligned}
\left\|\int_t^\infty e^{-i\omega_0 ks}T(s)Bu_kds\right\| &\leq \int_t^\infty \|e^{-i\omega_0 ks}T(s)Bu_k\|ds \leq \int_t^\infty \|T(s)\|\|B\|\|u_k\|ds\\
&\leq M\|B\|\|u_k\|\int_t^\infty e^{-cs}ds \leq \frac{M\|B\|\|u_k\|}{c}e^{-ct},
\end{aligned}$$

and thus

$$\|y_0(t)\| \leq \|CT(t)x_0\| + \sum_{k=-q}^{q} \|C\| \left\| \int_t^\infty e^{-i\omega_0 ks} T(s) B u_k ds \right\|$$
$$\leq M\|C\|\|x_0\|e^{-ct} + \|C\| \sum_{k=-q}^{q} \frac{M\|B\|\|u_k\|}{c} e^{-ct} \leq M\|C\| \left(\|x_0\| + \frac{\|B\|}{c} \sum_{k=-q}^{q} \|u_k\| \right) e^{-ct}$$

and thus $\|y_0(t)\| \to 0$ as $t \to \infty$ and the convergence is exponentially fast. □

We have for simplicity assumed in Theorem 5.2.1 that the system is exponentially stable. However, the output tracking probelm can also be solved in exactly the way in the case where the system is only strongly stable and $i\omega_0 k \in \rho(A)$ for every $k \in \{-q, \dots, q\}$. The assumption $i\omega_0 k \in \rho(A)$ is required to guarantee that the transfer function of the system is well-defined at these points. In this situation the tracking error $y(t) - y_{ref}(t)$ still converges to zero as $t \to \infty$, but the convergence is no longer exponentially fast.

Theorem 5.2.3. *Assume the semigroup $(T(t))_{t\geq 0}$ generated by A is strongly stable and $\{i\omega_0 k\}_{k=-q}^{q} \subset \rho(A)$. Let $y_{ref}(t)$ be the reference signal defined by* (5.1). *If we can choose $\{u_k\}_{k=-q}^{q} \subset \mathbb{C}^m$ in such a way that*

$$P(i\omega_0 k)u_k = a_k \qquad k \in \{-q, \dots, q\},$$

then the output of the system satisfies

$$\|y(t) - y_{ref}(t)\|_{\mathbb{C}^p} \to 0, \qquad \text{as} \quad t \to \infty.$$

Proof. Lemma 5.2.2 remains valid for strongly stable semigroups and for $\lambda \in \rho(A)$ with $\operatorname{Re}\lambda \geq 0$. This implies that for all $k \in \{-q, \dots, q\}$ the function $t \mapsto e^{i\omega_0 ks} T(s)x$ is in $L^1(0, \infty; X)$ and thus

$$\left\| \int_t^\infty e^{-i\omega_0 ks} T(s) B u_k ds \right\| \leq \int_t^\infty \|e^{-i\omega_0 ks} T(s) B u_k\| ds \to 0$$

as $t \to \infty$. Since we also have $\|CT(t)x\| \leq \|C\|\|T(t)x\| \to 0$ as $t \to \infty$, we have $y_0(t) \to 0$ as $t \to \infty$ similarly as in the proof of Theorem 5.2.1. □

Theorem 5.2.1 shows that we can design a control that solves the output tracking problem in particular if $p = m$ (the number of inputs is the same as the number of outputs) and the matrices $P(i\omega_0 k) \in \mathbb{C}^{p\times p}$ are nonsingular for every $k \in \{-q, \dots, q\}$. Then the unique choices of u_k are given by

$$u_k = P(i\omega_0 k)^{-1} a_k, \qquad k \in \{-q, \dots, q\}.$$

More generally, we can choose suitable vectors u_k if and only if $a_k \in \mathcal{R}(P(i\omega_0 k))$ for every k. In this situation the choices which result in vectors u_k with the smallest possible norms $\|u_k\|$ are given by

$$u_k = P(i\omega_0 k)^{\dagger} a_k, \qquad k \in \{-q, \dots, q\},$$

where $P(i\omega_0 k)^\dagger$ is the Moore–Penrose pseudoinverse of $P(i\omega_0 k)$. If $\mathcal{R}(P(i\omega_0 k)) = X$, then the pseudoinverse is given by the formula $P(i\omega_0 k)^\dagger = P(i\omega_0 k)^*(P(i\omega_0 k)P(i\omega_0 k)^*)^{-1}$.

In order to be able to solve the output tracking problem, it is necessary only required that we know the values $P(i\omega_0 k)$ of the transfer function $P(\lambda)$ of the system at the frequencies $\lambda = i\omega_0 k$ that are present in the reference signal $y_{ref}(t)$. The level of difficulty of computing the transfer function for a partial differential equation or for some other infinite-dimensional system depends heavily on the situation. There are known results and techniques for many situations, see for example [22, 8]. The following lemma shows that if the operator A is diagonalizable, then the transfer function $P(\lambda)$ always has a representation as an infinite series, and finite truncations of this series can be used as approximations of the values of the transfer function.

Lemma 5.2.4. *If A is diagonalizable so that $A = SA_DS^{-1}$ where $A_D = \operatorname{diag}(\lambda_k)_{k\in\mathbb{N}}$, then for every $\lambda \in \rho(A)$ we have*

$$P(\lambda)u = \sum_{k=1}^{\infty} \frac{\langle Bu, \psi_k\rangle C\phi_k}{\lambda - \lambda_k} + Du$$

where $\psi_k = (S^{-1})^* e_k$ and $\phi_k = Se_k$.

Proof. It follows from similarity that

$$(\lambda - A)^{-1} = (\lambda SS^{-1} - SA_DS^{-1})^{-1} = (S(\lambda - A_D)S^{-1})^{-1} = S(\lambda - A_D)^{-1}S^{-1},$$

where $(\lambda - A_D)^{-1}$ is a diagonal operator on $\ell^2(\mathbb{C})$ so that $(\lambda - A_D)^{-1} = \operatorname{diag}(\frac{1}{\lambda-\lambda_k})_{k=1}^{\infty}$. We thus have that for every $u \in U$

$$\begin{aligned}(\lambda - A)^{-1}Bu &= S(\lambda - A_D)^{-1}S^{-1}Bu = S\sum_{k=1}^{\infty} \frac{\langle S^{-1}Bu, e_k\rangle}{\lambda - \lambda_k} e_k = \sum_{k=1}^{\infty} \frac{\langle Bu, (S^{-1})^* e_k\rangle}{\lambda - \lambda_k} Se_k \\ &= \sum_{k=1}^{\infty} \frac{\langle Bu, \psi_k\rangle}{\lambda - \lambda_k}\phi_k,\end{aligned}$$

and thus for all $u \in U$

$$P(\lambda) = C(\lambda - A)^{-1}Bu + Du = \sum_{k=1}^{\infty} \frac{\langle Bu, \psi_k\rangle}{\lambda - \lambda_k} C\phi_k + Du.$$

□

5.2.1 Matlab Implementation

The construction of the control input that solves the output tracking problem for a stable system can be easily implemented using Matlab. The following function receives the frequencies and coefficient vectors of the reference signal and the function for computing the transfer function $P(\cdot)$ of the system as parameters, and based on this information constructs the appropriate control input $u(t)$.

```
function ufun = LinSysTrackStab(ref_w,ref_c,Pfun)
% function control = LinSysTrackStab(ref_w,ref_c,Pfun)
%
% Generates the control input "ufun" to achieve output tracking of the
% reference signal with frequencies given in "ref_w" (real values) and
% corresponding coefficient vectors given as columns of "ref_c". "Pfun" is
% a function handle that evaluates the transfer function of the system at a
% given point.
%
% Note that ufun can not be evaluated for a vector argument


% number of frequencies
N = length(ref_w);

% find out the number of inputs
m = size(Pfun(1i*ref_w(1)),2);

% store the coefficient vectors of the control output
ukvecs = zeros(m,N);

for ind = 1:N

    Pval = Pfun(1i*ref_w(ind));

    % If the coefficient vector ref_c(ind) is not in the range space of
    % P(iw), produce a warning
    if rank(Pval) < rank([Pval ref_c(ind)])
        warning('Tracking problem may not be solvable!')
    end

    % The operator "\" corresponds to the multiplication with the
    % pseudoinverse of P(iw)
    ukvecs(:,ind) = Pval\ref_c(ind);
end

% Construct the control function, u(t) = sum(exp(1i*wk*t)*uk,k=-q..q)
ufun = @(t) ukvecs*exp(1i*ref_w(:)*t);
```

In addition, we may want to implement a simple function that gives us a function handle for computing the values of the reference signal $y_{ref}(t)$.

```
function yref = LinSysTrackRef(ref_w,ref_c)
% function yref = LinSysTrackRef(ref_w,ref_c)
%
% Returns a function handle that computes the value of the reference signal
% y_ref at time t. The input arguments are the frequencies of the reference
% signal "ref_w" (real values) and the corresponding coefficient vectors
% given as columns of "ref_c".
%
% The function yref can not be evaluated for a vector.

yref = @(t) ref_c*exp(1i*ref_w(:)*t);
```

We can use the above functions to study output regulation for a linear system with parameters (A, B, C, D) in the following way.

```
% reference signal sin(2*pi*t)+1
ref_w = [-2*pi 0 2*pi];
ref_c = [1i/2 1 -1i/2];

% transfer function of the plant
Pfun = @(s) C*((s*eye(size(A))-A)\B)+D;

ufun = LinSysTrackStab(ref_w,ref_c,Pfun);
yref_fun = LinSysTrackRef(ref_w,ref_c);
```

The produced function `ufun` can now be used as an input to the system.

5.3 Stabilizability of Infinite-Dimensional Systems

By a stabilizable system we mean a system that can be made stable with a suitable state feedback of the form $u(t) = Kx(t) + \tilde{u}(t)$ where $\tilde{u}(t)$ is the new input to the system. It follows from the theory of semigroups that if A generates a strongly continuous semigroup and B and K are bounded operators, then also $A + BK$ generates a strongly continuous semigroup [11, Sec. III.1], [9, Sec. 3.2]. Stabilization of systems is an important control problem by itself, but it is also required in the study of output tracking for unstable systems in Section 5.4.

Definition 5.3.1. The system (A, B, C, D) is called *exponentially stabilizable* if there exists $K \in \mathcal{L}(X, U)$ such that the semigroup generated by $A + BK$ is exponentially stable.

Likewise, the system (A, B, C, D) is called *strongly stabilizable* if there exists $K \in \mathcal{L}(X, U)$ such that the semigroup generated by $A + BK$ is strongly stable.

The following theorem presents a condition for a diagonalizable unstable system to be stabilizable. The condition (5.3) in Theorem 5.3.2 is often referred to as the finite "unstable part" of the system being controllable. In the proof of the theorem we will see that we indeed then have a situation where the system splits into a finite-dimensional unstable part and an infinite-dimensional stable part, and the unstable part is controllable in the sense of controllability of finite-dimensional systems. For simplicity, we assume that the unstable eigenvalues of A are distinct. More generally, if $B \in \mathcal{L}(\mathbb{C}^m, X)$, it is under certain additional conditions possible to stabilize systems where the multiplicity of each unstable eigenvalue is at most m. The more general condition can be found in [9, Thm. 5.2.9].

Theorem 5.3.2. *Assume A is diagonalizable in such a way that $A = SA_DS^{-1}$ where $A_D = (\lambda_k)_{k\in\mathbb{N}}$. If there exists $\omega > 0$ and a set $I \subset \mathbb{N}$ of indices such that $\operatorname{Re}\lambda_k < -\omega < 0$ for all $k \in \mathbb{Z} \setminus I$, $\lambda_k \neq \lambda_l$ for all $k, l \in I$ with $k \neq l$, and*

$$(\langle b_1, \psi_k\rangle, \ldots, \langle b_m, \psi_k\rangle) \neq 0 \qquad \forall k \in I, \tag{5.3}$$

then the system (A, B, C, D) is exponentially stabilizable. In particular, there exists $K \in \mathcal{L}(U, X)$ such that the semigroup $(K(t))_{t\geq 0}$ generated by $A + BK$ satisfies $\|T_K(t)\| \leq Me^{-\omega t}$ for some constant $M \geq 1$ and for all $t \geq 0$.

Proof. We have $A + BK = SA_DS^{-1} + SS^{-1}BKSS^{-1} = S(A_D + S^{-1}BKS)S^{-1}$, where

$$S^{-1}B = \left(S^{-1}b_1, \ldots, S^{-1}b_m\right) \in \mathcal{L}(\mathbb{C}^m, \ell^2(\mathbb{C})).$$

By possibly changing the indexing of our matrices, we can assume that the set I of indices is equal to $I = \{1, \dots, N\}$. We can split the infinite diagonal matrix A_D and the operator $S^{-1}B \in \mathcal{L}(\mathbb{C}^m, \ell^2(\mathbb{C}))$ into parts as

$$A_D = \begin{pmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \lambda_N & \\ & & & \ddots \end{pmatrix} = \begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix},$$

$$S^{-1}B = \left(S^{-1}b_1, S^{-1}b_2, \dots, S^{-1}b_m\right) = \left(\begin{pmatrix} b_1^0 \\ b_1^1 \end{pmatrix}, \dots, \begin{pmatrix} b_m^0 \\ b_m^1 \end{pmatrix}\right) = \begin{pmatrix} B_0 \\ B_1 \end{pmatrix}$$

where $A_0 = (\lambda_k)_{k=1}^N \in \mathbb{C}^{N\times N}$, $A_1 = (\lambda_k)_{k=N+1}^\infty$, $B_0 \in \mathbb{C}^{N\times m}$ and $B_1 \in \mathcal{L}(\mathbb{C}^m, \ell^2(\mathbb{C}))$. Moreover, the matrix B_0 has the form

$$B_0 = \left(b_1^0, b_2^0, \dots, b_m^0\right) = \begin{pmatrix} \langle S^{-1}b_1, e_1\rangle & \dots & \langle S^{-1}b_m, e_1\rangle \\ \langle S^{-1}b_1, e_2\rangle & \dots & \langle S^{-1}b_m, e_2\rangle \\ \vdots & & \\ \langle S^{-1}b_1, e_N\rangle & \dots & \langle S^{-1}b_m, e_N\rangle \end{pmatrix} \tag{5.4a}$$

$$= \begin{pmatrix} \langle b_1, (S^{-1})^*e_1\rangle & \dots & \langle b_m, (S^{-1})^*e_1\rangle \\ \vdots & & \\ \langle b_1, (S^{-1})^*e_N\rangle & \dots & \langle b_m, (S^{-1})^*e_N\rangle \end{pmatrix} = \begin{pmatrix} \langle b_1, \psi_1\rangle & \dots & \langle b_m, \psi_1\rangle \\ \vdots & & \\ \langle b_1, \psi_N\rangle & \dots & \langle b_m, \psi_N\rangle \end{pmatrix}, \tag{5.4b}$$

since $\psi_k = (S^{-1})^*e_k$ for all $k \in \mathbb{N}$ by definition. The condition (5.3) implies that none of the row vectors of the matrix B_0 are zero.

We will now show that the finite-dimensional system (A_0, B_0) is controllable in the sense of Definition 2.1.1. This can be done by showing that the controllability matrix satisfies $\mathrm{rank}(B_0, A_0B_0, \dots, A_0^{N-1}B_0) = N$. If we denote

$$B_0 = \begin{pmatrix} d_1 \\ \vdots \\ d_N \end{pmatrix},$$

where $d_k \in \mathbb{C}^{1\times m}$ are the rows of the matrix B_0, we have

$$\left(B_0, A_0B_0, \dots, A_0^{N-1}B_0\right) = \begin{pmatrix} d_1 & \lambda_1 d_1 & \lambda_1^2 d_1 & \cdots & \lambda_1^{N-1}d_1 \\ d_2 & \lambda_2 d_2 & \lambda_2^2 d_2 & \cdots & \lambda_2^{N-1}d_2 \\ \vdots & \vdots & & & \vdots \\ d_N & \lambda_N d_N & \lambda_N^2 d_N & \cdots & \lambda_N^{N-1}d_N \end{pmatrix}$$

$$= \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_N \end{pmatrix} \begin{pmatrix} I & \lambda_1 I & \dots & \lambda_1^{N-1}I \\ I & \lambda_2 I & \dots & \lambda_2^{N-1}I \\ \vdots & \vdots & & \vdots \\ I & \lambda_N I & \dots & \lambda_N^{N-1}I \end{pmatrix}.$$

Here the first matrix is $N \times mN$, and its rank is equal to N since $d_k \neq 0$ for all $k \in \{1, \dots, N\}$ due to condition (5.3). Moreover, the second matrix has dimensions $mN \times mN$, and since

$\lambda_k \neq \lambda_l$ for all $k, l \in \{1, \dots, N\}$ with $k \neq l$, its determinant is equal to

$$\left(\prod_{1 \le k < l \le N} (\lambda_l - \lambda_k) \right)^m \neq 0$$

since the matrix is similar to a matrix $\operatorname{diag}(V, V, \dots, V) \in \mathbb{C}^{mN \times mN}$ where V is the *Vandermonde matrix*

$$V = \begin{pmatrix} 1 & \lambda_1 & \dots & \lambda_1^{N-1} \\ 1 & \lambda_2 & \dots & \lambda_2^{N-1} \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_N & \dots & \lambda_N^{N-1} \end{pmatrix}, \qquad \det(V) = \prod_{1 \le k < l \le N} (\lambda_l - \lambda_k) \neq 0.$$

Since the controllability matrix $\left(B_0, A_0B_0, \dots, A_0^{N-1}B_0\right)$ is a product of two matrices where the first one has rank N and the other is nonsingular, also the rank of the controllability matrix is N. Thus by Theorem 2.1.3 the system (A_0, B_0) is controllable. By [14, Thm. 4.2.4] we can choose $K_0 \in \mathbb{C}^{m \times N}$ such that the eigenvalues of the matrix $A_0 + B_0K_0$ all have real parts smaller than $-\omega < 0$.

We choose the operator $K \in \mathcal{L}(\mathbb{C}^m, X)$ in such a way that $K = \left(K_0,\ 0\right) S^{-1}$. Our aim is to show that the operator

$$A_D + S^{-1}BKS = \begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix} + \begin{pmatrix} B_0 \\ B_1 \end{pmatrix} \left(K_0,\ 0\right) = \begin{pmatrix} A_0 + B_0K_0 & 0 \\ B_1K_0 & A_1 \end{pmatrix}$$

generates an exponentially stable semigroup $(T_K(t))_{t \ge 0}$ on $\ell^2(\mathbb{C})$. To this end, we will show that there exists $M \ge 1$ such that $\|T_K(t)x_0\| \le Me^{-\omega t}\|x_0\|$. We know that $x(t) = T_K(t)x_0$ is the solution of the infinite-dimensional differential equation

$$\dot{x}(t) = (A_D + S^{-1}BKS)x(t), \qquad x(0) = x_0 \in X.$$

We can write $x(t) = (x_1(t)$
$x_2(t))^T \in X$ and $x_0 = (x_0^1, x_0^2)^T \in X$, where $x_1(t), x_0^1 \in \mathbb{C}^N$ and $x_2(t), x_2^0 \in \ell^2(\mathbb{C})$, and the above differential equation can be written in the form

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} A_0 + B_0K_0 & 0 \\ B_1K_0 & A_1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \qquad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} x_0^1 \\ x_2^0 \end{pmatrix}$$

$$\Leftrightarrow \qquad \begin{cases} \dot{x}_1(t) = (A_0 + B_0K_0)x_1(t), & x_1(0) = x_0^1 \\ \dot{x}_2(t) = A_1x_2(t) + B_1K_0x_1(t), & x_2(0) = x_0^2 \end{cases}$$

Since the eigenvalues of $A_0 + B_0K_0$ have real parts smaller than $-\omega < 0$, there exists $\omega_0 > \omega > 0$ and $M_0 \ge 1$ such that $\|e^{(A_0+B_0K_0)t}\| \le M_0e^{-\omega_0 t}$ for all $t \ge 0$. Moreover, since $A_1 = \operatorname{diag}(\lambda_k)_{k=N+1}^{\infty}$ where $\operatorname{Re}\lambda_k \le -\omega < 0$, the semigroup $(T_1(t))_{t \ge 0}$ generated by A_1 satisfies $T_1(t) = (e^{\lambda_k t})_{k=N+1}^{\infty}$ and $\|T_1(t)\| \le e^{-\omega t}$ for $t \ge 0$. Thus

$$\|x_1(t)\| \le \|e^{(A_0+B_0K_0)t}x_0^1\| \le M_0e^{-\omega_0 t}\|x_0^1\|$$

and the solution of the second differential equation satisfies

$$\begin{aligned}
\|x_2(t)\| &= \left\| T_1(t)x_0^2 + \int_0^t T_1(t-s)B_1K_0x_1(s)ds \right\| \\
&\leq \|T_1(t)x_0^2\| + \int_0^t \|T_1(t-s)\|\|B_1K_0\|\|x_1(s)\|ds \\
&\leq M_0e^{-\omega_0 t}\|x_0^2\| + M_0\|B_1K_0\|\|x_0^1\| \int_0^t e^{-\omega(t-s)}e^{-\omega_0 s}ds \\
&= M_0e^{-\omega_0 t}\|x_0^2\| + M_0\|B_1K_0\|\|x_0^1\|e^{-\omega t} \int_0^t e^{(\omega-\omega_0)s}ds \\
&\leq M_0e^{-\omega_0 t}\|x_0^2\| + M_0\|B_1K_0\|\|x_0^1\|e^{-\omega t} \int_0^\infty e^{(\omega-\omega_0)s}ds \\
&= M_0e^{-\omega_0 t}\|x_0^2\| + M_0\|B_1K_0\|\|x_0^1\|e^{-\omega t}\frac{1}{\omega_0-\omega}
\end{aligned}$$

since $\omega_0 > \omega$. Combining the above estimates (and using the scalar inequality $(a+b)^2 \leq 2a^2 + 2b^2$ for $a, b \geq 0$) we get

$$\begin{aligned}
\|T_K(t)x_0\|^2 &= \|x(t)\|^2 = \|x_1(t)\|^2 + \|x_2(t)\|^2 \\
&\leq M_0^2e^{-2\omega_0 t}\|x_0^2\|^2 + \left(M_0e^{-\omega_0 t}\|x_0^2\| + \frac{M_0\|B_1K_0\|\|x_0^1\|}{\omega_0-\omega}e^{-\omega t}\right)^2 \\
&\leq M_0^2e^{-2\omega_0 t}\|x_0^2\|^2 + 2M_0^2e^{-2\omega_0 t}\|x_0^2\|^2 + 2\frac{M_0^2\|B_1K_0\|^2\|x_0^1\|^2}{(\omega_0-\omega)}e^{-2\omega t} \\
&\leq \left(3M_0^2 + 2\frac{M_0^2\|B_1K_0\|^2}{(\omega_0-\omega)}\right) e^{-2\omega t}\max\{\|x_0^1\|^2, \|x_0^2\|^2\} \\
&\leq \left(3M_0^2 + 2\frac{M_0^2\|B_1K_0\|^2}{(\omega_0-\omega)}\right) e^{-2\omega t}\|x_0\|^2
\end{aligned}$$

which shows that $\|T_K(t)x_0\| \leq Me^{-\omega t}\|x_0\|$ for all $t \geq 0$ if we choose $M \geq 1$ such that $M^2 = 3M_0^2 + 2\frac{M_0^2\|B_1K_0\|^2}{(\omega_0-\omega)}$. □

The proof of Theorem 5.3.2 also tells us how to choose the stabilizing feedback operator $K \in \mathcal{L}(X, \mathbb{C}^m)$.

Corollary 5.3.3. *The stabilizing feedback operator in Theorem* 5.3.2 *is* $K = \begin{pmatrix} K_0, & 0 \end{pmatrix} S^{-1}$, *where* $K_0 \in \mathcal{L}(\mathbb{C}^N, \mathbb{C}^m)$ *is such that the eigenvalues* $A_0 + B_0K_0$ *have negative real parts. Here* $A_0 = \operatorname{diag}(\lambda_k)_{k\in I}$ *and*

$$B_0 = \begin{pmatrix} \langle b_1, \psi_{k_1}\rangle & \dots & \langle b_m, \psi_{k_1}\rangle \\ \vdots & & \\ \langle b_1, \psi_{k_N}\rangle & \dots & \langle b_m, \psi_{k_N}\rangle \end{pmatrix},$$

where $I = \{k_1, \dots, k_N\}$ *is the set of indices* k *for which the real part of the eigenvalue* λ_k *is larger than or equal to zero.*

Example 5.3.4. In this example we consider the heat equation on an interval $[0,1]$ with *Neumann boundary conditions* (as opposed to *Dirichlet boundary condition* in Section 3.3.2)

$$\begin{aligned}
\frac{\partial v}{\partial t}(\xi,t) &= \alpha\frac{\partial^2 v}{\partial \xi^2}(\xi,t) + b(\cdot)u(t), \qquad \xi \in (0,1)\\
\frac{\partial v}{\partial \xi}(0) &= 0, \quad \frac{\partial v}{\partial \xi}(1) = 0,\\
v(\xi,0) &= v_0(\xi),\\
y(t) &= \int_0^1 v(\xi,t)c(\xi)d\xi
\end{aligned}$$

where $\alpha > 0$ and $b(\cdot), c(\cdot) \in L^2(0,1;\mathbb{R})$. Similarly as in Section 3.3.2 we can show that the heat equation can be formulated as an infinite-dimensional linear system on the space $X = L^2(0,1)$ with the system operator

$$A = \alpha\frac{d^2}{d\xi^2}, \qquad \mathcal{D}(A) = \left\{\, f \in X \;\middle|\; f, f' \text{ abs. cont. } f'' \in L^2(0,1), \text{ and } f'(0) = f'(1) = 0 \,\right\}.$$

The operator A has eigenvalues $\{\lambda_k\}_{k=0}^\infty = \{-\alpha\pi^2k^2\}_{k=0}^\infty \subset \mathbb{R}$ with corresponding orthonormal eigenfunctions $\phi_k(\cdot) = \sqrt{2}\cos(k\pi\cdot)$ for $k \in \mathbb{N}$ and $\phi_0(\cdot) \equiv 1$. The operator A is diagonalizable in such a way that $A = SA_DS^{-1}$ where $A_D = \operatorname{diag}(\lambda_k)_{k=0}^\infty$ and $S \in \mathcal{L}(\ell^2(\mathbb{C}), X)$ is a boundedly invertible operator defined by

$$Se_k = \phi_k, \qquad k \in \mathbb{N}_0 = \{0,1,2,\ldots\}.$$

Since A has an eigenvalue $\lambda_0 = 0$ on the imaginary axis, the semigroup generated by A is unstable. We will now use Theorem 5.3.2 to study the stabilization of the heat equation with state feedback. Since the eigenvectors $\{\phi_k\}_{k=0}^\infty$ are orthonormal, we have $\psi_k = \phi_k$ for all $k \in \mathbb{N}_0$, and by Theorem 5.3.2 the system is stabilizable if

$$\langle b, \phi_0\rangle = \int_0^1 b(\xi)\cdot 1 d\xi \neq 0.$$

We first need to stabilize the pair (A_0, B_0). Since we now have $I = \{0\}$, the matrices A_0 and B_0 are both of size 1×1, and given by

$$A_0 = \lambda_0 = 0, \qquad B_0 = \langle b, \phi_0\rangle = \int_0^1 b(\xi)d\xi.$$

If we assume $\langle b, \phi_0\rangle \neq 0$, then for the choice $K_0 = -\frac{\alpha\pi^2}{\langle b,\phi_0\rangle}$ we have

$$A_0 + B_0K_0 = -\langle b, \phi_0\rangle\frac{\alpha\pi^2}{\langle b, \phi_0\rangle} = -\alpha\pi^2 < 0.$$

Finally, by Corollary 5.3.3 the exponentially stabilizing feedback operator for the original system is given by

$$K = \begin{pmatrix} K_0, & 0 \end{pmatrix} S^{-1} = \begin{pmatrix} -\frac{\alpha\pi^2}{\langle b,\phi_0\rangle}, & 0 \end{pmatrix} S^{-1}.$$

As in Section 3.3.2, we can also consider the diagonalized system

$$\dot{z}(t) = A_D z(t) + B_D u(t), \qquad z(0) = S^{-1}x_0 \in Z$$
$$y(t) = C_D z(t) + Du(t)$$

on the space $Z = \ell^2(\mathbb{C})$ with state $z(t) = S^{-1}x(t)$, and $A_D = S^{-1}AS = \mathrm{diag}(\lambda_k)_k$, $B_D = S^{-1}B$, and $C_D = CS$. Since the stabilizing input is

$$u(t) = Kx(t) = \big(K_0,\ 0\big)\, S^{-1}x(t) = \big(K_0,\ 0\big)\, z(t),$$

we have that the appropriate stabilizing feedback operator for the diagonalized system is $\big(K_0,\ 0\big) \in \mathcal{L}(\mathbb{C}^m, \ell^2(\mathbb{C}))$.

The behaviour of the unstable and the exponentially stabilized heat equation are illustrated in Figures 5.2 and 5.3 in the situation where $b(\cdot) = \xi_{[0,1/2]}(\cdot)$. In particular, the state of the unstable heat equation converges to a nonzero final state, and the state of the stabilized system converges to zero.

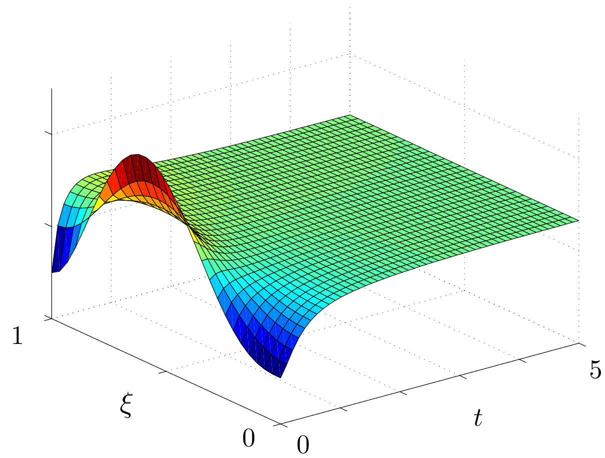

Figure 5.2: Unstable heat equation.

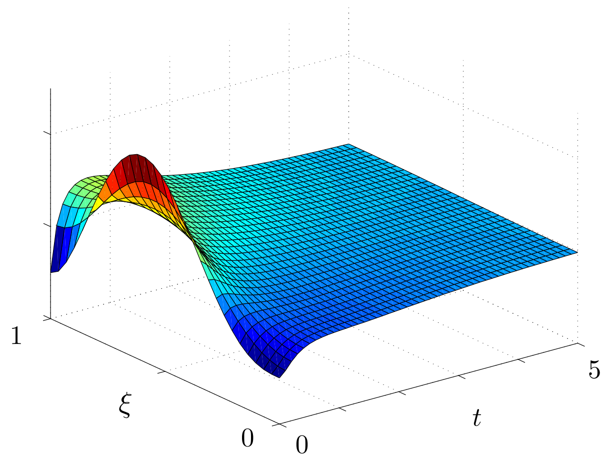

Figure 5.3: Stabilized heat equation.

◇

Theorem 5.3.2 presents conditions for exponential stabilizability of systems that have at most a finite number of "unstable eigenvalues" with nonnegative real parts. However, we saw in Section 4.4 that the spectrum of the system operator A of the controlled wave equation consists of an infinite number of points on the imaginary axis. Because of this, Theorem 5.3.2 can not be used in the stabilization of the wave equation. In fact, it turns out that the wave equation can not be made exponentially stable with bounded feedback.

Theorem 5.3.5. *Assume $B \in \mathcal{L}(\mathbb{C}^m, X)$ and assume the operator A is diagonalizable in such a way that $A = SA_DS^{-1}$ where $A_D = \mathrm{diag}(i\omega_k)_{k\in\mathbb{Z}}$ with $(\omega_k)_{k\in\mathbb{Z}} \subset \mathbb{R}$ and where $S \in \mathcal{L}(\ell^2(\mathbb{C}), X)$ is boundedly invertible. Then there exists no operator $K \in \mathcal{L}(X, \mathbb{C}^m)$ such that the semigroup generated by $A + BK$ is exponentially stable.*

Proof. Suppose that we can choose $K \in \mathcal{L}(X, \mathbb{C}^m)$ so that the semigroup generated by $A + BK$ is exponentially stable. Then we have from Theorem 4.2.4 that there exists $M > 0$ such that $\|R(i\omega, A + BK)\| \leq M$ for all $\omega \in \mathbb{R}$. This also implies that for all $x \in \mathcal{D}(A)$

$$\begin{aligned}\|x\| &= \|R(i\omega, A - BK)(i\omega - A - BK)x\| \leq \|R(i\omega, A - BK)\|\|(i\omega - A - BK)x\| \\ &\leq M\|(i\omega - A - BK)x\|\end{aligned}$$

and thus $\|(i\omega - A - BK)x\| \geq \frac{1}{M}\|x\|$ for all $\omega \in \mathbb{R}$ and $x \in \mathcal{D}(A)$. In particular, it should be impossible to find sequences $(s_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ and $(x_k)_{k\in\mathbb{N}} \subset X$ such that $\|x_k\| \geq c > 0$ for some constant $c > 0$ and $\|(is_k - A - BK)x_k\| \to 0$ as $k \to \infty$. We will show that such sequences do exist, and thus the semigroup generated by $A + BK$ is not exponentially stable.

We choose $s_k = \omega_k$ and $x_k = Se_k$ for all $k \in \mathbb{N}$. Since S is boundedly invertible, we have $1 = \|e_k\| = \|S^{-1}Se_k\| \leq \|S^{-1}\|\|x_k\|$, and thus $\|x_k\| \geq 1/\|S^{-1}\| > 0$ for all $k \in \mathbb{N}$. The operator K is such that

$$Kx_k = KSe_k = \begin{pmatrix} \langle Se_k, d_1\rangle \\ \vdots \\ \langle Se_k, d_m\rangle \end{pmatrix} = \begin{pmatrix} \langle e_k, S^*d_1\rangle \\ \vdots \\ \langle e_k, S^*d_m\rangle \end{pmatrix}$$

for some $d_j \in X$ for $j \in \{1, \ldots, m\}$. We have $S^*d_j \in \ell^2(\mathbb{C})$, which means that

$$\sum_{k\in\mathbb{Z}} |\langle S^*d_j, e_k\rangle|^2 < \infty.$$

In particular, this means that necessarily $|\langle S^*d_j, e_k\rangle| \to 0$ as $k \to \infty$ for all $j \in \{1, \ldots, m\}$. Since $Ax_k = SA_DS^{-1}Se_k = SA_De_k = i\omega_kSe_k = i\omega_kx_k$, a direct computation shows that

$$\|(i\omega_k - A - BK)x_k\| = \|BKx_k\| \leq \|B\|\|Kx_k\| = \|B\| \left(\sum_{j=1}^m |\langle e_k, S^*d_j\rangle|^2\right)^{1/2} \longrightarrow 0$$

as $k \to \infty$. As mentioned above, this contradicts the assumption that the semigroup generated by $A + BK$ is exponentially stable, and thus the proof is complete. □

The following theore shows that the wave equation, and more generally a diagonalizable system with eigenvalues on the imaginary axis, can be stabilized strongly whenever it is approximately controllable.

Theorem 5.3.6. *Assume $B \in \mathcal{L}(\mathbb{C}^m, X)$ and assume the operator A is diagonalizable in such a way that $A = SA_DS^{-1}$ where $A_D = \operatorname{diag}(i\omega_k)_{k\in\mathbb{Z}}$ for $(\omega_k)_{k\in\mathbb{Z}} \subset \mathbb{R}$ satisfying $\inf_{k\neq l}|\omega_k - \omega_l| > 0$, and where $S \in \mathcal{L}(\ell^2(\mathbb{C}), X)$ is boundedly invertible. If the pair (A, B) is approximately controllable, then the system can be stabilized strongly with state feedback*

$$u(t) = -\kappa B^*(S^{-1})^*S^{-1}x(t) + \tilde{u}(t), \qquad \kappa > 0.$$

Proof. The system operator after the state feedback is equal to $A - \kappa BB^*(S^{-1})^*S^{-1}$, and this operator satisfies

$$\begin{aligned}A - \kappa BB^*(S^{-1})^*S^{-1} &= SA_DS^{-1} - \kappa SS^{-1}BB^*(S^{-1})^*S^{-1} \\ &= S(A_D - \kappa S^{-1}B(S^{-1}B)^*)S^{-1} = S(A_D - B_DB_D^*)S^{-1}\end{aligned}$$

where $B_D = \sqrt{\kappa}S^{-1}B$. The system with the proposed state feedback is strongly stable if and only if the semigroup $(T_B(t))_{t\geq 0}$ generated by $A_D - B_D B_D^*$ on $\ell^2(\mathbb{C})$ is strongly stable. To verify that $(T_B(t))_{t\geq 0}$ is strongly stable, we will first use the Lumer–Phillips Theorem presented in 3.3.1 to show that $(T_B(t))_{t\geq 0}$ is a contraction semigroup, and therefore uniformly bounded. Subsequently, we will show that the imaginary axis is in the resolvent set of $A_D - B_D B_D^*$. By the so-called *Arendt–Batty–Lyubich–Vũ Theorem* [1, 16], these two properties are enough to guarantee that the semigroup generated by $A_D - B_D B_D^*$ is strongly stable.

Because the operator $-B_D B_D^*$ is bounded, we know that $A_D - B_D B_D^*$ generates a semigroup, and the contractivity of the semigroup using the Lumer–Phillips Theorem actually only requires verifying that $\operatorname{Re}\langle (A_D - B_D B_D^*)x, x\rangle \leq 0$ for all $x \in \mathcal{D}(A_D - B_D B_D^*) = \mathcal{D}(A_D)$. To this end, let $x \in \mathcal{D}(A)$ be arbitrary. Then

$$\begin{aligned}\operatorname{Re}\langle (A_D - B_D B_D^*)x, x\rangle &= \operatorname{Re}\langle A_D x, x\rangle - \operatorname{Re}\langle B_D B_D^* x, x\rangle \\ &= \operatorname{Re}\langle A_D x, x\rangle - \operatorname{Re}\langle B_D^* x, B_D^* x\rangle = -\|B_D^* x\|^2 \leq 0,\end{aligned}$$

since for any $x = (x_k)_{k\neq 0} \in \ell^2(\mathbb{C})$ a direct computation shows that

$$\operatorname{Re}\langle A_D x, x\rangle = \operatorname{Re}\langle (i\omega_k x_k)_{k\neq 0}, (x_k)_{k\neq 0}\rangle = \operatorname{Re}\sum_{k\neq 0} i\omega_k |x_k|^2 = 0.$$

Thus the Lumer–Phillips theorem tells us that the semigroup $(T_B(t))_{t\geq 0}$ generated by $A_D - B_D B_D^*$ satisfies $\|T_B(t)\| \leq 1$.

The proof of the theorem technically requires that we show that the imaginary axis $i\mathbb{R}$ belongs to the resolvent set of $A_D - B_D B_D^*$. This means that $i\omega - A_D + B_D B_D^*$ should have a bounded inverse for all $\omega \in \mathbb{R}$. In this proof we will only prove a weaker property that all the eigenvalues of $A_D - B_D B_D^*$ have strictly negative real parts. We do this because the proof of this property is more illustrative. Assume $\lambda \in \mathbb{C}$ is an eigenvalue of $A_D - B_D B_D^*$, and $x \neq 0$ is such that $(A_D - B_D B_D^*)x = \lambda x$. Our aim is to show that $\operatorname{Re}\lambda < 0$. We then have

$$\begin{aligned}\operatorname{Re}\lambda \|x\|^2 = \operatorname{Re}\lambda\langle x, x\rangle &= \operatorname{Re}\langle \lambda x, x\rangle = \operatorname{Re}\langle (A_D - B_D B_D^*)x, x\rangle \\ &= \operatorname{Re}\langle A_D x, x\rangle - \operatorname{Re}\langle B_D^* x, B_D^* x\rangle = -\operatorname{Re}\|B_D^* x\|^2\end{aligned}$$

since $\operatorname{Re}\langle A_D x, x\rangle = 0$ as shown above. Since $\|x\| > 0$, the above equation implies that $\operatorname{Re}\lambda < 0$ if $B_D^* x \neq 0$. However, if $B_D^* x = 0$, then

$$(A_D - B_D B_D^*)x = \lambda x \qquad \Leftrightarrow \qquad A_D x = \lambda x,$$

which means that $\lambda = i\omega_k$ and $x = \alpha e_k$ for some $k \in \mathbb{Z}\setminus\{0\}$ and $\alpha \in \mathbb{C}\setminus\{0\}$. Since $B = (b_1, \ldots, b_m)$ and $B_D = \sqrt{\kappa}S^{-1}B = \sqrt{\kappa}(S^{-1}b_1, \ldots, S^{-1}b_m)$ for $b_j \in X$, we have

$$0 = B_D^* x = B_D^* e_k = \sqrt{\kappa}\begin{pmatrix}\langle e_k, S^{-1}b_1\rangle \\ \vdots \\ \langle e_k, S^{-1}b_m\rangle\end{pmatrix} = \sqrt{\kappa}\begin{pmatrix}\langle (S^{-1})^* e_k, b_1\rangle \\ \vdots \\ \langle (S^{-1})^* e_k, b_m\rangle\end{pmatrix} = \sqrt{\kappa}\begin{pmatrix}\langle \psi_k, b_1\rangle \\ \vdots \\ \langle \psi_k, b_m\rangle\end{pmatrix},$$

but by Theorem 4.3.4 this contradicts the assumption that the system is approximately controllable. Because of this, the above computations imply that all the eigenvalues of $A_D - B_D B_D^*$ have negative real parts. □

Example 5.3.7. We can apply Theorem 5.3.6 to stabilize the wave equation in Section 4.4. We defined $S \in \mathcal{L}(\ell^2(\mathbb{C}), X)$ in such a way that $Se_k = \varphi_k$ for all $k \in \mathbb{Z} \setminus \{0\}$. We begin by showing that $S^* = S^{-1} \in \mathcal{L}(X, \ell^2(\mathbb{C}))$. Since $\{\varphi_k\}_{k\neq 0}$ is a basis of X, it is sufficient to show that $S^*\varphi_k = e_k = S^{-1}\varphi_k$ for all $k \neq 0$. To this end, let $y \in \ell^2(\mathbb{C})$ is arbitrary. Now for all $k \in \mathbb{Z} \setminus \{0\}$ we have

$$\begin{aligned}\langle S^*\varphi_k, y\rangle_{\ell^2} = \langle \varphi_k, Sy\rangle_X = \left\langle \varphi_k, \sum_{l\neq 0} \langle y, e_l\rangle Se_l \right\rangle_X = \sum_{l\neq 0} \overline{\langle y, e_l\rangle_{\ell^2}} \langle \varphi_k, Se_l\rangle_X \\ = \sum_{l\neq 0} \langle e_l, y\rangle_{\ell^2} \langle \varphi_k, \varphi_l\rangle_X = \langle e_k, y\rangle_{\ell^2}\end{aligned}$$

since the set $\{\varphi_k\}_{k\neq 0}$ is orthonormal. Since the above identity is true for all $y \in \ell^2(\mathbb{C})$, we must have that $S^*\varphi_k = e_k = S^{-1}\varphi_k$. Since $k \neq 0$ was arbitrary, we further have $S^* = S^{-1}$.

Using $(S^{-1})^* = (S^*)^* = S$ the strongly stabilizing feedback in Theorem 5.3.6 becomes

$$u(t) = -\kappa B^*(S^{-1})^* S^{-1} x(t) + \tilde{u}(t) = -\kappa B^* S S^{-1} x(t) + \tilde{u}(t) = -\kappa B^* x(t) + \tilde{u}(t),$$

which implies that the semigroup generated by $A - \kappa BB^*$ is strongly stable provided that the pair (A, B) is approximately controllable.

If the controlled wave equation has one input with

$$b(\xi) = \begin{cases} 1 & \frac{1}{3\sqrt{2}} \leq \xi \leq \frac{2}{3\sqrt{2}} \\ 0 & \text{otherwise} \end{cases}$$

then the system is approximately controllable, and it can be stabilized with the state feedback $u(t) = -\kappa B^* x(t) + \tilde{u}(t)$ where $\kappa > 0$. Figures 5.4 and 5.5 illustrate the behaviour of the orignal unstable wave equation and the strongly stabilized wave equation, respectively. In the simulation we chose $\kappa = 2$.

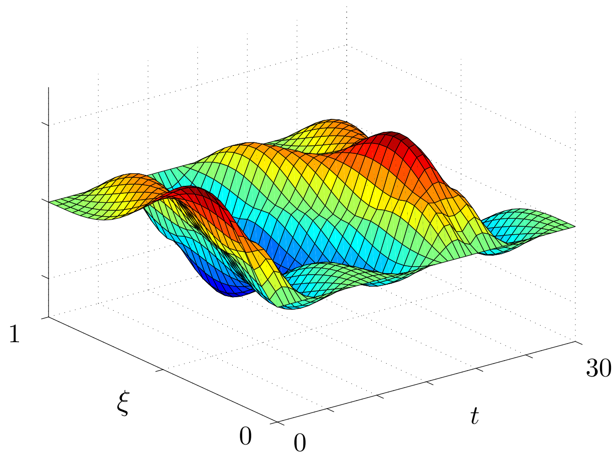

Figure 5.4: Unstable wave equation.

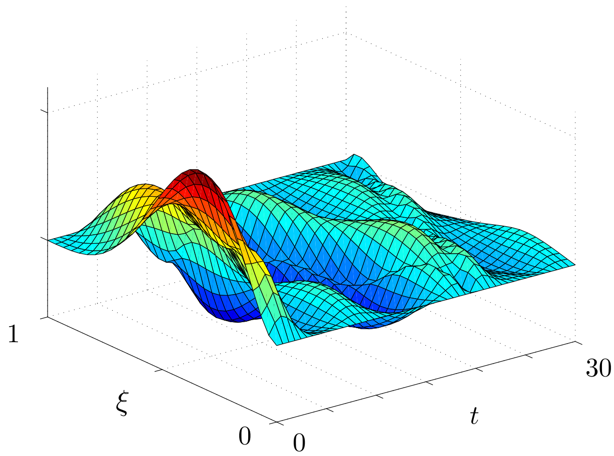

Figure 5.5: Stabilized wave equation.

◇

It should be noted that the choice for the operator K given in Theorem 5.3.6 is not the only choice for achieving strong stability for the wave equation and for similar systems. More generally, for a diagonalizable operator with its spectrum on the imaginary axis it is possible to use *pole placement of an infinite spectrum* [18, 21] to choose K in such a way that eigenvalues of the operator $A + BK$ are assigned to predetermined locations and the semigroup generated by $A+BK$ is strongly stable. A necessary requirement for the locations of the assigned eigenvalues is that the distances from the eigenvalues of A converge to zero [18], but this approach can nevertheless lead to considerably better stability properties for the stabilized system. See [21] for details on this procedure.

5.4 Output Tracking for Stabilizable Systems

In this section we consider the output tracking problem for unstable systems. The following theorem shows that the output tracking problem for a reference signal of the form

$$y_{ref}(t) = \sum_{k=-q}^{q} a_k e^{i\omega_0 kt}, \qquad \text{where} \qquad a_k \in \mathbb{C}^p,\ \omega_0 = \frac{2\pi}{\tau} \tag{5.5}$$

is solvable whenever the system is stabilizable. In particular, the result shows that we can use an input consisting of two parts: A stabilizing state feedback and a control input that solves the output tracking problem for the stabilized system.

Theorem 5.4.1. *Assume $K \in \mathcal{L}(X, \mathbb{C}^m)$ can be chosen in such a way that the semigroup generated by $A + BK$ is strongly stable and $\{i\omega_0 k\}_{k=-q}^{q} \subset \rho(A + BK)$. Let $y_{ref}(t)$ be the reference signal of the form* (5.5)*. If we can choose $\{u_k\}_{k=-q}^{q} \subset \mathbb{C}^m$ in such a way that*

$$P_K(i\omega_0 k)u_k = a_k \qquad k \in \{-q, \dots, q\},$$

where $P_K(\lambda) = (C + DK)R(\lambda, A + BK)B + D$, and if we choose the input of the system as

$$u(t) = Kx(t) + \sum_{k=-q}^{q} u_k e^{i\omega_0 kt}, \tag{5.6}$$

then the output of the system satisfies

$$\|y(t) - y_{ref}(t)\|_{\mathbb{C}^p} \to 0, \qquad \text{as} \quad t \to \infty.$$

Proof. The input (5.6) is of the form $u(t) = Kx(t) + \tilde{u}(t)$, and with this input the system becomes

$$\begin{aligned} \dot{x}(t) &= (A + BK)x(t) + B\tilde{u}(t), \qquad x(0) = x_0 \in X \\ y(t) &= (C + DK)x(t) + D\tilde{u}(t). \end{aligned}$$

Since the function $P_K(\cdot)$ where $P_K(\lambda) = (C + DK)R(\lambda, A + BK)B + D$ is the transfer function of the above system, we have from Theorem 5.2.3 that the choice

$$\tilde{u}(t) = \sum_{k=-q}^{q} u_k e^{i\omega_0 kt}$$

the output of the system satisfies $\|y(t) - y_{ref}(t)\|_{\mathbb{C}^p} \to 0$ as $t \to \infty$. □

For finite-dimensional the transfer function $P_K(\lambda)$ of the stabilized plant can simply be computed using the matrices $(A+BK, B, C+DK, D)$ of the stabilized system. For infinite-dimensional systems one can use again use a numerical approximation, or if the original system is stabilized by moving a finite number of unstable eigenvalues to the half-plane $\mathbb{C}^-$ as in Theorem 5.3.2, then we can use the formula in the following lemma.

Lemma 5.4.2. *If a system (A, B, C, D) is stabilized as in Theorem 5.3.2, then the transfer function $P_K(\lambda)$ has the form*

$$P_K(\lambda) = (C_0 + DK_0)R(\lambda, A_0 + B_0K_0)B_0 + D + P_1(\lambda)(I + K_0R(\lambda, A_0 + B_0K_0)B_0).$$

where $CS = (C_0,\ C_1)$ and $P(\lambda) = C_1R(\lambda, A_1)B_1$ is such that for all $\lambda \in \rho(A_1)$

$$\left(\sum_{k=N+1}^{\infty} \frac{\langle b_j, \psi_k\rangle\langle c_l, \phi_k\rangle}{\lambda - \lambda_k} \right)_{lj} \in \mathbb{C}^{p\times m}, \qquad \phi_k = Se_k,\ \psi_k = (S^{-1})^* e_k.$$

Proof. We have

$$\begin{aligned}\lambda - A - BK = S(\lambda - A_D - S^{-1}BKS)S^{-1} &= S\left(\lambda - \begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix} - \begin{pmatrix} B_0 \\ B_1 \end{pmatrix} (K_0,\ 0)\right) S^{-1} \\ &= S \begin{pmatrix} \lambda - A_0 - B_0K_0 & 0 \\ -B_1K_0 & \lambda - A_1 \end{pmatrix} S^{-1},\end{aligned}$$

where the inverse of the block-triangular operator is given by

$$\begin{pmatrix} \lambda - A_0 - B_0K_0 & 0 \\ -B_1K_0 & \lambda - A_1 \end{pmatrix}^{-1} = \begin{pmatrix} R(\lambda, A_0 + B_0K_0) & 0 \\ R(\lambda, A_1)B_1K_0R(\lambda, A_0 + B_0K_0) & R(\lambda, A_1) \end{pmatrix}.$$

This is indeed true since direct computations show that

$$\begin{aligned}&\begin{pmatrix} \lambda - A_0 - B_0K_0 & 0 \\ -B_1K_0 & \lambda - A_1 \end{pmatrix} \begin{pmatrix} R(\lambda, A_0 + B_0K_0) & 0 \\ R(\lambda, A_1)B_1K_0R(\lambda, A_0 + B_0K_0) & R(\lambda, A_1) \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ -B_1K_0R(\lambda, A_0 + B_0K_0) + B_1K_0R(\lambda, A_0 + B_0K_0) & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}&\begin{pmatrix} R(\lambda, A_0 + B_0K_0) & 0 \\ R(\lambda, A_1)B_1K_0R(\lambda, A_0 + B_0K_0) & R(\lambda, A_1) \end{pmatrix} \begin{pmatrix} \lambda - A_0 - B_0K_0 & 0 \\ -B_1K_0 & \lambda - A_1 \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ R(\lambda, A_1)B_1K_0 - R(\lambda, A_1)B_1K_0 & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.\end{aligned}$$

Denote $CS = (C_0,\ C_1)$. Since $\lambda - A - BK = S(\lambda - A_D - B_D(K_0,\ 0))S^{-1}$, where $B_D = S^{-1}B = \begin{pmatrix} B_0 \\ B_1 \end{pmatrix}$, we also have $R(\lambda, A + BK) = SR(\lambda, A_D + B_D(K_0,\ 0))S^{-1}$, and

$$\begin{aligned}P_K(\lambda) &= (C + DK)R(\lambda, A + BK)B + D = (C + DK)SR(\lambda, A_D + B_D(K_0,\ 0))S^{-1}B + D \\ &= (C_0 + DK_0,\ C_1) \begin{pmatrix} R(\lambda, A_0 + B_0K_0) & 0 \\ R(\lambda, A_1)B_1K_0R(\lambda, A_0 + B_0K_0) & R(\lambda, A_1) \end{pmatrix} \begin{pmatrix} B_0 \\ B_1 \end{pmatrix} + D \\ &= (C_0 + DK_0)R(\lambda, A_0 + B_0K_0)B_0 + C_1R(\lambda, A_1)B_1(I + K_0R(\lambda, A_0 + B_0K_0)B_0) + D.\end{aligned}$$

Similarly as in (5.4) we can see that

$$B_1 = \left(b_1^1, \dots, b_m^1\right), \qquad b_j^1 = \begin{pmatrix} \langle b_j, \psi_{N+1}\rangle \\ \langle b_j, \psi_{N+2}\rangle \\ \vdots \end{pmatrix}$$

$$C_1 x = \begin{pmatrix} \langle x, c_1^1\rangle_{\ell^2} \\ \dots \\ \langle x, c_p^1\rangle_{\ell^2} \end{pmatrix}, \qquad c_j^1 = \begin{pmatrix} \langle c_j, \phi_{N+1}\rangle \\ \langle c_j, \phi_{N+2}\rangle \\ \vdots \end{pmatrix}$$

Since $A_1 = \operatorname{diag}(\lambda_k)_{k=N+1}^{\infty}$ is the "stable part" of the original system, we have that for any $u = (u_1, \dots, u_m)^T \in \mathbb{C}^m$

$$C_1 R(\lambda, A_1) B_1 u = C_1 R(\lambda, A_1) \sum_{j=1}^m b_j^1 u_j = \sum_{j=1}^m C_1 R(\lambda, A_1) b_j^1 u_j = \sum_{j=1}^m \begin{pmatrix} \sum_{k=N+1}^{\infty} \frac{\langle b_j, \psi_k\rangle \langle c_1, \phi_k\rangle}{\lambda - \lambda_k} \\ \vdots \\ \sum_{k=N+1}^{\infty} \frac{\langle b_j, \psi_k\rangle \langle c_p, \phi_k\rangle}{\lambda - \lambda_k} \end{pmatrix} u_j.$$

This means that for every $\lambda \in \rho(A_1)$ the operator $C_1 R(\lambda, A_1) B_1$ is a $p \times m$ matrix with elements

$$\sum_{k=N+1}^{\infty} \frac{\langle b_j, \psi_k\rangle \langle c_l, \phi_k\rangle}{\lambda - \lambda_k}, \qquad j \in \{1, \dots, m\},\ l \in \{1, \dots, p\}.$$

□

Example 5.4.3. We will now consider output tracking for the heat equation with Neumann boundary conditions in studied in Example 5.3.4. Assume the system has one input and one measured output with

$$b(\xi) = \chi_{[0,1/2]}(\xi) = \begin{cases} 1 & 0 \le \xi \le 1/2 \\ 0 & \text{otherwise} \end{cases} \qquad c(\xi) = \chi_{[1/2,1]}(\xi) = \begin{cases} 1 & 1/2 \le \xi \le 1 \\ 0 & \text{otherwise} \end{cases}$$

We saw that the equation is unstable due to one eigenvalue $0 \in \sigma_p(A)$, but it can be stabilized with state feedback. We have $\langle b, \phi_0\rangle_{L^2} = \int_0^{1/2} 1 d\xi = 1/2$, and thus the appropriate stabilizing feedback operator $K \in \mathcal{L}(X, \mathbb{C})$ is given by

$$K = \left(K_0,\ 0\right) S^{-1} = \left(-2\alpha\pi^2,\ 0\right) S^{-1}.$$

We can use the formula in Lemma 5.4.2 to compute $P_K(\lambda)$. Since $C_0 = \langle c, \phi_0\rangle = 1/2$, $D = 0$ and

$$B_0 = \frac{1}{2}, \qquad K_0 = -2\alpha\pi^2, \qquad A_0 + B_0 K_0 = -\alpha\pi^2, \qquad R(\lambda, A_0 + B_0 K_0) = \frac{1}{\lambda + \alpha\pi^2},$$

we have

$$\begin{aligned} P_K(\lambda) &= C_0 R(\lambda, A_0 + B_0 K_0) B_0 + P_1(\lambda)(I + K_0 R(\lambda, A_0 + B_0 K_0) B_0) \\ &= \frac{1}{4(\lambda + \alpha\pi^2)} + (1 - \frac{\alpha\pi^2}{\lambda + \alpha\pi^2}) \sum_{k=1}^{\infty} \frac{\langle b, \phi_k\rangle \langle \phi_k, c\rangle}{\lambda + \alpha\pi^2 k^2}, \end{aligned}$$

where $\phi_k = \sqrt{2}\cos(k\pi\cdot)$ for $k \in \mathbb{N}$. By truncating the infinite series we can easily get numerical approximations for the values $P_K(\lambda)$ of the transfer function.

We consider the output tracking of a periodic triangle signal depicted in Figure 5.1 with $\tau = 2$. We can use a Fourier approximation

$$y_{ref}(t) = \sum_{k=-q}^{q} a_k e^{i\omega_0 kt}, \qquad \text{where} \qquad a_k \in \mathbb{C},\ \omega_0 = \frac{2\pi}{\tau} = \pi.$$

with $q = 7$. The coefficients a_k have the formula

$$a_k = \frac{(-1)^k - 1}{k^2\pi^2} = \begin{cases} -\frac{2}{k^2\pi^2} & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

Figure 5.6 depicts the behaviour of the state and output of the controlled system. Due to exponential stability of the semigroup generated by $A + BK$, the convergence of the output to the reference signal is guaranteed to be exponentially fast.

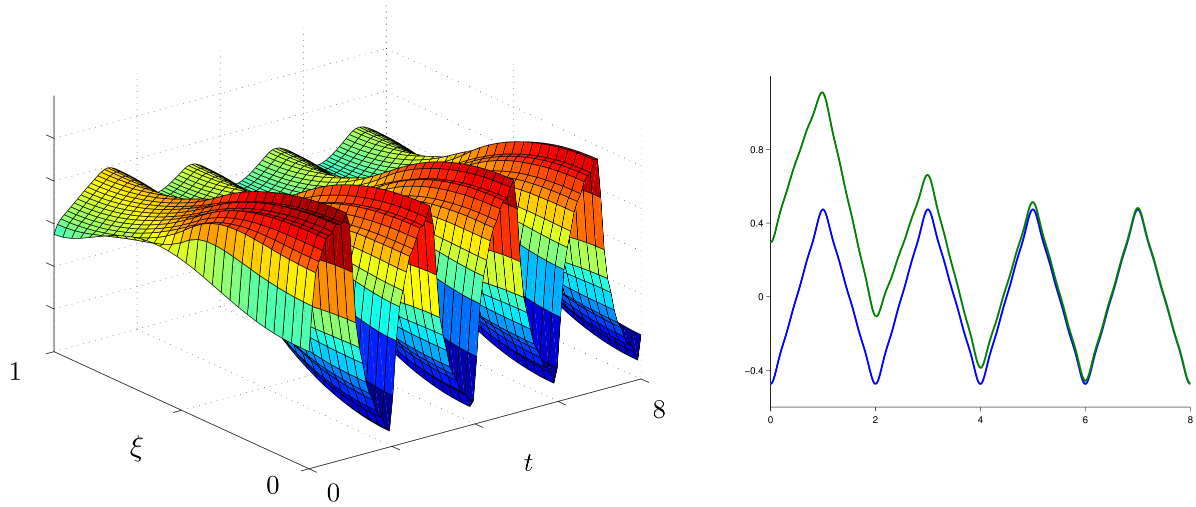

Figure 5.6: Tracking of a periodic triangle signal.

◇

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A. Finite-Dimensional Differential Equations

A.1 The Matrix Exponential Function

In this appendix we review some basic properties of the *matrix exponential function* e^{tA}, where $A \in \mathbb{R}^{n\times n}$ and $t \in \mathbb{R}$. This function plays a crucial role in studying systems of linear differential equations. We will see that the matrix exponential function can be computed conveniently using the Jordan canonical form.

It should be noted that the usefulness of the matrix exponential function in studying differential equations is mainly of theoretical nature: Numerical computation of an exponential matrix is very difficult, and therefore the differential equations should rather be solved numerically using other approaches, such as the *Runge-Kutta-methods*.

For a scalar $a \in \mathbb{R}$ the exponential function e^{ta} can be expressed using the series representation

$$e^{ta} = \sum_{k=0}^{\infty} \frac{(ta)^k}{k!}.$$

This same series representation can be used to define the exponential of a matrix. In view of the applications to solving differential equation, we define the exponential of a matrix directly for a matrix tA, where $t \in \mathbb{R}$.

Definition A.1.1. Matrix exponential function. Let $A \in \mathbb{R}^{n\times n}$. We define e^{tA} as the matrix

$$e^{tA} = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} \in \mathbb{R}^{n\times n}. \tag{A.1}$$

Remark A.1.2. In order for the definition to be sensible, it is important to ensure that the series in (A.1) is convergent. We, however, omit the proof in these lecture notes.

Exercise A.1.3. Use the definition to compute e^{tA}, when $t \in \mathbb{R}$, and (a) when $A = \alpha I \in \mathbb{C}^{n\times n}$ and $\alpha \in \mathbb{C}$ (b) when $A = O \in \mathbb{C}^{n\times n}$ (use the convention that $O^0 = I$). ◇

A.2 Linear Systems of Differential Equations

The most important application of the matrix exponential function is the solutions of linear systems of differential equations can be expressed using the matrix function e^{tA}. Let us consider a homogenic first order initial value problem

$$\begin{cases} \frac{d}{dt}x_1(t) &= a_{11}x_1(t) + a_{12}x_2(t) + a_{13}x_3(t) + \cdots + a_{1n}x_n(t) \\ \frac{d}{dt}x_2(t) &= a_{21}x_1(t) + a_{22}x_2(t) + a_{23}x_3(t) + \cdots + a_{2n}x_n(t) \\ &\vdots \\ \frac{d}{dt}x_n(t) &= a_{n1}x_1(t) + a_{n2}x_2(t) + a_{n3}x_3(t) + \cdots + a_{nn}x_n(t) \end{cases}, \qquad \begin{cases} x_1(0) = x_1^0, \\ x_2(0) = x_2^0, \\ \vdots \\ x_n(0) = x_n^0, \end{cases}$$

with n equations and n unknown functions $x_1(t), \ldots x_n(t)$. The initial values $x_1^0, \ldots, x_n^0 \in \mathbb{R}$ are known. The system of equations can be written as a homogenic first order matrix differential equation

$$\frac{d}{dt}\boldsymbol{x}(t) = A\boldsymbol{x}(t), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{A.2}$$

for all $t \geq 0$, where $\boldsymbol{x}(t) = (x_1(t), \ldots, x_n(t))^T \in \mathbb{R}$ is an unknown vector-valued function. The differentiation of $\boldsymbol{x}(t)$ with respect to t is understood component-wise, i.e.,

$$\frac{d}{dt}\boldsymbol{x}(t) = \left[\frac{d}{dt}x_1(t), \ldots, \frac{d}{dt}x_n(t)\right]^T.$$

The initial value of the equation (A.2) is the vector $\boldsymbol{x}_0 = (x_1^0, x_2^0, \ldots, x_n^0)^T \in \mathbb{R}^n$.

The following theorem states that the solution of the matrix differential equation (A.2) can be expressed using the matrix exponential function.

Theorem A.2.1. *The differential of the matrix exponential function with respect to* t *satisfies*

$$\frac{d}{dt}e^{tA} = Ae^{tA} = e^{tA}A. \tag{A.3}$$

The initial value problem (A.2) *has a unique solution*

$$\boldsymbol{x}(t) = e^{tA}\boldsymbol{x}_0.$$

Proof. We omit the proof of the differentiation formula (A.3). It can be proved using the series expression in (A.1), but this requires detailed consideration for the convergences of all the series involved.

We will first show that the function $\boldsymbol{x}(t) = e^{tA}\boldsymbol{x}_0$ is a solution of the initial value problem (A.2). It is immediate from the definition of the matrix exponential function that $e^{0 \cdot A} = e^O = I$. This implies that the function $\boldsymbol{x}(t)$ satisfies the initial condition $\boldsymbol{x}(0) = e^{0 \cdot A}\boldsymbol{x}_0 = I\boldsymbol{x}_0 = \boldsymbol{x}_0$. Using the differentiation formula (A.3) we can also see that for all $t > 0$ we have

$$\frac{d}{dt}\boldsymbol{x}(t) = \frac{d}{dt}\left(e^{tA}\boldsymbol{x}_0\right) = \left(\frac{d}{dt}e^{tA}\right)\boldsymbol{x}_0 = \left(Ae^{tA}\right)\boldsymbol{x}_0 = A\left(e^{tA}\boldsymbol{x}_0\right) = A\boldsymbol{x}(t).$$

This concludes that $\boldsymbol{x}(t)$ is a solution of the initial value problem (A.2).

To prove the uniqueness of the solution, let us assume $\boldsymbol{y}(t)$ is a solution to the initial value problem (A.2). Our aim is to show that $\boldsymbol{y}(t) = e^{tA}\boldsymbol{x}_0$ for all $t \geq 0$.

Let us consider the derivative of the difference $\boldsymbol{z}(t) = \boldsymbol{y}(t) - e^{tA}\boldsymbol{x}_0$. Using the knowledge that $\boldsymbol{y}(t)$ is a solution of (A.2) we get

$$\frac{d}{dt}\boldsymbol{z}(t) = \frac{d}{dt}\boldsymbol{y}(t) - \frac{d}{dt}\left(e^{tA}\boldsymbol{x}_0\right) = A\boldsymbol{y}(t) - Ae^{tA}\boldsymbol{x}_0 = A\left(\boldsymbol{y}(t) - e^{tA}\boldsymbol{x}_0\right) = A\boldsymbol{z}(t)$$

and $\boldsymbol{z}(0) = \boldsymbol{y}(0) - e^{0\cdot A}\boldsymbol{x}_0 = \boldsymbol{x}_0 - \boldsymbol{x}_0 = \mathbf{0}$. This implies that $\boldsymbol{z}(t)$ is a solution of the initial value problem

$$\frac{d}{dt}\boldsymbol{z}(t) = A\boldsymbol{z}(t), \qquad \boldsymbol{z}(0) = \mathbf{0}. \tag{A.4}$$

Let $t > 0$ be arbitrary. Define a function $\boldsymbol{u}(s) = e^{(t-s)A}\boldsymbol{z}(s)$ for $0 \le s \le t$. Using the differentiation rules for the product of two functions and for composition of functions we can see that

$$\begin{aligned}\frac{d}{ds}\boldsymbol{u}(s) &= \frac{d}{ds}\left(e^{(t-s)A}\boldsymbol{z}(s)\right) = \left(\frac{d}{ds}e^{(t-s)A}\right)\boldsymbol{z}(s) + e^{(t-s)A}\left(\frac{d}{ds}\boldsymbol{z}(s)\right)\\ &= (-1)e^{(t-s)A}A\boldsymbol{z}(s) + e^{(t-s)A}A\boldsymbol{z}(s) = \mathbf{0}.\end{aligned}$$

This implies that $(u_1(s), \ldots, u_n(s))^T = \boldsymbol{u}(s) = (0, \ldots, 0)^T$, and therefore $\boldsymbol{u}(s)$ is a constant function. In particular, we can see using the initial condition in (A.4) that

$$\boldsymbol{z}(t) = e^{(t-t)A}\boldsymbol{z}(t) = \boldsymbol{u}(t) = \boldsymbol{u}(0) = e^{(t-0)A}\boldsymbol{z}(0) = e^{tA}\mathbf{0} = \mathbf{0}.$$

Because $t > 0$ was arbitrary, we have shown that $\boldsymbol{z}(t) = \mathbf{0}$ for all $t \ge 0$. This immediately implies that $\boldsymbol{y}(t) = e^{tA}\boldsymbol{x}_0$ for all $t \ge 0$. □

A.3 Computing the Matrix Exponential Function e^{tA}

The matrix exponential function e^{At} can be computed conveniently using the Jordan canonical form $A = SJS^{-1}$ of the matrix A. If we consider a single term in the series (A.1), we then have

$$\begin{aligned}\frac{(tA)^k}{k!} &= \frac{t^k}{k!}\overbrace{AA\cdots A}^{k\text{ kpl}} = \frac{t^k}{k!}(SJS^{-1})(SJS^{-1})\cdots(SJS^{-1}) = \frac{t^k}{k!}SJS^{-1}SJS^{-1}\cdots SJS^{-1}\\ &= \frac{t^k}{k!}SJ^kS^{-1} = S\begin{bmatrix}(t^k/k!)J_1^k & 0 & \cdots & 0\\ 0 & (t^k/k!)J_2^k & \cdots & 0\\ \vdots & & \ddots & \vdots\\ 0 & \cdots & 0 & (t^k/k!)J_p^k\end{bmatrix}S^{-1}.\end{aligned}$$

Because of this, the matrix exponential function e^{tA} can be written in the form (omitting the considerations for the convergence of the series)

$$e^{tA} = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} = \sum_{k=0}^{\infty} \frac{St^kJ^kS^{-1}}{k!} = S\,\mathrm{diag}\left(\sum_{k=0}^{\infty} \frac{(tJ_1)^k}{k!}, \sum_{k=0}^{\infty} \frac{(tJ_2)^k}{k!}, \ldots, \sum_{k=0}^{\infty} \frac{(tJ_p)^k}{k!}\right) S^{-1}$$

$$= S \begin{bmatrix} e^{tJ_1} & 0 & \cdots & 0 \\ 0 & e^{tJ_2} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & e^{tJ_p} \end{bmatrix} S^{-1}$$

This way, computing e^{tA} is reduced to computing the exponential matrices e^{tJ_j} of the individual blocks of J. Since the blocks J_j are of particular forms, the following theorem covers all possible situations.

Theorem A.3.1. *The matrix exponential functions of the blocks J_j satisfy the following.*

- *If $J_j = \lambda \in \mathbb{R}^{1\times1}$, then $e^{tJ_j} = e^{t\lambda}$.*
- *If*

$$J_j = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \in \mathbb{R}^{2\times2}, \qquad \textit{then} \qquad e^{tJ_j} = e^{t\alpha} \begin{bmatrix} \cos(\beta t) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t) \end{bmatrix}.$$

- *If $J_j = \left[\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right] \in \mathbb{R}^{2\times2}$, then $e^{tJ_j} = e^{t\lambda} \left[\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right]$.*
- *If*

$$J_j = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \in \mathbb{R}^{3\times3}, \qquad \textit{then} \qquad e^{tJ_j} = e^{t\lambda} \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}.$$

- *If*

$$J_j = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & \cdots & \lambda & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{bmatrix} \in \mathbb{R}^{q\times q}, \quad \textit{then} \quad e^{tJ_j} = e^{t\lambda} \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{q-1}}{(q-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{q-2}}{(q-2)!} \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & \cdots & 1 & t \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}.$$

B. Some Elements of Functional Analysis

B.1 Infinite-Dimensional Vector Spaces

To make terminology more precise, we call a vector space finite-dimensional if it has a finite *basis* $\{q_1, \ldots, q_n\} \subset X$ where $n \in \mathbb{N}$ and $X = \operatorname{span}\{q_1, \ldots, q_n\}$. Also an infinite-dimensional vector space X may have a countably infinite basis $\{q_k\}_{k\in\mathbb{N}} \subset x$, but this is not always the case. Most of the spaces that we consider are relatively "nice" and have countable bases.

A *vector space* X over the field $\mathbb{C}$ of scalars is a set that is closed under the addition of two of its elements, i.e., $x + y \in X$ whenever $x, y \in X$, and closed under multiplication by scalar, i.e., $\alpha x \in X$ whenever $x \in X$ and $\alpha \in \mathbb{C}$. The computation rules of a vector space are the same as for vectors in the usual finite-dimensional spaces $\mathbb{C}^n$ and $\mathbb{R}^n$.

Example B.1.1. Some vector spaces:

(a) The space $X = C(0, 1)$ of complex-valued functions $f : [0, 1] \to \mathbb{C}$ that are continuous on the interval $[0, 1]$ is a vector space. Indeed, if f and g are continuous on $[0, 1]$ and if $\alpha \in \mathbb{C}$, then also the functions $f + g$ and αf are continuous on $[0, 1]$.

(b) The function space

$$L^2(0, 1; \mathbb{C}) = \left\{ f : (0, 1) \to \mathbb{C} \;\middle|\; \int_0^1 |f(\xi)|^2 d\xi < \infty \right\}.$$

is a vector space since if $f, g \in L^2(0, 1; \mathbb{C})$ and $\alpha \in \mathbb{C}$, then also $f + g \in L^2(0, 1; \mathbb{C})$ since

$$\int_0^1 |f(\xi) + g(\xi)|^2 d\xi \le 2 \int_0^1 |f(\xi)|^2 + 2 \int_0^1 |g(\xi)|^2 d\xi < \infty$$

and $\alpha f \in L^2(0, 1; \mathbb{C})$ since

$$\int_0^1 |\alpha f(\xi)|^2 d\xi = |\alpha|^2 \int_0^1 |f(\xi)|^2 < \infty.$$

(c) The space of *infinite sequences* (or infinite vectors)

$$X = \left\{ (x_1, x_2, x_3, \ldots) \;\middle|\; x_k \in \mathbb{C} \text{ for all } k \in \mathbb{N} \right\}$$

is a vector space. The addition and scalar multiplication of two vectors are defined as for vectors of finite lengths,

$$\begin{aligned} x + y &= (x_1 + y_1, x_2 + y_2, x_3 + y_3, \ldots) \\ \alpha x &= (\alpha x_1, \alpha x_2, \alpha x_3, \ldots). \end{aligned}$$

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In addition X is a *normed linear space* $(X, \|\cdot\|)$ if there is a function $\|\cdot\| : X \to [0, \infty)$ with the properties

(1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.

(2) $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in X$ and $\alpha \in \mathbb{C}$.

(3) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

The function $\|\cdot\|$ is then a *norm* on the space X.

The space X is an *inner product space space* $(X, \langle\cdot,\cdot\rangle)$ if there is a function $\langle\cdot, :\rangle X \times X \to \mathbb{C}$ with the properties

(1) $\langle x, x\rangle \geq 0$ for all $x \in X$ and $\langle x, x\rangle = 0$ if and only if $x = 0$.

(2) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ for all $x, y, z \in X$.

(2) $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$ for all $x, y \in X$ and $\alpha \in \mathbb{C}$.

(3) $\langle x, y\rangle = \overline{\langle y, x\rangle}$ for all $x, y \in X$.

The function $\langle\cdot,\cdot\rangle$ is an *inner product* on the space X. An inner product $\langle\cdot,\cdot\rangle$ can always be used to define a norm $\|\cdot\|$ such that $\|x\| = \sqrt{\langle x, x\rangle}$. This particular norm on the space X is called the *norm induced by the inner product* $\langle\cdot,\cdot\rangle$.

Definition B.1.2. A normed vector space $(X, \|\cdot\|)$ is called a *Banach space* if X is *complete* with respect to the norm $\|\cdot\|$. An inner product space $(X, \langle\cdot,\cdot\rangle)$ is called a *Hilbert space* if it is complete with respect to the norm induced by the inner product $\langle\cdot,\cdot\rangle$.

We recall that a vector space being "complete" means that every *Cauchy-sequence* $(x_k)_{k\in\mathbb{N}} \subset X$ converges in X. That is, if $(x_k)_{k\in\mathbb{N}} \subset X$ is such that $\lim_{k,l\to\infty}\|x_k - x_l\| = 0$, then there exist $x \in X$ such that $\lim_{k\to\infty}\|x_k - x\| = 0$.

Example B.1.3. We return to our earlier examples.

(a) The space $X = C(0, 1)$ of complex-valued functions $f : [0, 1] \to \mathbb{C}$ is a Banach space with the norm $\|\cdot\|$ defined by

$$\|f\| = \sup_{\xi\in[0,1]} |f(\xi)|.$$

It is also a normed linear space if we choose another norm $\|\cdot\|_{L^2}$ defined by

$$\|f\|_{L^2}^2 = \int_0^1 |f(\xi)|^2 d\xi,$$

but in this case the space is not complete.

(b) The function space $L^2(0,1;\mathbb{C})$ is a Banach space with the norm $\|\cdot\|_{L^2}$ defined above. In fact, it is a Hilbert space, because the norm $\|\cdot\|_{L^2}$ is induced by the inner product $\langle\cdot,\cdot\rangle_{L^2}$ defined by

$$\langle f, g\rangle_{L^2} = \int_0^1 f(\xi)\overline{g(\xi)}d\xi, \qquad f, g \in L^2(0,1).$$

Indeed, for every $f \in L^2(0,1)$ we have

$$\langle f, f\rangle_{L^2} = \int_0^1 f(\xi)\overline{f(\xi)}d\xi = \int_0^1 |f(\xi)|^2 d\xi, = \|f\|_{L^2}^2.$$

(c) The space of infinite sequences can be made into a Hilbert space if we only include elements $x = (x_1, x_2, \ldots) \in X$ for which the sum of the squares of the absolute values of the elements are finite, i.e.

$$X = \left\{\, x = (x_1, x_2, \ldots) \;\Big|\; \sum_{k=1}^{\infty} |x_k|^2 < \infty \,\right\}.$$

We can now define an inner product $\langle\cdot,\cdot\rangle$ and the corresponding induced norm by

$$\langle x, y\rangle = \sum_{k=1}^{\infty} x_k \overline{y_k}, \qquad \text{and} \qquad \|x\|^2 = \sum_{k=1}^{\infty} |x_k|^2.$$

This Hilbert space is commonly denoted by $\ell^2(\mathbb{C})$ (“small L-two”).

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Translations of Important Terms

Abstract Cauchy problem. Abstrakti Cauchy-ongelma
Adjoint operator. Adjugaatti-operaattori
Asymptotically stable. Asymptoottisesti stabiili

Banach space. Banach-avaruus
Basis (of a subspace). (Aliavaruuden) kanta

Control. Ohjaus
Controllability matrix. Ohjattavuusmatriisi
Controller. Säätäjä

Detectable. Havaittava
Diagonal. Diagonaalinen
Diagonalizable. Diagonalisoituva
Diagonalization. Diagonalisointi
Differential equation. Differentiaaliyhtälö
Distributed parameter system. Jakautunut järjestelmä
Disturbance rejection. Häirösignaalin vaimentaminen
Domain (of an operator). (Operaattorin) määrittelyjoukko

Eigenfunction. Ominaisfunktio
Eigenvalue. Ominaisarvo
Eigenvector. Ominaisvektori
Exponentially stable. Eksponentiaalisesti stabiili

Feedback. Takaisinkytkentä
Finite-dimensional. Äärellisulotteinen
Function space. Funktioavaruus

Half-plane $\mathbb{C}_\pm$. Puolitaso $\mathbb{C}_\pm$
Heat equation. Lämpöyhtälö
Hilbert space. Hilbert-avaruus

Infinite-dimensional. Ääretönulotteinen
Inner product. Sisätulo
Input. Sisääntulo, ohjaus

Jordan canonical form. Jordanin kanoninen muoto

Linear. Lineaarinen
Linear system. Lineaarinen järjestelmä

Matrix exponential function. Matriisieksponenttifunktio

Nonlinear. Epälineaarinen
Norm. Normi

Observable. Tarkkailtava
Observer. Tarkkailija
Operator. Operaattori
Optimal control. Optimisäätö
Output. Mittaus, ulostulo

Partial differential equation. Osittaisdifferentiaaliyhtälö
Plant. Järjestelmä

Robust. Robusti
Robust output regulation. Robusti regulointi
Robustness. Robustisuus

Semigroup. Puoliryhmä
Space. Avaruus
Stabilizable. Stabiloituva
Stable. Stabiili
State. Tila
State feedback. Tilatakaisinkytkentä
State Space. Tila-avaruus
Strongly continuous semigroup. Vahvasti jatkuva puoliryhmä
Strongly stable. Vahvasti stabiili (= asymptoottisesti stabiili)
Subspace. Aliavaruus
System. Järjestelmä

Transfer function. Siirtofunktio

Unbounded. Ei-rajoitettu
Uniformly bounded. Tasaisesti rajoitettu
Uniformly continuous. Tasaisesti jatkuva

Vector space. Vektoriavaruus

Wave equation. Aaltoyhtälö