Advanced Functional Analysis

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Contents

List of Notation (to be expanded)

A^{-1} – (algebraic) inverse of an operator
A^* – adjoint of an operator
$\|x\|$ – norm of an element x
$\langle x, y\rangle$ – inner product on an inner product space
$\mathcal{L}(X, Y)$ – The space of bounded linear operators $A : X \to Y$
$\mathcal{L}(X)$ – The space of bounded linear operators $A : X \to X$
$\mathcal{R}(A)$ – range of an operator
$\mathcal{N}(A)$ – kernel (or null-space) of an operator
$\mathcal{G}(A)$ – The graph of an operator
$L^p(a, b)$ – the Lebesgue space of functions $f : (a, b) \to \mathbb{C}$
$L^p(a, b; X)$ – the Lebesgue space of functions $f : (a, b) \to X$
$C([a, b])$ – space of continuous functions $f : [a, b] \to \mathbb{C}$
$C([a, b]; X)$ – space of continuous functions $f : [a, b] \to X$
$C(\Omega; X)$ – space of continuous functions $f : \Omega \subset \mathbb{R}^n \to X$
$C_b(\Omega; X)$ – continuous functions $f : \Omega \subset \mathbb{R}^n \to X$ with $\|f\|_\infty := \sup_{\xi\in\Omega}\|f\|_X < \infty$
$\sigma(A)$ – spectrum of an operator
$\rho(A)$ – the resolvent set of an operator
$\mathcal{F}f$ – Fourier transform of a function f
$\mathcal{F}^{-1}g$ – Inverse Fourier transform of a function g

1. Closed Operators

In this chapter we focus on an important class of unbounded linear operators $A : \mathcal{D}(A) \subset X \to Y$, namely *closed operators*. We begin by studying the characteristic and fundamental properties of this class, and study the *spectrum* of unbounded operators on Banach spaces.

Later on the course these results are applied in the study of linear ordinary and partial differential equations. Such equations can often be written in an abstract form

$$Af = g$$

where A is a closed operator on a suitable function space. The questions of existence, uniqueness and other properties of the solution f can then be studied using the properties of the operator A.

1.1 Closed Linear Operators

Closed linear operators $A : \mathcal{D}(A) \subset X \to Y$ form a strictly larger class than bounded operators between two Banach spaces X and Y.

Definition 1.1.1. An operator $A : \mathcal{D}(A) \subset X \to Y$ between two Banach spaces X and Y is *closed* if it has the following property:

> If $(x_n)_n \subset \mathcal{D}(A)$ is a sequence such that $x_n \to x$ and $Ax_n \to y$ as $n \to \infty$ for some $x \in X$ and $y \in Y$, then we have $x \in \mathcal{D}(A)$ and $Ax = y$.

An operator $A : \mathcal{D}(A) \subset X \to Y$ is closed if and only if the so-called *graph of* A defined as

$$\mathcal{G}(A) := \{ (x,y) \in X \times Y \mid x \in \mathcal{D}(A) \text{ and } y = Ax \}$$

is a closed subspace of $X \times Y$. (Here the norm on $X \times Y$ is defined as $\|(x,y)\|_{X\times Y} = \|x\|_X + \|y\|_Y$).

Exercise 1.1.1. Show that the above claim holds. ◇

Exercise 1.1.2. Assume that the operator $A : \mathcal{D}(A) \subset X \to Y$ has an algebraic inverse $A^{-1} : \mathcal{D}(A^{-1}) \subset Y \to X$ with $\mathcal{D}(A^{-1}) = \mathcal{R}(A)$. Show that A is closed if and only if A^{-1} is closed. ◇

Often the closedness of an operator $A : \mathcal{D}(A) \subset X \to Y$ can be shown indirectly, by showing that it has a bounded inverse.

Lemma 1.1.2. *Let X and Y be Banach spaces. If $A : \mathcal{D}(A) \subset X \to Y$ has a bounded inverse $A^{-1} \in \mathcal{L}(Y, X)$, then A is a closed operator.*

Exercise 1.1.3. Prove Lemma 1.1.2. ◇

Theorem 1.1.3 (Closed Graph Theorem)**.** *Let X and Y be Banach spaces and let $A : \mathcal{D}(A) \subset X \to Y$ be a closed operator. If $\mathcal{D}(A)$ is a closed subset of X, then A is a bounded operator, i.e., there exists $M > 0$ such that*

$$\|Ax\|_Y \le M\|x\|_X, \qquad \forall x \in \mathcal{D}(A).$$

One of the important consequences of the Closed Graph Theorem is that closed and bijective operators between two Banach spaces are "boundedly invertible".

Theorem 1.1.4. *Let X and Y be Banach spaces and let $A : \mathcal{D}(A) \subset X \to Y$ be a closed operator. If A is bijective (injective and surjective), then A has a bounded inverse $A^{-1} \in \mathcal{L}(Y, X)$.*

Exercise 1.1.4. Prove that Theorem 1.1.4 follows from the Closed Graph Theorem. ◇

Example 1.1.5. Consider an interval $[a, b] \subset \mathbb{R}$ and let $X = C[a, b]$ with norm $\|\cdot\|_\infty$. We can define a first order differential operator on X by

$$(Af)(\xi) = f'(\xi), \qquad f \in C^1[a, b] := \{\, f \in C[a, b] \mid f' \in C[a, b] \,\}.$$

If we restrict the domain of the operator in such a way that it $f \in \mathcal{D}(A)$ satisfy a *boundary condition* $f(a) = 0$, i.e.,

$$f \in \mathcal{D}(A) := \{\, f \in C[a, b] \mid f' \in C[a, b],\ f(a) = 0 \,\},$$

then A has a bounded inverse $A^{-1} \in \mathcal{L}(X)$. Indeed, we can find a formula for A^{-1} by letting $g \in X$ be arbitrary and looking for an element $f \in \mathcal{D}(A)$ such that $Af = g$. Taking into account the boundary condition, this operator equation is equivalent to the *boundary value problem*

$$\begin{aligned} f'(\xi) &= g(\xi), \qquad \xi \in [a, b], \\ f(a) &= 0. \end{aligned}$$

The differential equation on the first line has the general solution $f(\xi) = c_0 + \int_a^\xi g(s)ds$, where $c_0 \in \mathbb{C}$ is a constant, and the the boundary condition $f(a) = 0$ implies $c_0 = 0$. Because of this, the equation $Af = g$ has a unique solution which clearly satisfies $f \in \mathcal{D}(A)$. Thus A has an algebraic inverse and $\mathcal{R}(A) = X$. Finally, to show that $A^{-1}f$ is bounded, we can let $g \in X$ be arbitrary and estimate

$$\|A^{-1}g\|_\infty = \max_{\xi \in [a,b]} |(A^{-1}g)(\xi)| \le \max_{\xi \in [a,b]} \int_a^\xi |g(s)|ds = \int_a^b |g(s)|ds \le (b - a)\|g\|_\infty.$$

The fact that $A^{-1} \in \mathcal{L}(X)$ implies that $A : \mathcal{D}(A) \subset X \to X$ is a closed operator.

Also the operator $A_1 : \mathcal{D}(A_1) \subset X \to X$ defined as $A_1 f = f'$ with the larger domain $\mathcal{D}(A_1) = C^1[a,b]$ is a closed operator on X. This can be seen directly using the definition. Indeed, if $(f_n)_{n\in\mathbb{N}} \subset C^1[a,b]$ is such that $\|f_n - f\|_\infty \to 0$ and $\|f_n' - g\|_\infty = \|A_1 f_n - g\|_X \to 0$ as $n \to \infty$, then the sequences of functions $(f_n)_n$ and $(f_n')_n$ are *uniformly convergent*, and it follows that $f \in C^1[a,b]$ and $A_1 f = f' = g$. Thus A_1 is a closed operator. Note that A_1 is *an extension* of A, since $\mathcal{D}(A) \subset \mathcal{D}(A_1)$ and $A_1 f = Af$ for all $f \in \mathcal{D}(A)$. ◇

1.2 Spectral Theory for Closed Operators

The *spectrum* of a linear operator $A : \mathcal{D}(A) \subset X \to X$ is an important concept in functional analysis. In the following we define the parts of the spectrum $\sigma(A)$ and the *resolvent set* $\rho(A) = \mathbb{C} \setminus \sigma(A)$ are defined for unbounded operators $A : \mathcal{D}(A) \subset X \to X$.

Definition 1.2.1. Let X be a Banach space and let $A : \mathcal{D}(A) \subset X \to X$.

The *resolvent set* $\rho(A)$ of A is defined as

$$\rho(A) = \{\, \lambda \in \mathbb{C} \mid \text{The operator } \lambda - A \text{ has a bounded inverse } (\lambda - A)^{-1} \in \mathcal{L}(X) \,\}.$$

The set $\sigma(A) = \mathbb{C} \setminus \rho(A)$ is called the *spectrum* of A, and can be divided into the disjoint parts — the *point spectrum* $\sigma_p(A)$, the *continuous spectrum* $\sigma_c(A)$, and the *residual spectrum* $\sigma_r(A)$ — defined as

$$\sigma_p(A) = \{\, \lambda \in \mathbb{C} \mid \text{The operator } \lambda - A \text{ is not injective, i.e., } \mathcal{N}(\lambda - A) \neq \{0\} \,\}$$

$$\sigma_c(A) = \{\, \lambda \in \mathbb{C} \mid \lambda - A \text{ injective, but } \overline{\mathcal{R}(\lambda - A)} = X \text{ but } \mathcal{R}(\lambda - A) \neq X \,\}$$

$$\sigma_r(A) = \{\, \lambda \in \mathbb{C} \mid \lambda - A \text{ injective, but } \overline{\mathcal{R}(\lambda - A)} \neq X \,\}.$$

In the definitions of $\sigma_c(A)$ and $\sigma_r(A)$ the notation $\overline{\mathcal{R}(\lambda - A)}$ denotes the closure of the subspace $\mathcal{R}(\lambda - A)$ in X. The points $\lambda \in \sigma_p(A)$ are called *eigenvalues* of A, and by definition there exists an *eigenvector* $x \in \mathcal{D}(A)$ such that $x \neq 0$ satisfying $Ax = \lambda x$. This corresponds exactly to the case of matrices, whose spectra consist entirely of eigenvalues, i.e., $\sigma(A) = \sigma_p(A)$ for all matrices $A \in \mathbb{C}^{n\times n}$ or $A \in \mathbb{R}^{n\times n}$.

It should be noted that while $\sigma(A)$, $\sigma_p(A)$ and $\rho(A)$ always defined in the same way, the division of the other parts of the spectrum $\sigma(A)$ may be done in a different way in different references. Especially it is imporant to be careful with the definition of the residual spectrum $\sigma_r(A)$!

Exercise 1.2.1. Show that $\mathbb{C} = \rho(A) \cup \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A)$ and the sets $\sigma_p(A)$, $\sigma_c(A)$, and $\sigma_r(A)$ are *mutually disjoint* (i.e., their pairwise intersections are empty sets). ◇

Definition 1.2.2. Let X be a Banach space and let $A : \mathcal{D}(A) \subset X \to X$. The *resolvent operator* of A is defined as $(\lambda - A)^{-1}$ for $\lambda \in \rho(A)$

For a scalar $\lambda \in \mathbb{C}$, the notation "$\lambda - A$" is a short-hand expression for "$\lambda I - A$", i.e., $(\lambda - A)x = \lambda x - Ax$ for $x \in \mathcal{D}(A)$.

Many of the familiar properties of the spectrum and the resolvent of a bounded operator continue to hold for closed operators. In particular, $\rho(A)$ is always an open subset of $\mathbb{C}$, and therefore $\sigma(A) = \mathbb{C} \setminus \rho(A)$ is closed. However, there are differences as well: If the operator is bounded, i.e., $A \in \mathcal{L}(X)$, then both $\sigma(A)$ and $\rho(A)$ are non-empty subsets of $\mathbb{C}$, and $\sigma(A)$ is contained in a circle centered at $0 \in \mathbb{C}$ with radius $\|A\|$. If A is unbounded, either one of $\sigma(A)$ or $\rho(A)$ can in general be empty sets.

Exercise 1.2.2. Show that if $A : \mathcal{D}(A) \subset X \to X$ and $\lambda \in \rho(A)$, then $\mathcal{R}((\lambda - A)^{-1}) \subset \mathcal{D}(A)$ and $A(\lambda - A)^{-1}x = (\lambda - A)^{-1}Ax$ for all $x \in \mathcal{D}(A)$. Thus $(\lambda - A)^{-1}$ *commutes with* A. ◇

Exercise 1.2.3. Let X be a Banach space and $A : \mathcal{D}(A) \subset X \to X$. Show that the resolvent operator satisfies the *resolvent identity*

$$(\lambda - A)^{-1} - (\mu - A)^{-1} = (\mu - \lambda)(\lambda - A)^{-1}(\mu - A)^{-1}, \qquad \lambda, \mu \in \rho(A).$$

Deduce that $(\lambda - A)^{-1}$ and $(\mu - A)^{-1}$ commute for all $\lambda, \mu \in \rho(A)$. ◇

Exercise 1.2.4. Show that if $\lambda \in \rho(A)$ and $\mu \in \mathbb{C}$ is such that $|\lambda - \mu|\|(\lambda - A)^{-1}\| < 1$, then $\mu \in \rho(A)$ and the resolvent operator $(\mu - A)^{-1}$ is given by the *Neumann series*

$$(\mu - A)^{-1} = \sum_{n=0}^{\infty} (\lambda - \mu)^n (\lambda - A)^{-n-1}.$$

Deduce that $\rho(A)$ is an open subset of $\mathbb{C}$ and that $\rho \mapsto (\lambda - A)^{-1} \in \mathcal{L}(X)$ is continuous on $\rho(A)$. ◇

Exercise 1.2.5. Let $X = C_b(\mathbb{R}^2) := \{ f \in C^2(\mathbb{R}) \mid \|f\|_\infty < \infty \}$ and consider the operator $A : \mathcal{D}(A) \subset X \to X$ defined as

$$(Af)(x, y) = (x + iy)f(x, y), \qquad (x, y) \in \mathbb{R}^2,$$
$$\mathcal{D}(A) = \{ f \in C_b(\mathbb{R}^2) \mid \sup_{(z,y)\in\mathbb{R}^2} |(x + iy)f(x, y)| < \infty \}.$$

Show that $\sigma(A) = \mathbb{C}$ (and thus $\rho(A) = \varnothing$). ◇

Exercise 1.2.6. Let $X = L^2(\mathbb{R}^2)$ and consider the operator $A : \mathcal{D}(A) \subset X \to X$ defined as

$$(Af)(x, y) = (x + iy)f(x, y), \qquad (x, y) \in \mathbb{R}^2,$$
$$\mathcal{D}(A) = \{ f : \mathbb{R}^2 \to \mathbb{C} \mid \int_{\mathbb{R}^2} |(x + iy)f(x, y)|^2 dxdy < \infty \}.$$

Show that $\sigma(A) = \mathbb{C}$ (and thus $\rho(A) = \varnothing$). ◇

Theorem 1.2.3. *Let X be a Banach space assume $A : \mathcal{D}(A) \subset X \to X$ is such that $\rho(A) \neq \varnothing$. The mapping $\lambda \mapsto (\lambda - A)^{-1} \in \mathcal{L}(X)$ is an analytic function on $\rho(A)$ and for all $n \in \mathbb{N}$*

$$\frac{d^n}{d\lambda^n}(\lambda - A)^{-1} = (-1)^n n!(\lambda - A)^{-(n+1)}, \qquad n \in \mathbb{N}.$$

Here the property that a function $f : \Omega \subset \mathbb{C} \to X$ for a Banach space X and $\Omega \subset \mathbb{C}$ open is said to be *locally analytic* if for every point $\lambda_0 \in \Omega$ there exists an neighbourhood of λ_0 such that the function has a *derivative* $f'(\lambda_0)$ defined by

$$\left\| \frac{f(\lambda) - f(\lambda_0)}{\lambda - \lambda_0} - f'(\lambda_0) \right\| \to 0, \qquad \text{as} \quad |\lambda - \lambda_0| \to 0.$$

The term *"locally" analytic* takes into account that the resolvent $\rho(A)$ takes into account that the resolvent $\rho(A)$ does not need to be a connected subset of $\mathbb{C}$, and therefore also the derivative of a locally analytic function can be different in the different "connected components" of the domain.

Exercise 1.2.7. Prove Theorem 1.2.3. ◇

1.3 Adjoints of Closed Operators on Hilbert Spaces

On Hilbert spaces we can define the concept of an *adjoint* also for unbounded operators. Throughout this section X is a Hilbert space.

Definition 1.3.1. Let $A : \mathcal{D}(A) \subset X \to Y$ with Hilbert spaces X and Y and assume $\mathcal{D}(A)$ is dense in X. The *adjoint of* A is an operator $A^* : \mathcal{D}(A^*) \subset Y \to X$ defined so that $y \in \mathcal{D}(A^*)$ if and only if there exists $z \in X$ satisfying

$$\langle Ax, y \rangle_Y = \langle x, z \rangle_X \qquad \forall x \in \mathcal{D}(A).$$

In this case we define $A^* y = z$.

The assumption that $\mathcal{D}(A)$ is dense in X is required in order for A^* to be well-defined as an operator $A^* : \mathcal{D}(A) \subset Y \to X$ (otherwise there may be several $z \in Z$ which satisfy $A^* y = z$). We say that such operators $A : \mathcal{D}(A) \subset X \to Y$ are *densely defined*.

Exercise 1.3.1. Show that the adjoint A^* of $A : \mathcal{D}(A) \subset X \to Y$ is a closed operator. ◇

Lemma 1.3.2. *Assume X and Y are Hilbert spaces and $A : \mathcal{D}(A) \subset X \to Y$ is densely defined. The adjoint operator has the following fundamental properties.*

(a) *If $B \in \mathcal{L}(X, Y)$, then $(A+B)^* = A^* + B^*$, which means that $\mathcal{D}((A+B)^*) = \mathcal{D}(A^*)$ and*

$$(A + B)^* y = A^* y + B^* y \qquad \forall y \in \mathcal{D}(A^*).$$

(b) *If A has a bounded inverse $A^{-1} \in \mathcal{L}(Y, X)$, then also $A^* : \mathcal{D}(A^*) \subset Y \to X$ has a bounded inverse and $(A^*)^{-1} = (A^{-1})^* \in \mathcal{L}(X, Y)$.*

Exercise 1.3.2. Prove Lemma 1.3.2. ◇

Definition 1.3.3. Let X be a Hilbert space. The operator $A : \mathcal{D}(A) \subset X \to X$ is *symmetric* if

$$\langle Ax, y\rangle_X = \langle x, Ay\rangle_X \qquad \forall x, y \in \mathcal{D}(A).$$

It is *self-adjoint* if $\mathcal{D}(A^*) = \mathcal{D}(A)$ and $A^*x = Ax$ for all $x \in \mathcal{D}(A)$.

In a more compact form, the self-adjointess of $A : \mathcal{D}(A) \subset X \to X$ be expressed as a condition that $A^* = A$.

Exercise 1.3.3. Describe the relation between A^* and A for a symmetric operator. Show that every self-adjoint operator is symmetric. Is the converse true? ◇

Self-adjoint operators have the following very special spectral properties.

Theorem 1.3.4. *Let $A : \mathcal{D}(A) \subset X \to X$ with a Hilbert space X. If A is self-adjoint, then $\sigma(A)$ is not empty and $\sigma(A) \subset \mathbb{R}$. Moreover, the norm of the resolvent operator satisfies*

$$\|(\lambda - A)^{-1}\| \le \frac{1}{|\operatorname{Im}\lambda|}, \qquad \lambda \notin \mathbb{R}. \tag{1.1}$$

Exercise 1.3.4. Let $A : \mathcal{D}(A) \subset X \to X$ be symmetric. Show that for every $\lambda \in \mathbb{C}$ and $x \in \mathcal{D}(A)$ we have

$$\|(\lambda - A)x\|^2 = \|(\operatorname{Re}\lambda - A)x\|^2 + (\operatorname{Im}\lambda)^2\|x\|^2.$$

Deduce that $\|(\lambda - A)x\| \ge |\operatorname{Im}\lambda|\|x\|$ for all $\lambda \in \mathbb{C}$ and $x \in \mathcal{D}(A)$. ◇

Exercise 1.3.5. Let $A : \mathcal{D}(A) \subset X \to X$ be self-adjoint. Show that

(a) $\lambda \notin \sigma_p(A)$ for all $\lambda \notin \mathbb{R}$. This part is also true under the weaker assumption that A is symmetric.
Hint: *Use Exercise* 1.3.4.

(b) $\lambda \notin \sigma_r(A)$ for all $\lambda \notin \mathbb{R}$.
Hint: *Show the equivalent property $\mathcal{R}(\lambda - A)^\perp = \{0\}$ if $\lambda \notin \mathbb{R}$. Here you can employ self-adjointness and part* (a).

(c) $\lambda \notin \sigma_c(A)$ for all $\lambda \notin \mathbb{R}$. This part is also true under the weaker assumption that A is closed and symmetric.
Hint: *Begin by showing that if $\lambda \notin \mathbb{R}$, then $\mathcal{R}(\lambda - A)$ is necessarily a closed subset of X because of the estimate $\|(\lambda - A)x\| \ge |\operatorname{Im}\lambda|\|x\|$. Here you in particular need the fact that A is a closed operator. Combined with part* (b)*, the closedness of $\mathcal{R}(\lambda - A)$ implies the claim.*

(d) Prove that (1.1) is satisfied. **Hint:** *Use the inequality in Exercise* 1.3.4.

◇

Theorem 1.3.5. *Assume X is a Hilbert space and $A : \mathcal{D}(A) \subset X \to X$ is a closed and symmetric operator. If both $\mathcal{R}(i - A)$ and $\mathcal{R}(-i - A)$ are dense in X, then A is self-adjoint.*

Exercise 1.3.6. Prove Theorem 1.3.5. **Hint:** *This proof is not long, but it's not at all easy to figure out where to start! You should find this proof in the literature, and use that as a reference (repeat the proof with details here). Since A is symmetric, it's self-adjointness can be proved by showing that $\mathcal{D}(A^*) \subset \mathcal{D}(A)$. What is typically done is that the proof is begun by justifying that for any $y \in \mathcal{D}(A^*)$ it is possible (based on one of the assumptions and part* (c) *of Exercise* 1.3.5*) to find $x \in \mathcal{D}(A)$ such that $(i - A)x = (i - A^*)y$. After this, the symmetry, the other assumption, and part* (a) *of Exercise* 1.3.5 *it is possible to finally conclude that $y = x \in \mathcal{D}(A)$. If you find another proof, or if you are able to find some kind of an intuitive justification for how this proof is begun, let me know! I'm interested to see it!* ◇

Lemma 1.3.6. *If $A : \mathcal{D}(A) \subset X \to X$ is symmetric and $\rho(A) \cap \mathbb{R} \neq \varnothing$, then A is self-adjoint.*

Exercise 1.3.7. Prove Lemma 1.3.6. **Hint:** *Use Theorem* 1.3.5 *for an operator $B = \frac{1}{\varepsilon}(A - \lambda_0)$ with $\mathcal{D}(B) = \mathcal{D}(A)$ for suitable choices of $\lambda_0 \in \mathbb{R}$ and $\varepsilon > 0$. Begin by showing that B is symmetric whenever A is symmetric.* ◇

Lemma 1.3.7. *Let X be a Hilbert space. If $A : \mathcal{D}(A) \subset X \to X$ satisfies*

$$\langle Ax, x\rangle \in \mathbb{R}, \qquad \forall x \in \mathcal{D}(A),$$

then A is symmetric.

Exercise 1.3.8. Prove Lemma 1.3.7. **Hint:** *Study the condition $\langle Ax, x\rangle \in \mathbb{R}$ in the case where $x = y + \alpha z$ with arbitrary fixed elements $y, z \in \mathcal{D}(A)$ and for two different suitable choices of scalars $\alpha \in \mathbb{C}$, for example $\alpha = 1$ and $\alpha = i$, to show that $\mathrm{Re}\langle Az, y\rangle = \mathrm{Re}\langle z, Ay\rangle$ and $\mathrm{Im}\langle Az, y\rangle = \mathrm{Im}\langle z, Ay\rangle$. Some of you already proved this as part of the second preliminary exercise problems.* ◇

In the following two examples we show two cases where the self-adjointness can be shown in a "direct" manner by showing that $\mathcal{D}(A^*) \subset \mathcal{D}(A)$. These examples are in no way meant to be easy, but their purpose is only to illustrate that these types of arguments are also sometimes possible.

Example 1.3.8. In this example we consider a *multiplication operator*. To this end, let $g \in C((-\infty, \infty), \mathbb{R})$ be a fixed function, and define the multiplication operator $M_g : \mathcal{D}(M_g) \subset X \to X$ on $X = L^2(-\infty, \infty; \mathbb{C})$ so that for every $f \in \mathcal{D}(M_g) := \{ f \in X \mid g(\cdot)f(\cdot) \in L^2(-\infty, \infty; \mathbb{C}) \}$ we define

$$(M_g f)(\xi) = g(\xi)f(\xi), \qquad \text{for almost every } \xi \in \mathbb{R},$$

We will show that the fact that the values $g(\xi)$ are real for every $\xi \in \mathbb{R}$ implies that M_g is a self-adjoint operator. Before considering the main properties, we should note that the domain $\mathcal{D}(M_g)$ is dense in X, since it includes the subspace $C_0^\infty(\mathbb{R}; \mathbb{C})$ of smooth functions with compact support, which is dense in X, and thus M_g does have a well-defined adjoint. We will begin by showing that A is symmetric using Lemma 1.3.7. If $f \in \mathcal{D}(M_g)$, then

$$\langle M_g f, f\rangle_X = \int_{\mathbb{R}} (M_g f)(\xi)\overline{f(\xi)} d\xi = \int_{\mathbb{R}} g(\xi) f(\xi)\overline{f(\xi)} d\xi = \int_{\mathbb{R}} g(\xi)|f(\xi)|^2 d\xi \in \mathbb{R},$$

since we assumed $g(\xi) \in \mathbb{R}$ for all $\xi \in \mathbb{R}$, and thus M_g is symmetric by Lemma 1.3.7. It remains to show that $\mathcal{D}(M_g^*) \subset \mathcal{D}(M_g)$. To this end, let $h \in \mathcal{D}(M_g^*)$ be arbitrary. Then by definition we have that for all $f \in \mathcal{D}(M_g)$ we must have

$$\begin{aligned} 0 &= \langle M_g f, h\rangle_X - \langle f, M_g^* h\rangle_X = \int_{\mathbb{R}} g(\xi) f(\xi)\overline{h(\xi)} d\xi - \int_{\mathbb{R}} f(\xi)\overline{(M_g^* h)(\xi)} d\xi \\ &= \int_{\mathbb{R}} f(\xi)\overline{\left[g(\xi)h(\xi) - (M_g^* h)(\xi)\right]} d\xi. \end{aligned}$$

Since this holds for any $f \in \mathcal{D}(M_g)$, our aim is to make suitable choices of f to show that in fact $(M_g^* h)(\xi) = g(\xi)h(\xi)$ for (almost) all $\xi \in \mathbb{R}$. Since $M_g^* h \in X$, this will immediately imply that $g(\cdot)h(\cdot) = M_g^* h \in L^2(\mathbb{R};\mathbb{C})$, which shows that in fact $h \in \mathcal{D}(M_g)$, and thus the proof will be complete. For any $N > 0$ we can choose

$$f_N(\xi) = \begin{cases} g(\xi)h(\xi) - (M_g^* h)(\xi), & \text{for (almost) all } \xi \in [-N, N] \\ 0 & \text{if } |\xi| > N. \end{cases}$$

Since $f_N(\xi) = 0$ if $|\xi| > N$, it is easy to see that $f_N \in \mathcal{D}(M_g)$ for any $N > 0$. Now the above identity implies that

$$\begin{aligned} 0 &= \langle M_g f_N, h\rangle_X - \langle f_N, M_g^* h\rangle_X = \int_{\mathbb{R}} f_N(\xi)\overline{\left[g(\xi)h(\xi) - (M_g^* h)(\xi)\right]} d\xi \\ &= \int_{-N}^{N} |g(\xi)h(\xi) - (M_g^* h)(\xi)|^2 d\xi. \end{aligned}$$

This implies that $g(\xi)h(\xi) - (M_g^* h)(\xi) = 0$ for almost every $\xi \in [-N, N]$. However, since this holds for arbitrary $N > 0$, we have that $g(\xi)h(\xi) = (M_g^* h)(\xi)$ for almost all $\xi \in \mathbb{R}$. As argued above, this shows that in fact $h \in \mathcal{D}(M_g)$ and the proof is complete. $\diamond$

Example 1.3.9. In this example we consider a second order *differential operator* $A : \mathcal{D}(A) \subset X \to X$ on $X = L^2(0,1)$ defined as

$$\begin{aligned} &(Af)(\xi) = \beta f''(\xi), \\ &f \in \mathcal{D}(A) := \left\{ f \in X \;\middle|\; f, f' \text{ absolutely continuous and } f(0) = f(1) = 0 \right\}, \end{aligned}$$

where $\beta > 0$ is a constant. The property that f is "absolutely continuous" is discussed in greater detail in the next chapter, but for now it is sufficient to think of it as a stronger form of continuity, which guarantess that f has a derivative $f' \in L^2(0,1)$ which is defined "in the L^2-sense".

In this example we will show that A is self-adjoint, i.e., $A^* = A$. We begin by showing that A is symmetric. To this end, let $f \in \mathcal{D}(A)$. We then have $f(0) = f(1) = 0$, and using integration by parts we can see that

$$\begin{aligned}\langle Af, f\rangle_{L^2} &= \beta\int_0^1 f''(\xi)\overline{f(\xi)}d\xi = \beta\left(f'(1)\overline{f(1)} - f'(0)\overline{f(0)}\right) - \beta\int_0^1 f'(\xi)\overline{f'(\xi)}d\xi \\ &= -\beta\int_0^1 |f'(\xi)|^2 d\xi \in \mathbb{R}.\end{aligned}$$

Thus A is symmetric by Lemma 1.3.7. It remains to show that $\mathcal{D}(A^*) \subset \mathcal{D}(A)$. To this end, let $g \in \mathcal{D}(A^*)$ be arbitrary. Then for any $f \in \mathcal{D}(A)$ we have $\langle Af, g\rangle = \langle f, h\rangle$ where $h = A^*g \in L^2(0,1)$. If we choose $f = \phi_n := 2\sin(n\pi\cdot) \in \mathcal{D}(A)$, then $Af = A\phi_n = \phi_n'' = -n^2\pi^2 \cdot 2\sin(n\pi\cdot) = -n^2\pi^2\phi_n$ and

$$\hat{g}(n) := \langle g, \phi_n\rangle = 2\int_0^1 g(\xi)\sin(n\pi\xi)d\xi, \qquad \hat{h}(n) := \langle h, \phi_n\rangle = 2\int_0^1 h(\xi)\sin(n\pi\xi)d\xi$$

are the coefficients of the Fourier sine series of g and h, respectively. The identity $\langle Af, g\rangle = \langle f, h\rangle$ implies $-n^2\pi^2\hat{g}(n) = \hat{h}(n)$. Since $h \in L^2(0,1)$, the theory of Fourier series implies $(\hat{h}(n))_{n\in\mathbb{N}} \in \ell^2(\mathbb{C})$. Thus also $(n^2\hat{g}(n))_{n\in\mathbb{N}} \in \ell^2(\mathbb{C})$. Long story short, the Fourier theory implies that g and g' are absolutely continuous and $g'' \in L^2(0,1)$. To show that $g(0) = g(1) = 0$, it suffices to note that since g was shown to be continuously differentiable, its Fourier sine series

$$g(\xi) = \sum_{n=1}^{\infty} \hat{g}(n)\phi_n(\xi)$$

converges uniformly on $[0,1]$ (and in particular pointwise), and thus $\phi_n(0) = \phi_n(1) = 0$ for all $n \in \mathbb{N}$ implies $g(0) = g(1) = 0$ as well.

In this example, we could have also used Lemma 1.3.6 to show that the symmetric operator A is also self-adjoint by showing that $0 \in \rho(A)$. In fact, the formula for the inverse A^{-1} can be derived in a similar way as in Example 1.1.5 by noting that the equation $Af = g$ is equivalent to the boundary value problem

$$\begin{aligned}\beta f''(\xi) &= g(\xi), \qquad \xi \in [0,1] \\ f(0) &= f(1) = 0.\end{aligned}$$

These kinds of boundary value problems will be studied in greater detail in the next chapter. ◇

2. Analysis of Linear Differential Equations

One of our main interests on this course is to use functional analysis and operator theory to study *differential operators* and associated linear *(partial) differential equations*. In this section we learn how differential equations can be formulated as *abstract linear equations* with a differential operator $A : \mathcal{D}(A) \subset X \to X$, and how the properties of A can be used to study the solvability and properties of the original equations.

Perhaps the simplest examples of linear differential equations (with only a single variable $\xi \in [a,b]$) are the "boundary value problems"

$$f'(\xi) = g(\xi), \qquad \forall \xi \in (a,b) \tag{2.1a}$$
$$f(a) = 0 \tag{2.1b}$$

and

$$f'(\xi) = g(\xi), \qquad \forall \xi \in (a,b) \tag{2.2a}$$
$$f(b) = 0 \tag{2.2b}$$

which are both first order ordinary differential equations. In the literature (and on this course) the derivative is sometimes alternatively denoted by $f'(\xi)$, $\frac{df}{d\xi}(\xi)$, or $f_\xi(\xi)$.

A particular case of a second order differential equation is

$$-(\beta f')'(\xi) = g(\xi), \qquad \forall \xi \in (a,b) \tag{2.3a}$$
$$f(a) = f(b) = 0. \tag{2.3b}$$

Here $\beta : (a,b) \to \mathbb{R}$ is a function, and if the derivatives $\beta'(\xi)$ and $f''(\xi)$ are well-defined (which is not always the case!), we have $(\beta f')'(\xi) = \beta'(\xi) f'(\xi) + \beta(\xi) f''(\xi)$. In the equations that we consider we assume that the values of $\beta(\cdot)$ are uniformly positive, i.e., there exists a constant $\beta_0 > 0$ such that $\beta(\xi) \geq \beta_0 > 0$ for all $\xi \in [a,b]$. On the other hand, *partial differential equations* involve differentiation with respect to two or more variables. On this course study partial differential equations of the form

$$-\beta_1 \frac{\partial^2 f}{\partial \xi_1^2}(\xi_1,\xi_2) - \beta_2 \frac{\partial^2 f}{\partial \xi_2^2}(\xi_1,\xi_2) = g(\xi_1,\xi_2), \qquad (\xi_1,\xi_2) \in \Omega \subset \mathbb{R}^2 \tag{2.4a}$$
$$f(\xi_1,\xi_2) = 0, \qquad \text{for} \quad (\xi_1,\xi_2) \in \partial\Omega \tag{2.4b}$$

with constants $\beta_1, \beta_2 > 0$ Here $\Omega \subset \mathbb{R}^2$ is the *spatial domain* of the equation, and $f : \overline{\Omega} \subset \mathbb{R}^2 \to \mathbb{R}$ (or sometimes the values of f can be complex). The additional equation (2.4b) is the *boundary condition* of the equation, and $\partial\Omega$ denotes the boundary of the set Ω. This type of a boundary condition is a *Dirichlet boundary condition*. One particularly important special case of (2.4) is the *Poisson equation*, where $\beta_1 = \beta_2 = -1$, and which is typically written in the form

$$\begin{aligned} -\Delta f &= g, \qquad &\text{on } \Omega \\ f &= 0, \qquad &\text{on } \partial\Omega, \end{aligned}$$

where the *Laplace operator* is defined as $\Delta = \frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2}$.

Even *linear* partial differential equations form a complex area of mathematics. We focus here on *Elliptic equations*[1], and even there we focus on a special class of equations with at most two variables, and in the case of $\Omega \subset \mathbb{R}^2$ (i.e., for (2.4)) we assume β_1 and β_2 are constants. While studying this chapter you should also study additional material on elliptic equations (for example from the books by Evans, Brezis, or Renardy & Rogers). Specific things you should learn are the following.

- Elliptic equations can in general have coefficients β_1 and β_2 which depend on the spatial variables. How are these expressed in the general form?
- Elliptic equations can also have terms with differentiation with respect to both variables in the same term.
- Most results require the spatial domain $\Omega \subset \mathbb{R}^2$ to be "sufficiently nice". You can find out what kind of things are typically used. Throughout this course we assume in particular that $\Omega \subset \mathbb{R}^2$ is open, bounded, convex, it has a smooth boundary, and Ω is located "on one side of its boundary $\partial\Omega$". One particular type of domain satisfying all the requirements is a disk $\Omega = \{\, (\xi_1, \xi_2) \in \mathbb{R}^2 \mid \xi_1^2 + \xi_2^2 < R^2 \,\}$ for some $R > 0$. Figure 2.1 lists a few different types of domains to illustrate the different types of assumptions on Ω and $\partial\Omega$.

Figure 2.1: Different types of domains Ω on $\mathbb{R}^2$. The disk has smooth boundary, while the rectangle does not (though it is "piecewise smooth"). In the third picture a single segment has been removed from the disk, and the resulting Ω is not "on one side of $\partial\Omega$". The fourth domain has corners, so the boundary $\partial\Omega$ is not smooth, and Ω is also not convex.

Exercise 2.0.1. Write down the general form of the elliptic partial differential equation on $\Omega \subset \mathbb{R}^2$ in the case where the coefficients are allowed to depend on the variables $(\xi_1, \xi_2) \in \Omega$. Also write down the condition under which the equation is called *elliptic*. **Hint:** *You can find these in the literature. Most general form is for equations on n-dimensional spaces (i.e., with $f : \mathbb{R}^n \to \mathbb{R}$), and now you are interested in the case where $n = 2$.* ◇

[1] Other typical classes of PDEs are *parabolic equations* and *hyperbolic equations*, both of which include time t as one variable.

2.1 Well-Posedness of Differential Equations

The main questions related to differential equations are of the following form:

(i) Under what assumptions (on g and $\beta(\cdot)$ in (2.3) or β_1, β_2 in (2.4)) do the equations have solutions f?

(ii) Are these solutions unique?

(iii) What are the properties of the solutions, for example, are they continuously differentiable?

In particular, a differential equation is called *well-posed* (in the sense of Hadamard) if

> for any g (in a suitable class of functions) the differential equation has a unique solution f, and for another $\tilde{g}$ which is "close" to g the corresponding solution $\tilde{f}$ is "close" to f.

Well-posedness can therefore be summarised as requiring

- **existence** of solution,
- **uniqueness** of the solution, and
- **continuous dependence** of the solution on g.

These questions can be studied by writing the above differential equations as abstract linear equations on a suitable function space X.

Definition 2.1.1. Let X be a linear space, $A : \mathcal{D}(A) \subset X \to X$, and $g \in X$. An *abstract linear equation* has the form

$$Af = g, \qquad f \in \mathcal{D}(A). \tag{2.5}$$

For the equations (2.1)–(2.3) the operator A represents the differentiation of the function f of the left-hand side of the equations. The choice of X is done partly based on the properties of g, and the desired properties of the solution f. For example, if the considered functions g are continuous on $[a, b]$, one may choose $X = C([a, b])$ with the norm $\|g\|_X = \max_{\xi\in[a,b]}|g(\xi)|$. In this case also the solutions $f \in X$ will be continuous functions. The domain $\mathcal{D}(A)$ contains the differentiability requirements on f such that $Af \in X$, and the *boundary conditions* (2.1b), (2.2b), (2.3b), and (2.4b) are included in $\mathcal{D}(A)$ as well.

Exercise 2.1.1. Formulate the differential equation (2.3) as abstract linear equation on $X = C([a, b])$ in the case where $\beta \in C^1([a, b])$ (i.e., $\beta(\cdot)$ is continuously differentiable). Justify your answer. **Hint:** *This involves defining the operator A and finding an appropriate domain $\mathcal{D}(A)$ in such a way that $Af \in X$ whenever $f \in \mathcal{D}(A)$.* ◇

Exercise 2.1.2. Show that if $A : \mathcal{D}(A) \subset X \to X$ has a bounded inverse, then the abstract linear equation (2.5) is well-posed. In particular show that the continuous dependence of f on g is satisfied in the sense that there exists a constant $M > 0$ such that if $A\tilde{f} = \tilde{g}$, then $\|f - \tilde{f}\|_X \le M\|g - \tilde{g}\|_X$ (thus $\|f - \tilde{f}\|_X$ converges to zero if $\|g - \tilde{g}\|_X$ goes to zero, which is indeed "continuity"). In what sense does the "continuous dependence" of f from g hold? What are the more precise correspondences between the different parts of well-posedness and the invertibility properties (i.e., injectivity, surjectivity, and boundedness of the inverse) of A? Justify your answer. **Hint:** *If A has an inverse, then you can easily solve f from the equation* (2.5). ◇

Exercise 2.1.3. Show that the differential equation

$$\begin{aligned} f'(\xi) &= g(\xi), \qquad \forall \xi \in (a,b) \\ f(a) &= f(b) = 0 \end{aligned}$$

(similar to (2.2) but with two boundary conditions) is not well-posed (when considering for example continuous functions $g \in C[a,b]$). Also show that if the boundary condition (2.2b) is removed completely, the resulting differential equation is not well-posed either. **Hint:** *The differential equation $f'(\xi) = g(\xi)$ has an explicit solution, which you can find easily. In the first part you can note that the solution does not exist for arbitrary functions g (only under additional conditions).* $\diamond$

Very often the choice $X = C([a,b])$ (or more $X = C(\Omega)$ in the case of (2.4)) is too restrictive, and it is better to instead consider the above differential equations for functions g that are not necessarily continuous, but instead satisfy[2] $g \in L^2(\Omega)$. In this case it is natural to choose $X = L^2(\Omega)$. In order to answer the question of "when do we have $Af \in X$ for some $f \in X$", we need to take a longer detour into how derivatives are defined for functions on Lebesgue spaces. In the next section we will see that the correct choice for the domain $\mathcal{D}(A)$ in this case is an appropriate *Sobolev space*.

2.2 Distributional Derivatives and Sobolev Spaces

In this section we will define what we mean by a "derivative" of a function which is not differentiable in the classical sense. In particular, we will define the concept of a *weak derivative* (or a *distributional derivative*), which is defined using integration by parts in a suitable way. The only downside of this approach is that the weak derivative of a function is not necessarily a "function" at all in a typical sense. Perhaps the most classical example of this that the weak derivative of the Heaviside step function $H \in L^2(-1,1)$ defined by

$$H(\xi) = \begin{cases} 0 & -1 \le \xi < 0 \\ 1 & 0 \le \xi \le 1 \end{cases}$$

is the "Dirac delta" $\delta_0(\xi)$, which has the property that $\int_a^b f(\xi)\delta(\xi)d\xi = f(0)$ if $a < 0 < b$. However, $\delta(\xi)$ is not a function (since its values for $\xi \in [-1,1]$ cannot be defined) but instead it is a generalised function or a *distribution*. Using weak derivatives, it indeed turns out that every function will have well-defined (partial) derivatives of all orders in the larger class of *distributions*. In addition, for any function $f \in C^1(\Omega)$ the weak derivatives with respect to all variables ξ_j coincide with $\frac{\partial f}{\partial \xi_j}$. Using the concepts of distributional derivatives, we can define *Sobolev spaces* as spaces of functions $f \in L^2(\Omega)$ whose distributional derivatives of sufficiently high orders are also functions in $L^2(\Omega)$ (in addition to being distributions).

Distributional derivatives and Sobolev spaces would deserve a chapter or two by themselves, but here we will focus on covering the minimal required properties of these concept that we will use in defining studying differential equations of the form (2.3) and (2.4). For additional properties of these mathematical objects and for concrete examples on distributional derivatives you should study, for example Chapter 5 of Evans, Chapters 5–6 of Renardy & Rogers, and Chapters 8–9 of Brezis.

[2]More generally, $g \in L^p(\Omega)$ can be studied, but on this course we mostly restrict our attention to $p = 2$.

Definition 2.2.1. Let $\Omega \subset \mathbb{R}^n$ be open and bounded. The set of smooth functions $\phi \in C^\infty(\Omega)$ whose support $\operatorname{supp}\phi := \overline{\{\, \xi \in \Omega \mid \phi(\xi) \neq 0 \,\}}$ satisfies $\operatorname{supp}\phi \subset \Omega$ and is compact (as a subset of $\mathbb{R}^n$) is denoted $C_c^\infty(\Omega)$. The functions $\phi \in C_c^\infty(\Omega)$ are called *test functions*.

Note that since Ω is assumed to be an open set in $\mathbb{R}^n$ and since the support of a function is by definition a closed set, the condition $\operatorname{supp}\phi \subset \Omega$ causes the function $\phi \in C_c^\infty(\Omega)$ to have identically zero values near the boundary of the set Ω. It's also worth noting that since Ω is assumed to be a bounded set, the compactness of $\operatorname{supp}\phi$ as a subset of $\mathbb{R}^n$ follows automatically from the condition $\operatorname{supp}\phi \subset \Omega$ and the fact that $\operatorname{supp}\phi$ is closed by definition. Nevertheless, we state this requirement explicitly, since in the more general case where Ω may be *unbounded*, the compactness requirement is an important part of the definition of test functions.

In the following we use a "multi-index notation" $D^\alpha f$ for the partial derivatives of a function $f : \mathbb{R}^n \to \mathbb{R}$. More precisely, for a n-tuple $\alpha := (\alpha_1, \dots, \alpha_n)$ where $\alpha_j \in \mathbb{N} \cup \{0\}$ we define $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ and denote

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial^{\alpha_1}\xi_1 \cdots \partial^{\alpha_n}\xi_n}(\xi_1, \dots, \xi_n)$$

(i.e., $\alpha_j \in \mathbb{N}$ signifies how many times the function f is differentiated with respect to ξ_j in $D^\alpha f$, and $\alpha_j = 0$ means that $D^\alpha f$ does not contain differentiation with respect to ξ_j). For brevity we also denote $f(\xi_1, \dots, \xi_n)$ by $f(\xi)$ with $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$.

Exercise 2.2.1. Write the expressions on the left-hand sides of the differential equations (2.3) and (2.4) using the multi-index notation, for example using $D^{(k,l)} f$ with $k, l \in \mathbb{N} \cup \{0\}$. (In the one-dimensional case you can simply denote Df in place of $D^1 f$.) $\diamond$

Definition 2.2.2. Let $\Omega \subset \mathbb{R}^n$ be open and bounded and let $\alpha = (\alpha_1, \dots, \alpha_n)$. The function $f \in L^2(\Omega)$ is said to have an α^{th}*-weak derivative* if there exists $g \in L^2(\Omega)$ such that

$$\int_\Omega f(\xi)(D^\alpha \phi)(\xi) d\xi = (-1)^{|\alpha|} \int_\Omega g(\xi)\phi(\xi) d\xi, \qquad \forall \phi \in C_c^\infty(\Omega).$$

If such $g \in L^2(\Omega)$ exists, we denote $D^\alpha f = g$.

In addition to the notation $D^\alpha f$, we will also continue to use the other notations for derivatives, for example $f'(\xi)$, $f^{(k)}(\xi)$, and $\frac{\partial^k f}{\partial \xi_j^k}(\xi)$, which will also be interpreted as weak derivatives whenever the function f is not differentiable in the classical sense. We also note that in a more general version of Definition 2.2.2, f and g could be allowed to be in $L^1_{\text{loc}}(\Omega)$.

The definition of the weak derivative is motivated by integration by parts. Indeed, if $f \in C(\Omega)$ and if $\frac{\partial f}{\partial \xi_j}$ exists and is continuous for some $1 \leq j \leq n$, then $\phi \in C_c^\infty(\Omega)$ implies that $\phi^{(j)}(\xi) = 0$ for all $j \in \mathbb{N} \cup \{0\}$ and for ξ near the boundary $\partial\Omega$. The usual integration by parts therefore implies that for every $\phi \in C_c^\infty(\Omega)$ we have

$$\int_\Omega f(\xi) \frac{\partial \phi}{\partial \xi_j}(\xi) d\xi = -\int_\Omega \frac{\partial f}{\partial \xi_j}(\xi)\phi(\xi) d\xi.$$

Repeating this procedure with respect to the different variables also shows that if $f \in C(\Omega)$ is such that the defivative $D^\alpha f$ exists in the classical sense, then we indeed have

$$\int_\Omega f(\xi)(D^\alpha \phi)(\xi)d\xi = (-1)^{|\alpha|} \int_\Omega (D^\alpha f)(\xi)\phi(\xi)d\xi, \qquad \forall \phi \in C_c^\infty(\Omega).$$

Weak derivatives have the same typical properties as the classical derivatives, for example the operation is linear, differentiation of products is done in the same way, see for example Section 5.2.3 in Evans.

The *Sobolev spaces* $H^k(\Omega)$ are defined in the following[3].

Definition 2.2.3. Let $\Omega \subset \mathbb{R}^n$ be open and bounded set. For $k \in \mathbb{N} \cup \{0\}$ we define the *Sobolev space* of order k as

$$H^k(\Omega) = \{ f \in L^2(\Omega) \mid D^\alpha f \in L^2(\Omega) \text{ whenever } |\alpha| \le k \}.$$

The norm on $H^k(\Omega)$ is defined so that

$$\|f\|_{H^k}^2 = \sum_{0 \le |\alpha| \le k} \|D^\alpha f\|_{L^2}^2, \qquad f \in H^k(\Omega).$$

Note that the norm also includes the term $\|D^\alpha f\|_{L^2}^2$ with $\alpha = (0, \dots, 0)$ (corresponding to $|\alpha| = 0$), which is exactly $\|D^\alpha f\|_{L^2}^2 = \|f\|_{L^2}^2$. Because of this we always have $\|f\|_{L^2} \le \|f\|_{H^k}$. We also have by definition that $H^0(\Omega) = L^2(\Omega)$.

Theorem 2.2.4. *The space $H^k(\Omega)$ with the norm $\|\cdot\|_{H^k}$ is a Hilbert space.*

The proof of the above result can be found for example in the books by Evans or Renardy & Rogers.

Exercise 2.2.2. Write down the expression for the inner product $\langle f, g\rangle_{H^k}$ corresponding to the norm $\|\cdot\|_{H^k}$. ◇

Note that the space of test functions $C_c^\infty(\Omega)$ is a subspace of $H^k(\Omega)$ for every $k \in \mathbb{N} \cup \{0\}$. The space $C_c^\infty(\Omega)$ is dense in $L^2(\Omega)$, but it is *not dense* in $H^k(\Omega)$ for any $k \ge 1$. This is actually of a sign of the property that an arbitrary function $f \in L^2(\Omega)$ does not need to have any meaningful "values" at the boundary $\partial\Omega$. Even in the one-dimensional case, for example the function $f(\xi) = \sin(\pi/\xi)$ plotted in Figure 2.2, which satisfies $f \in L^2(0,1)$, but clearly does not have any sensible value at $\xi = 0$. The following theorem regarding the *trace operator* shows on the contrary all functions $f \in H^1(\Omega)$ on the other hand have a well-defined behaviour on the boundary.

[3] In this section we have restricted our attention in several ways. For example, we always assume Ω is bounded, while considering unbounded domains, most notably $\Omega = \mathbb{R}^n$, would be possible as well. Moreover, we only define weak derivatives and Sobolev spaces for functions on $f \in L^2(\Omega)$, even though analogous definitions are possible for $f \in L^p(\Omega)$ for all $1 \le p \le \infty$. For $1 \le p \le \infty$ with $p \ne 2$ the Sobolev spaces are Banach spaces and they are typically denoted by $W^{1,p}(\Omega)$.

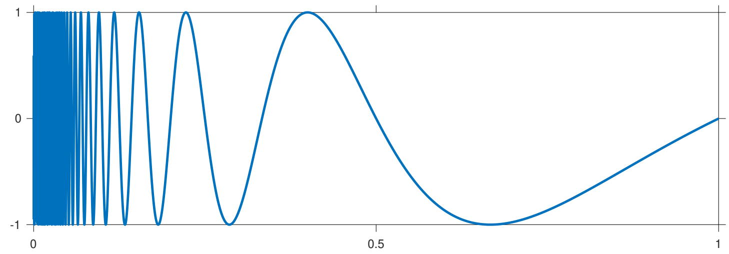

Figure 2.2: Plot of the function $f(\xi) = \sin(\pi/\xi)$.

Theorem 2.2.5 (Trace Theorem). *Assume $\Omega \subset \mathbb{R}^n$ is open and bounded and has smooth boundary $\partial\Omega$. There exists a bounded linear operator $T \in \mathcal{L}(H^1(\Omega), L^2(\partial\Omega))$ such that*

$$Tf = f|_{\partial\Omega}, \qquad \text{for all} \quad f \in H^1(\Omega) \cap C(\overline{\Omega}). \tag{2.6}$$

Then $Tf \in L^2(\partial\Omega)$ is called the trace *of $f \in H^1(\Omega)$ on $\partial\Omega$.*

Here $f|_{\partial\Omega} : \partial\Omega \to \mathbb{C}$ denotes the restriction of the function $f \in C(\overline{\Omega})$ to $\partial\Omega$, i.e. $f|_{\partial\Omega}(\xi) = f(\xi)$ for all $\xi \in \partial\Omega$. In the condition (2.6) the considered functions f are assumed to be continuous also on the boundary of Ω due to the assumption $f \in C(\overline{\Omega})$. For other functions $f \in H^1(\Omega)$ we have $Tf \in L^2(\partial\Omega)$, and therefore the values of the trace Tf do not need to be defined at each point $\xi \in \partial\Omega$, but nevertheless the trace Tf can be interpreted to describe the behaviour of f on the boundary $\partial\Omega$ in a "generalised sense". The theorem in particular means that there exists a constant $C > 0$ (depending only on Ω) such that

$$\|Tf\|_{L^2(\partial\Omega)} \le C\|f\|_{H^1(\Omega)}. \tag{2.7}$$

for any $f \in H^1(\Omega)$.

Traces are very important in our study of partial differential equations, because we can use them to describe the boundary conditions. In particular, the Dirichlet boundary conditions (2.3b) and (2.4b) can be compactly written as "$Tf = 0$" for the solutions f of the differential equations. Because the functions $f \in H^1(\Omega)$ satisfying $Tf = 0$, we make the following definition.

Definition 2.2.6. Assume $\Omega \subset \mathbb{R}^n$ is open and bounded and has smooth boundary $\partial\Omega$. We define

$$H_0^1(\Omega) = \{\, f \in H^1(\Omega) \mid Tf = 0 \,\}.$$

As mentioned before, the space $C_c^\infty(\Omega)$ of test functions is not dense in $H^1(\Omega)$. In fact, the space $H_0^1(\Omega)$ (which is a smaller space than $H^1(\Omega)$) is the *closure* of $C_c^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{H^1}$. This is the way the space is $H_0^1(\Omega)$ usually defined (it is simpler, since it does not require the concept of the trace). Note that $H_0^1(\Omega)$ is a closed subspace of $H^1(\Omega)$, since it is the kernel of the bounded operator $T \in \mathcal{L}(H^1(\Omega), L^2(\partial\Omega))$.

Finally, in addition to derivatives, it is useful for us to define the idea or "integral functions", or *primitives*, of functions in $L^2(\Omega)$. We will do this only in the case $\Omega = (a,b) \subset \mathbb{R}$, where the proofs are quite simple.

Lemma 2.2.7. *Let $\Omega = (a,b) \subset \mathbb{R}$. If $f \in L^2(\Omega)$ and $c_0 \in \mathbb{C}$, then the function $g : [a,b] \to \mathbb{C}$ defined by*

$$g(\xi) = c_0 + \int_a^\xi f(t)dt$$

is (uniformly) continuous on $[a,b]$, $g \in H^1(\Omega)$, and $Dg = f$.

The above lemma in particular implies that with the above notation we have

$$g(\xi) = c_0 + \int_a^\xi (Dg)(t)dt, \qquad \xi \in [a,b].$$

The function $g(\cdot)$ is indeed uniformly continuous (since it is continuous on a closed interval), but it has an even better continuity property, called *absolute continuity*. Absolute continuity has alternative equivalent definitions, but this definition is the most intuitive for our purposes.

Definition 2.2.8. A function $f \in L^2(a,b)$ is defined to be *absolutely continuous* if there exists $g \in L^2(a,b)$ such that

$$f(\xi) = f(a) + \int_a^\xi g(t)dt, \qquad \forall \xi \in [a,b].$$

In the one-dimensional case $\Omega = (a,b)$, the space $H^1(\Omega)$ is in fact exactly the set of absolutely continuous functions (the fact that every absolutely continuous function is in $H^1(\Omega)$ can be proved using the same argument that is also used to prove Lemma 2.2.7).

2.3 Abstract Formulations of Elliptic Differential Equations

Using the above definitions of weak derivatives and Sobolev space, we are finally a position to formulate the differential equations (2.3) and (2.4), and the associated differential operators on the space $X = L^2(\Omega)$.

Exercise 2.3.1. Assume $g \in L^2(a,b)$ and $\beta \in C[a,b]$. Formulate the boundary value problem (2.3) as an abstract linear equation of the form $Af = g$ on the space $X = L^2(\Omega)$ with $\Omega = (a,b) \subset \mathbb{R}$. **Hint:** *The operator A and its domain are similar to those in Example* 1.3.9 *but require some modifications due to the fact that $\beta(\cdot)$ is now a function. You can define the domain $\mathcal{D}(A)$ using the Sobolev spaces $H^1(\Omega)$ and the trace operator (or directly the space $H_0^1(\Omega)$).* ◇

Exercise 2.3.2. Let $\Omega \subset \mathbb{R}^2$ be bounded and open with smooth boundary. Assume $g \in L^2(\Omega)$ and $\beta_1, \beta_2 > 0$. Formulate the boundary value problem (2.4) as an abstract linear equation of the form $Af = g$ on the space $X = L^2(\Omega)$. **Hint:** *This can be done similarly as in the previous exercise, and $\mathcal{D}(A)$ can be defined using the Sobolev spaces $H^2(\Omega)$ and the trace operator (or directly the space $H_0^1(\Omega)$). If you would like more hints, you can first read the next couple of pages as well.* ◇

We can also rewrite the differential equations (2.3) and (2.4) in a common "general form". This can be done conveniently by using the weak derivatives to define the *gradient* $\nabla f : \Omega \to \mathbb{C}^n$ and *divergence* $\nabla \cdot g$ in the usual way. In particular, for $\Omega \subset \mathbb{R}^n$ and for $f \in H^1(\Omega)$ and $g \in H^1(\Omega; \mathbb{C}^n)$ (i.e. $g = [g_1, \dots, g_n]^T : \Omega \to \mathbb{C}^n$ with $g_k \in H^1(\Omega)$ for all $k \in \{1, \dots, n\}$) we define

$$\nabla f = \begin{bmatrix} \frac{df}{d\xi_1} \\ \vdots \\ \frac{df}{d\xi_n} \end{bmatrix} \qquad \text{and} \qquad \nabla \cdot g = \nabla \cdot \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix} = \frac{dg_1}{d\xi_1} + \dots + \frac{dg_n}{d\xi_n}.$$

In particular, for the domains $\Omega = (a, b) \subset \mathbb{R}$ and $\Omega \subset \mathbb{R}^2$ we then have

$$(\nabla f)(\xi) = f'(\xi), \qquad \text{for} \quad \xi \in \Omega = (a, b) \subset \mathbb{R}$$

$$(\nabla f)(\xi_1, \xi_2) = \begin{bmatrix} \frac{df}{d\xi_1}(\xi_1, \xi_2) \\ \frac{df}{d\xi_1}(\xi_1, \xi_2) \end{bmatrix}, \qquad \text{for} \quad (\xi_1, \xi_2) \in \Omega \subset \mathbb{R}^2.$$

Using the divergence and gradient the differential equations (2.3) and (2.4) can be written in the forms

$$-\nabla \cdot (\beta \nabla f) = g, \qquad \text{equation (2.3)}$$

$$-\nabla \cdot \left(\begin{bmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{bmatrix} \nabla f \right) = g, \qquad \text{equation (2.4)}.$$

Exercise 2.3.3. Expand the derivatives on the left-hand sides of the above equations (using the definitions of the divergence and gradient) to verify that the equations indeed match (2.3) and (2.4). ◇

Exercise 2.3.4. Show that if $\Omega \subset \mathbb{R}^n$ and $f \in H^1(\Omega)$, then

$$\|f\|_{H^1}^2 = \|f\|_{L^2}^2 + \|\nabla f\|_{L^2(\Omega;\mathbb{C}^n)}^2$$

(in particular $\|\nabla f\|_{L^2(\Omega;\mathbb{C}^n)} \leq \|f\|_{H^1}$). ◇

We originally assumed that in the partial differential equation (2.4) the coefficients $\beta_1, \beta_2 > 0$ were constants, but motivated by the above expressions we can consider a larger class of second order partial differential equations also in the case of $\Omega \subset \mathbb{R}^2$, and at the same time consider similar equations on higher-dimensional spatial domains $\Omega \subset \mathbb{R}^n$ with $n \geq 1$. Indeed, the above forms for (2.3) and (2.4) show that the common "general form" is the following.

Definition 2.3.1. Assume $\Omega \subset \mathbb{R}^n$ with $n \in \mathbb{N}$ is open and bounded with smooth boundary $\partial\Omega$. For $\beta \in C(\overline{\Omega}; \mathbb{R}^{n \times n})$ and $g : \Omega \to \mathbb{C}$, consider the (partial) differential equation

$$-\left[\nabla \cdot \left(\beta \nabla f\right)\right](\xi) = g(\xi), \qquad \forall \xi \in \Omega, \tag{2.8a}$$

$$f(\xi) = 0, \qquad \forall \xi \in \partial\Omega. \tag{2.8b}$$

The equation (2.8) is called *elliptic (partial) differential equation* if $\beta(\xi) \in \mathbb{R}^{n \times n}$ is positive definite for every $\xi \in \overline{\Omega}$.

The concept of "elliptic partial differential equations" is even more general than stated in Definition 2.3.1. In particular, elliptic equations include differential equations of higher order, and the left-hand side of (2.8a) can in general include additional terms. Equations of this type are used, for example, in describing the distribution of electric potentials, and steady states of temperature distributions. Finally, different types of boundary conditions are possible as well, and our case is in particular called the *Dirichlet problem* due to the Dirichlet-type boundary condition (2.8b).

Since we assume that the function $\beta : \overline{\Omega} \to \mathbb{R}^{n\times n}$ is continuous on the bounded and closed set $\overline{\Omega}$, the ellipticity also implies that β in fact also satisfies a stronger condition called *uniform ellipticity*. In particular, there exists a constant $\beta_0 > 0$ such that

$$v^T\beta(\xi)v \geq \beta_0\|v\|^2 > 0, \qquad \forall v \in \mathbb{R}^n,\ \xi \in \overline{\Omega}. \tag{2.9}$$

This means that the matrix $\beta(\xi) - \beta_0 I \in \mathbb{R}^{n\times n}$ is positive semidefinite for every $\xi \in \overline{\Omega}$. This also agrees with our original assumption for the equation (2.3), where we assumed that $\beta \in C([a,b];\mathbb{R})$ is such that for some $\beta_0 > 0$ we have $\beta(\xi) \geq \beta_0 > 0$ for all $\xi \in [a,b]$ (which is precisely condition (2.9) in the case $n = 1$).

Exercise 2.3.5. If $\beta \in C^1(\overline{\Omega};\mathbb{R}^{n\times n})$, then (2.8a) can be written in the form

$$\sum_{0\leq|\alpha|\leq 2} \tilde{\beta}_\alpha(\xi)(D^\alpha f)(\xi) = g(\xi)$$

(where α are multi-indices) for suitable functions $\tilde{\beta}_\alpha \in C(\overline{\Omega};\mathbb{R})$. Prove this result in the special case where $n = 2$. **Hint:** *Write*

$$\beta(\cdot) = \begin{bmatrix} \beta_{11}(\cdot) & \beta_{12}(\cdot) \\ \beta_{21}(\cdot) & \beta_{22}(\cdot) \end{bmatrix}$$

with functions $\beta_{kj} \in C^1(\overline{\Omega};\mathbb{R})$ *and expand the left-hand side of* (2.8a) *using the product rule.*
$\diamond$

If $g \in L^2(\Omega)$, then the well-posedness and properties of solutions of the elliptic equation (2.8) can be studied by formulating the equation as an abstract linear equation on $X = L^2(\Omega)$. In particular, if $\beta \in C^1(\overline{\Omega};\mathbb{R}^{n\times n})$, then Exercise 2.3.5 shows that the correct condition on f such that the left-hand side belongs to $X = L^2(\Omega)$ *seems* to be[4] $f \in H^2(\Omega)$. Because of this, under these assumptions the equation (2.8) can be written in the abstract form $Af = g$ on $X = L^2(\Omega)$ in terms of the operator $A : \mathcal{D}(A) \subset X \to X$ defined by

$$Af = -\nabla \cdot \big(\beta\nabla f\big), \qquad f \in \mathcal{D}(A) \tag{2.10}$$

and the suitable choice of the domain $\mathcal{D}(A)$ (including the boundary condition (2.8b)) is

$$\begin{aligned} f \in \mathcal{D}(A) &:= \{\, f \in H^2(\Omega) \mid f(\xi) = 0 \text{ for (almost) all } \xi \in \partial\Omega \,\} \\ &= H^2(\Omega) \cap H_0^1(\Omega). \end{aligned}$$

Note that since $H^1(\Omega) \subset H^2(\Omega)$, the intersection $H^2(\Omega) \cap H_0^1(\Omega)$ consists of precisely those functions $f \in H^2(\Omega)$ which have the boundary trace $Tf = 0$. The operator A in (2.10) is likewise called an *elliptic differential operator* if $\beta(\xi)$ is positive definite for each $\xi \in \overline{\Omega}$.

[4]This choice is intuitive, and it is in fact exactly correct if the geometry properties of Ω and its boundary $\partial\Omega$ are sufficiently "nice" (in particular if $\partial\Omega$ is smooth, or alternatively if Ω is convex and bounded). In this case $\nabla \cdot (\beta\nabla f) \in L^2(\Omega)$ if and only if $f \in H^2(\Omega)$.

Exercise 2.3.6. Show that the elliptic differential operator A in (2.10) with domain $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$ is symmetric. **Hint:** *You can use Lemma* 1.3.7*. In verifying the condition on* $\langle Af, f\rangle$*, you can use the multidimensional integration by parts formula*

$$\int_\Omega u(\xi)(\nabla \cdot \boldsymbol{v})(\xi) d\xi = \int_{\partial\Omega} u(\xi)\boldsymbol{v}(\xi) \cdot \nu(\xi) d\xi - \int_\Omega (\nabla u)(\xi) \cdot \boldsymbol{v}(\xi) d\xi$$

for $u \in H^1(\Omega)$ *and* $\boldsymbol{v} \in H^1(\Omega; \mathbb{R}^n)$*, where* $\nu(\cdot)$ *is the unit outward normal of* Ω *at* $\xi \in \partial\Omega$*. The computation is similar as the integration by parts argument in the next section.* ◇

2.3.1 Different Types of Solutions

In this section we define in which different types of functions can be considered to be "solutions" of the elliptic differential equation (2.8). Throughout the section and the rest of this chapter (if not otherwise mentioned) we assume $\Omega \subset \mathbb{R}^n$ with $n \in \mathbb{N}$ to be open and bounded with smooth boundary $\partial\Omega$. The strongest (and the most conservative) form of solutions are *classical solutions*, where the differential equation (2.8a) holds pointwise.

Definition 2.3.2. Assume $\beta \in C^1(\overline{\Omega}; \mathbb{R}^{n\times n})$ and $g \in C_b(\Omega)$ (continuous and bounded). A function

$$f \in C^2(\Omega) \cap C^1(\overline{\Omega})$$

is a *classical solution* of (2.8) if (2.8a) holds pointwise for all $\xi \in \Omega$ and $f(\xi) = 0$ for all $\xi \in \partial\Omega$.

In the case where $g \in L^2(\Omega)$ it is appropriate to consider the following concept of *strong solutions*. Note that in the definition the boundary condition $f|_{\partial\Omega} = 0$ is satisfied in the sense of boundary traces.

Definition 2.3.3. Assume $\beta \in C^1(\overline{\Omega}; \mathbb{R}^{n\times n})$ and $g \in L^2(\Omega)$. A function

$$f \in H^2(\Omega) \cap H_0^1(\Omega)$$

is a *strong solution* of (2.8) if

$$-\nabla \cdot (\beta \nabla f) = g \tag{2.11}$$

(equality in the sense of $L^2(\Omega)$-functions).

Finally, we can define *weak solutions* of the elliptic equation (2.8). This concept is based on the observation that we can relax the sense in which the equality "$-\nabla \cdot (\beta\nabla f) = g$" holds even further. Indeed, we can first note that $f \in H^2(\Omega) \cap H_0^1(\Omega)$ is a strong solution of the elliptic equation if and only if[5]

$$-\langle h, \nabla \cdot (\beta\nabla f)\rangle_{L^2} = \langle h, g\rangle_{L^2}, \qquad \text{for all} \quad h \in H_0^1(\Omega). \tag{2.12}$$

[5]This is due to the fact that $H_0^1(\Omega)$ is dense in $L^2(\Omega)$. This property, on the other hand, follows from the fact that $C_c(\Omega)$ is dense in $L^2(\Omega)$ and $C_c(\Omega) \subset H_0^1(\Omega) \subset L^2(\Omega)$.

We can rewrite the inner product $\langle h, \nabla\cdot(\beta\nabla f)\rangle_{L^2}$ on the left-hand side of the equation using the multi-dimensional integration by parts formula

$$\int_\Omega u(\xi)(\nabla\cdot\boldsymbol{v})(\xi)d\xi = \int_{\partial\Omega} u(\xi)\boldsymbol{v}(\xi)\cdot\nu(\xi)d\xi - \int_\Omega (\nabla u)(\xi)\cdot\boldsymbol{v}(\xi)d\xi$$

which is valid for $u \in H^1(\Omega)$ and $\boldsymbol{v} \in H^1(\Omega;\mathbb{C}^n)$ on $\Omega \subset \mathbb{R}^n$. More precisely, we have

$$\begin{aligned}
-\langle h, \nabla\cdot(\beta\nabla f)\rangle_{L^2} &= -\int_\Omega h(\xi)\overline{[\nabla\cdot(\beta\nabla f)]\,(\xi)}d\xi \\
&= -\int_{\partial\Omega} h(\xi)\beta(\xi)\overline{(\nabla f)(\xi)}\cdot\nu(\xi)d\xi + \int_\Omega (\nabla h)(\xi)\cdot\beta(\xi)\overline{(\nabla f)(\xi)}d\xi \\
&= \int_\Omega \langle(\nabla h)(\xi), \beta(\xi)(\nabla f)(\xi)\rangle_{\mathbb{C}^n} d\xi \\
&=: B[h,f].
\end{aligned}$$

The integral over the boundary $\partial\Omega$ is zero due to the property that $h|_{\partial\Omega} = 0$ in the sense of traces. The last integral expression defines a *sesquilinear form*[6] $B[h,f]$. Because of the above computation and (2.12), a function $f \in H^2(\Omega)\cap H^1_0(\Omega)$ is a strong solution of (2.8) if and only if $B[h,f] = \langle h, g\rangle_{L^2}$ for all $h \in H^1_0(\Omega)$. However, we should note that the integral expression which determines $B[h,f]$ is well-defined also for any $f \in H^1(\Omega)$ and $h \in H^1_0(\Omega)$. In this way, we can consider the equation (2.8) also for functions which are not assumed to have well-defined or square integrable second order derivatives, i.e., $f \notin H^2(\Omega)$. Functions satisfying the differential equation in this generalised sense are called *weak solutions* of (2.8). Note that the requirement $f \in H^1_0(\Omega)$ again guarantees that the boundary condition (2.8b) is satisfied in the sense of boundary traces. In the definition we on purpose also drop the assumption of the smoothness of the boundary $\partial\Omega$.

Definition 2.3.4. Assume $\Omega \subset \mathbb{R}^n$ with $n \in \mathbb{N}$ is open and bounded, $\beta \in C(\overline{\Omega};\mathbb{R}^{n\times n})$, and $g \in L^2(\Omega)$. A function $f \in H^1_0(\Omega)$ is a *weak solution* of (2.8) if

$$B[h,f] = \langle h, g\rangle_{L^2} \qquad \forall h \in H^1_0(\Omega), \tag{2.13}$$

where $B[\cdot,\cdot] : H^1_0(\Omega)\times H^1_0(\Omega) \to \mathbb{C}$ is defined by

$$B[h,f] = \int_\Omega \langle(\nabla h)(\xi), \beta(\xi)(\nabla f)(\xi)\rangle_{\mathbb{C}^n} d\xi, \qquad f,h \in H^1_0(\Omega). \tag{2.14}$$

In the next section we will use the fundamental properties of the sesquilinear form $B[\cdot,\cdot]$ to establish the existence of a unique weak solution of the elliptic equation. The following exercises begin the verification of these properties.

Exercise 2.3.7. Assume $\beta \in C(\overline{\Omega};\mathbb{R}^{n\times n})$. Show that the sesquilinear form $B[\cdot,\cdot] : H^1_0(\Omega)\times H^1_0(\Omega) \to \mathbb{C}$ defined in (2.14) is bounded, i.e., there exists $M > 0$ such that

$$|B[h,f]| \le M\|h\|_{H^1}\|f\|_{H^1}.$$

[6]Meaning $B[c_1h_1 + c_2h_2, f] = c_1B[h_1,f] + c_2B[h_2,f]$ and $B[h, c_1f_1 + c_2f_2] = \overline{c_1}B[h,f_1] + \overline{c_2}B[h,f_2]$. The form is *bilinear* if we consider real-valued functions and constants, in which case the complex conjugates are not needed.

Hint: *Since* β *is continuous on* $\overline{\Omega}$*, it is also bounded in the sense that* $\max\|\beta(\xi)\|_{\mathbb{C}^{n\times n}} < \infty$. *You can use Cauchy–Schwarz inequalities for the inner product on* $\mathbb{C}^n$ *and for the integral, and also employ the estimate in Exercise* 2.3.4. ◇

Exercise 2.3.8. Assume $\beta \in C(\overline{\Omega};\mathbb{R}^{n\times n})$ and assume the differential equation (2.8) is elliptic. Show that the sesquilinear form $B[\cdot,\cdot] : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{C}$ in (2.14) satisfies $B[h,f] = \overline{B[f,h]}$. Show that there exists $\tilde{c}_0 > 0$ such that

$$B[f,f] \geq \tilde{c}_0 \|\nabla f\|_{L^2}^2, \qquad f \in H_0^1(\Omega). \tag{2.15}$$

Hint: *These properties follow fairly easily from the basic assumptions and ellipticity.* ◇

2.4 Existence of Weak Solutions

In this section we show that under suitable (and very reasonable) assumptions the elliptic equation has a unique weak solution $f \in H_0^1(\Omega)$ for every $g \in L^2(\Omega)$. Our first main result, the *Lax–Milgram Theorem*, introduces a general condition on a sesquilinear form $B[\cdot,\cdot]$ such that the abstract equation (2.13) has a solution $f \in H_0^1(\Omega)$. After this, we show that under suitable conditions the particular form in (2.14) associated to the elliptic equation satisfies these conditions.

Theorem 2.4.1. *Let* X *be a (real or complex) Hilbert space. Assume the sesquilinear form* $B[\cdot,\cdot] : X \times X \to \mathbb{C}$ *is* bounded *and* coercive *in the sense that there exist constants* $c_1, c_0 > 0$ *such that*

$$|B[x,y]| \leq c_1 \|x\|_X \|y\|_X, \qquad x,y \in X \qquad \textit{(boundedness)}$$

and

$$B[x,x] \geq c_0 \|x\|_X^2 \qquad x \in X \qquad \textit{(coercivity)}.$$

Then for any $\varphi \in X^* := \mathcal{L}(X,\mathbb{C})$ *(the dual space of* X*) there exists a unique* $y_\varphi \in X$ *such that*

$$B[x,y_\varphi] = \varphi(x), \qquad \forall x \in X. \tag{2.16}$$

Moreover, there exists $M > 0$ *(independent of* φ*) such that* $\|y_\varphi\|_X \leq M\|\varphi\|_{X^*}$.

One thing to observe is that the conclusion of the Lax–Milgram Theorem is almost identical to the familiar *Riesz Representation Theorem*, namely (2.16) means that

> "*every linear functional* $\varphi \in X^*$ *has the form* $\varphi(x) = B[x,y]$ *for all* $x \in X$ *for some unique element* $y \in X$."

Indeed, the Lax–Milgram Theorem is a generalisation of the Riesz Representation Theorem due to the fact that bounded and coercive forms $B[\cdot,\cdot]$ are a strictly more general concept than inner products. However, if a bounded and coercive form has the additional property that it is *symmetric* in the sense that $B[x,y] = \overline{B[y,x]}$ for every $x,y \in X$, then the mapping $(x,y) \mapsto B[x,y]$ in fact *does* define a new inner product on the space X, and the Lax–Milgram Theorem follows fairly directly from the Riesz Representation Theorem. On this course we will prove Theorem 2.4.1 only in this situation by showing that $B[\cdot,\cdot]$ is indeed an inner product on X. The proof of the more general version can be found in the literature, and the Riesz Representation Theorem plays a key role in the proof also in the case of non-symmetric forms. The proof of Theorem 2.4.1 in the case $B[x,y] = \overline{B[y,x]}$ is covered in the homework problems.

In exercise 2.3.7 we already saw that $B[\cdot,\cdot] : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{C}$ defined in (2.14) is bounded, and by Exercise 2.3.8 $B[\cdot,\cdot]$ is also symmetric and *non-negative*, meaning that $B[f,f] \geq 0$ and for all $f \in H_0^1(\Omega)$. The estimate in Exercise 2.3.8 is not yet as strong as coercivity required in the Lax–Milgram Theorem, but (2.15) can be combined with the very useful *Poincarè's Inequality* to show the coercivity. We can proceed this way because we consider a special class of elliptic equations with derivatives of order 2 and no "lower order terms". In a more general situation the coercivity of $B[\cdot,\cdot]$ in (2.14) is established by the so-called *Gårding's Inequality*.

Theorem 2.4.2 (Poincarè's Inequality). *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Then there exists $c > 0$ such that*

$$\|f\|_{L^2} \leq c\|\nabla f\|_{L^2} \qquad \forall f \in H_0^1(\Omega).$$

In particular, $\|f\|_{H^1} \leq \sqrt{c^2+1}\|\nabla f\|_{L^2}$ for all $f \in H_0^1(\Omega)$.

Proof. Since $\Omega \subset \mathbb{R}^n$ is bounded, there exists $d > 0$ such that if $\xi = (\xi_1, \ldots, \xi_n)^T \in \Omega$, then $|\xi_1| \leq d$. If $f \in C_c^\infty(\Omega)$, then we note that

$$\frac{\partial}{\partial \xi_1}|f(\xi)|^2 = \frac{\partial f}{\partial \xi_1}(\xi)\overline{f(\xi)} + f(\xi)\overline{\frac{\partial f}{\partial \xi_1}(\xi)} = 2\,\mathrm{Re}\left[\frac{\partial f}{\partial \xi_1}(\xi)\overline{f(\xi)}\right]$$

and thus

$$\left|\frac{\partial}{\partial \xi_1}|f(\xi)|^2\right| \leq 2\left|\frac{\partial f}{\partial \xi_1}(\xi)\right||f(\xi)|.$$

Using integration by parts (and the fact that $\operatorname{supp} f$ is compact in Ω) we get that

$$\begin{aligned}
\|f\|_{L^2}^2 &= \int_\Omega 1 \cdot |f(\xi)|^2 d\xi = \left|\int_\Omega \xi_1 \cdot \frac{\partial}{\partial \xi_1}|f(\xi)|^2 d\xi\right| \leq 2\int_\Omega |\xi_1| \cdot \left|\frac{\partial f}{\partial \xi_1}(\xi)\right||f(\xi)|d\xi \\
&\leq 2d\int_\Omega \left|\frac{\partial f}{\partial \xi_1}(\xi)\right||f(\xi)|d\xi \leq 2d\left\|\frac{\partial f}{\partial \xi_1}\right\|_{L^2}\|f\|_{L^2} \\
&\leq 2d\|\nabla f\|_{L^2(\Omega;\mathbb{C}^n)}\|f\|_{L^2},
\end{aligned}$$

which implies the claim for $c = 2d$. The second estimate follows directly from $\|f\|_{H^1}^2 = \|f\|_{L^2}^2 + \|\nabla f\|_{L^2}^2 \leq (c^2+1)\|\nabla f\|_{L^2}^2$ (Exercise 2.3.4). □

Exercise 2.4.1. Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Show that the $B[\cdot,\cdot] : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{C}$ in (2.14) is coercive, i.e., there exists $c_0 > 0$ such that

$$B[f,f] \geq c_0\|f\|_{H^1}^2, \qquad f \in H_0^1(\Omega).$$

Hint: *Combine the estimate in Exercise* 2.3.8 *and Poincarè's Inequality.* ◇

Exercise 2.4.2. Show that the form in (2.14) is not coercive on $X = H^1(\Omega)$. **Hint:** *It is very easy to find a nonzero function $f \in H^1(\Omega)$ such that $B[f,f] = 0$.* ◇

With the help of the Lax–Milgram Theorem and the properties of $B[\cdot,\cdot]$ proved in the previous exercises we can prove the following theorem on the existence of weak solutions for the elliptic partial differential equation.

Theorem 2.4.3. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and assume $\beta \in C(\overline{\Omega};\mathbb{R}^{n\times n})$. Then for every $g \in L^2(\Omega)$ the elliptic differential equation* (2.8) *has a unique weak solution $f \in H_0^1(\Omega)$. Moreover, there exists a constant $M > 0$ such that the solutions satisfy*

$$\|f\|_{H^1} \leq M\|g\|_{L^2}.$$

Exercise 2.4.3. Prove Theorem 2.4.3. **Hint:** *Apply the Lax–Milgram Theorem on the Hilbert space $X = H_0^1(\Omega)$. The boundedness and coercivity of $B[\cdot,\cdot]$ on X follow from the above exercises. What you simply need to prove is that for every $g \in L^2(\Omega)$ the mapping $h \mapsto \langle h, g\rangle_{L^2}$ belongs to the dual space X^* and the corresponding functional φ satisfies $\|\varphi\|_{X^*} \leq \|g\|_{L^2}$.* ◇

In the proof of Theorem 2.4.3 we showed that if $X = H_0^1(\Omega)$, then every $g \in L^2(\Omega)$ defines a functional $\varphi \in X^*$. However, the space $(H_0^1(\Omega))^*$ is strictly larger than $L^2(\Omega)$[7]. In particular, the *point evaluation* at $\xi_0 \in \Omega$ defined by

$$\varphi(f) = f(\xi_0)$$

is a bounded linear functional on $H_0^1(\Omega)$. The Lax–Milgram Theorem gives us the existence of f also for $\varphi \in X^*$, and indeed in the literature the "weak solutions" of (2.8) are usually defined as functions $f \in H_0^1(\Omega)$ satisfying

$$B[h,f] = \varphi(h), \qquad h \in H_0^1(\Omega)$$

for a fixed $\varphi \in (H_0^1(\Omega))^*$. The dual space $(H_0^1(\Omega))^*$ has a special name and notation as well.

Definition 2.4.4. For $\Omega \subset \mathbb{R}^n$ the Sobolev space of order -1 is defined as $H^{-1}(\Omega) = (H_0^1(\Omega))^*$.

Sobolev spaces of negative orders have several uses in analysis of partial differential equations, and the spaces $H^{-k}(\Omega)$ can be defined similarly.

2.5 Regularity of Solutions – From Weak to Strong Solutions

In the previous section we saw that under very mild assumptions the elliptic equation has a unique weak solution $f \in H_0^1(\Omega)$ for every $g \in L^2(\Omega)$. In this section we are interested in the question of when this unique weak solution turns out to be a strong solution as well. The definitions imply that a weak solution $f \in H_0^1(\Omega)$ is also a strong solution if we in addition have $f \in H^2(\Omega)$. This property (existence and square integrability of additional derivatives) is called *regularity* of the solution. For example in the case where $\beta(\xi) \equiv I$, the form

$$-\Delta f = g$$

[7] In the sense that the functionals which have the form $\varphi_g = \langle \cdot, g\rangle_{L^2}$ for some $g \in L^2(\Omega)$ are a proper subspace of $(H_0^1(\Omega))^*$.

of the elliptic differential equation (which in this case is the *Poisson problem*) would seem to give a strong indication that if $g \in L^2(\Omega)$, we could expect the solution f to be in the space $H^2(\Omega)$. However, this is in general not true, and especially spatial domains $\Omega \subset \mathbb{R}^n$ with corners can cause the weak solution (which does exist whenever Ω is open and bounded) not to have the additional square integrable second derivatives and thus $f \notin H^2(\Omega)$ (see for example Renardy & Rogers, Example 8.52 for a counter-example).

The results in this section on *global regularity* show that if the domain Ω has a smooth boundary[8] and $\beta \in C^1(\overline{\Omega}; \mathbb{R}^{n\times n})$, then every weak solution of the elliptic equation is also a strong solution. The proof of this result is quite long and technical, and its details do not really fit into the main theme of the course. Because of this, we simply state this result.

Theorem 2.5.1 (Global Regularity). *Assume $\Omega \subset \mathbb{R}^n$ is open and bounded with smooth boundary $\partial\Omega$ and assume $\beta \in C^1(\overline{\Omega}; \mathbb{R}^{n\times n})$ is such that $\beta(\xi)$ is positive definite for all $\xi \in \overline{\Omega}$. If $g \in L^2(\Omega)$ and if $f \in H_0^1(\Omega)$ is the unique weak solution of the elliptic equation* (2.8)*, then $f \in H^2(\Omega)$ and there exists $M > 0$ (independent of $g \in L^2(\Omega)$) such that*

$$\|f\|_{H^2} \le M\|g\|_{L^2}.$$

Theorem 2.5.1 allows us to finally prove that under the stated additional assumptions the elliptic equation has a unique *strong* solution.

Theorem 2.5.2. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded with smooth boundary $\partial\Omega$ and $\beta \in C^1(\overline{\Omega}; \mathbb{R}^{n\times n})$. For every $g \in L^2(\Omega)$ the elliptic equation*

$$\begin{aligned} -\nabla \cdot (\beta\nabla f) &= g \qquad && \text{in } \Omega \\ f &= 0 && \text{on } \partial\Omega \end{aligned}$$

has a unique strong solution $f \in H^2(\Omega) \cap H_0^1(\Omega)$, and there exists $M > 0$ (independent of g) such that

$$\|f\|_{H^2} \le M\|g\|_{L^2}.$$

Exercise 2.5.1. Prove Theorem 2.5.2. **Hint:** *Use Theorem* 2.5.1 *to show that the mild solution of the elliptic equation is also a strong solution.* ◇

The existence of strong solutions of the elliptic equation gives us valuable information about the elliptic differential operator

$$Af = -\nabla \cdot (\beta\nabla f), \tag{2.17}$$

as shown in the following Exercise.

Exercise 2.5.2. Let the assumptions of Theorem 2.5.1 be satisfied. Show that the elliptic differential operator $A : \mathcal{D}(A) \subset X \to X$ in (2.17) on $X = L^2(\Omega)$ with domain $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$ is boundedly invertible. Conclude that A is also self-adjoint. Finally, show that the inverse A^{-1} has the property

$$A^{-1} \in \mathcal{L}(L^2(\Omega), H^2(\Omega)).$$

[8]The literature offers much sharper results, for example a "C^2-boundary" more than enough, but for simplicity we only consider smooth boundaries on this course.

Hint: *The symmetry of A was already shown in Exercise* 2.3.6. *You can use the Theorem* 2.5.2 *to show that $0 \in \rho(A)$, and then use results in Section* 1.3. *The claims regarding the inverse A^{-1} follow quite directly from Theorem* 2.5.2. ◇

The property $A^{-1} \in \mathcal{L}(L^2(\Omega), H^2(\Omega))$ shown in Exercise 2.5.2 is a strictly stronger property than $A^{-1} \in \mathcal{L}(X)$, since the norm on $H^2(\Omega)$ is "stronger" than the norm on $X = L^2(\Omega)$. This boundedness property of the inverse plays an important role in our investigation of the spectrum of the operator A in the next chapter.

3. Spectral Properties of Elliptic Differential Operators

In this section we focus on the study of the spectral properties of the elliptic differential operator $A : \mathcal{D}(A) \subset X \to X$ defined by

$$Af = -\nabla \cdot \big(\beta \nabla f\big), \qquad f \in \mathcal{D}(A) \tag{3.1a}$$

with domain

$$\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega) \tag{3.1b}$$

on the space $X = L^2(\Omega)$. Throughout the section (unless otherwise stated) we assume that $\Omega \subset \mathbb{R}^n$ is open and bounded with smooth boundary $\partial\Omega$, $\beta \in C^1(\overline{\Omega}; \mathbb{R}^{n\times n})$ and there exists $\beta_0 > 0$ such that the matrix $\beta(\xi) - \beta_0 I$ is positive semidefinite for every $\xi \in \overline{\Omega}$. We already saw in Exercise 2.3.6 that A is a symmetric operator even with weaker assumptions on Ω, and in Exercise 2.5.2 we also saw that under the above assumptions A is a self-adjoint operator. In this chapter we will see that we can say a lot about the spectrum of the operator A even without any other assumptions on n, Ω or β. Indeed, we will learn that under the above assumptions the whole spectrum of A consists of eigenvalues, each with at most finite number of independent eigenfunctions, and the eigenvalues do not have any accumulation points. All of these properties are based on the fact that the operator A has *compact resolvents*. As we saw in the homework problems, the operator A with this property has a *spectral representation* (or *eigenfunction expansion*)

$$Af = \sum_{k=1}^{\infty} \lambda_k \langle f, \phi_k \rangle_X \phi_k,$$

$$\mathcal{D}(A) = \Big\{ f \in X \;\Big|\; \sum_{k=1}^{\infty} |\lambda_k|^2 |\langle f, \phi_k \rangle_X|^2 < \infty \Big\}$$

where $\lambda_k \in \mathbb{R}$ are the eigenvalues of A and $\phi_k \in H^2(\Omega) \cap H_0^1(\Omega)$ are the corresponding orthonormal eigenfunctions. This representation can be used in the analysis of the elliptic differential equation and also in the numerical approximation of the solution $f = A^{-1}g$ of the elliptic equation. Moreover, the eigenfunctions ϕ_k describe the characteristic features of the elliptic equation and because of this they are often of independent interest. For example in time-dependent *evolution equations* the eigenvalues can often correspond to some *natural frequencies* of a vibrating system, and the corresponding eigenfunctions are the system's *(normal) modes*. On the other hand, the eigenvalues of the elliptic operator in the Schrödinger equation describe the natural energy levels of a quantum mechanical system.

Since the eigenvalue $\lambda \in \mathbb{R}$ and the corresponding eigenfunction $\phi \in \mathcal{D}(A)$ satisfy $A\phi = \lambda\phi$, the function $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$ is the (strong) solution of the differential equation

$$-\left[\nabla \cdot \left(\beta \nabla \phi\right)\right](\xi) = \lambda\phi(\xi), \qquad \forall \xi \in \Omega, \tag{3.2a}$$
$$\phi(\xi) = 0, \qquad \forall \xi \in \partial\Omega. \tag{3.2b}$$

This is called the *eigenvalue problem* corresponding to the elliptic equation. It is easy to see that $\lambda \in \mathbb{R}$ is an eigenvalue of A if and only if the equation (3.2) has a nonzero solution ϕ, and in this case ϕ is an eigenfunction of A corresponding to the eigenvalue λ.

Before moving on to more challenging parts of the spectral analysis, we can prove a few basic properties of the spectrum of the elliptic operator. We first note that besides being symmetric and self-adjoint, the operator A is also *positive*, meaning that $\langle Af, f\rangle > 0$ for all $f \in \mathcal{D}(A)$ with $f \neq 0$. In fact, this follows from the integration by parts argument in Section 2.3.1, and Exercise 2.4.1, which together imply that

$$\langle Af, f\rangle_X = B[f, f] \geq c_0 \|f\|_{H^1}^2 > 0$$

for some constant $c_0 > 0$ and for all $f \in \mathcal{D}(A)$ with $f \neq 0$.

Exercise 3.0.1. In this problem we show that $\sigma(A) \subset (0, \infty)$. This property is a consequence of the following parts (recall that we have already shown $0 \in \rho(A)$ in Exercise 2.5.2).

(a) Show that $\sigma_p(A) \subset (0, \infty)$. **Hint:** *This is easy to do using the property that A is a positive operator.*

(b) Show that $\sigma_r(A) \cap \mathbb{R} = \varnothing$. **Hint:** *This property holds for all self-adjoint operators. Use self-adjointness to show that if $\mathcal{R}(\lambda - A)$ is not dense for some $\lambda \in \mathbb{R}$, then necessarily $\lambda \in \sigma_p(A)$.*

(c) Show that if $\lambda < 0$, then $\|(\lambda - A)f\| \geq |\lambda|\|f\|$ for all $f \in \mathcal{D}(A)$. **Hint:** *This is analogous to Exercise* 1.3.4.

(d) Use part (c) to show that $\lambda \notin \sigma_c(A)$ for all $\lambda < 0$. **Hint:** *This can be completed similarly as part* (d) *of Exercise* 1.3.5. *If you cannot figure out the complete proof, you could justify in detail why the same arguments should be applicable.*

◇

3.1 Embedding Theorems for Sobolev Spaces

The compactness of the resolvents of the elliptic operator A are based on fundamental properties of Sobolev spaces. For convenience, we begin by recalling the definition of a compact operator.

Definition 3.1.1. Let X and Y be normed linear spaces. An operator $K \in \mathcal{L}(X, Y)$ is compact if for every bounded sequence $(x_n)_{n\in\mathbb{N}} \subset X$ the sequence $(Kx_n)_{n\in\mathbb{N}} \subset Y$ has a convergent subsequence.[a]

[a] In brief form the condition for $K \in \mathcal{L}(X, Y)$ to be compact can be characterised as the property that "*the image of the unit ball of X under K is precompact in Y*".

In the following we define the concept of *embeddings* of spaces (suom. *upotus*).

Definition 3.1.2. Let X be a normed linear space. A subspace Y (with norm $\|\cdot\|_Y$) is said to be *continuously embedded in* X if the identity map $J : Y \to X$ defined by $Jx = x \in X$ for all $x \in Y$ satisfies $J \in \mathcal{L}(Y, X)$. Equivalently, Y is continuously embedded in X if there exists $M > 0$ such that

$$\|x\|_X \le M\|x\|_Y, \qquad \forall x \in Y. \tag{3.3}$$

The subspace Y of X is said to be *compactly embedded in* X if the identity map $J \in \mathcal{L}(Y, X)$ is a compact operator.

Embeddings take place especially in situations where the subspace Y of X is defined using a norm $\|\cdot\|_Y$ as $Y = \{\, x \in X \mid \|x\|_Y < \infty \,\}$. Here it is essential that the norm $\|\cdot\|_Y$ is "stronger" than the natural norm $\|\cdot\|_X$ on X in the sense that condition (3.3) is satisfied for some constant $M > 0$, but a corresponding inequality in the reverse direction does not hold[1].

Exercise 3.1.1. Assume $\Omega \subset \mathbb{R}^n$ is open and bounded. **Hint:** *The required estimates are very straightforward, so do not make your answers too long!*

(a) Show that $C(\overline{\Omega})$ is continuously embedded in $L^p(\Omega)$ whenever $1 \le p \le \infty$. **Hint:** *Use the definition directly. Consider the case $p = \infty$ separately.*

(b) Show that $H^k(\Omega)$ is continuously embedded in $H^j(\Omega)$ whenever $k, j \in \mathbb{N} \cup \{0\}$ are such that $k \ge j \ge 0$.

(c) Show that $L^p(\Omega)$ is continuously embedded in $L^q(\Omega)$ whenever $1 \le q \le p \le \infty$. **Hint:** *Use Hölder's inequality with suitable exponents for the product $|f(\xi)|^q = 1 \cdot |f(\xi)|^q$ (this is a standard technique in analysis of Lebesgue spaces). Consider the case $p = \infty$ separately.*

◇

For our purposes the most important use of embeddings between spaces lies in showing compactness of operators. In particular, we will later use the following lemma in showing that the elliptic operator A has compact resolvents.

Lemma 3.1.3. *Let X and Y be normed linear spaces and assume Y is compactly embedded in X. If an operator $B \in \mathcal{L}(X)$ in addition satisfies $B \in \mathcal{L}(X, Y)$, then B is compact (as an operator $B : X \to X$).*

In the above lemma the property $B \in \mathcal{L}(X, Y)$ means that $\mathcal{R}(B) \subset Y$ and there exists $M_1 > 0$ such that $\|Bx\|_Y \le M_1\|x\|_X$ for all $x \in X$.

Exercise 3.1.2.

(a) Prove Lemma 3.1.3. **Hint:** *Since the identity mapping $J : Y \to X$ is compact, you can express the operator $B : X \to X$ as a composition of a bounded operator and a compact operator, and use the general properties of compact operators to complete the proof.*

[1] Otherwise the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ would be equivalent, and the situation would be trivial from the point of view of the embeddings.

(b) Prove that if Y is compactly embedded in X and X is continuously embedded in Z, then Y is compactly embedded in Z. Also prove that if Y is continuously embedded in X and X is compactly embedded in Z, then Y is compactly embedded in Z. **Hint:** *Write the identity map $J : Y \to Z$ as a composition of a continuous and compact operator. Both of the properties can be proved quite conveniently at the same time.*

◇

In order to show that the elliptic operator A has compact resolvents, we need results on embeddings between Sobolev spaces. For more general versions, you can see the books by Brezis (Section 9.3) and Evans (Section 5.6).

Theorem 3.1.4 (Rellich–Kondrachov). *Assume $\Omega \subset \mathbb{R}^n$ is open and bounded with smooth boundary $\partial\Omega$. Then the following hold.*

- *If $n < 2$, then $H^1(\Omega)$ is compactly embedded in $C(\overline{\Omega})$.*
- *If $n = 2$, then $H^1(\Omega)$ is compactly embedded in $L^q(\Omega)$ for all $q \geq 2$.*
- *If $n > 2$, then $H^1(\Omega)$ is compactly embedded in $L^q(\Omega)$ for all $1 \leq q < \frac{2n}{n-2}$.*

We omit the full proof of the Rellich–Kondrachov theorem for now (we will come back to it at the end of the course). It can be proved using using direct arguments (see Evans or Brezis) or the *Fourier transform* (see Rogers–Renardy).

Corollary 3.1.5. *Assume $\Omega \subset \mathbb{R}^n$ is open and bounded with smooth boundary $\partial\Omega$. Then $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$.*

Exercise 3.1.3.

(a) Prove Corollary 3.1.5. **Hint:** *Go through the different cases for $n \in \mathbb{N}$ in the Rellich–Kondrachov Theorem. In the case $n = 1$ you will also need earlier exercises.*

(b) Prove that if $\Omega \subset \mathbb{R}^n$ is open and bounded with smooth boundary $\partial\Omega$, then $H^2(\Omega)$ is compactly embedded in $L^2(\Omega)$. **Hint:** *Use earlier exercises.*

◇

We will only prove the Rellich–Kondrachov Theorem in the special case where $n = 1$ and $\Omega = (a, b) \subset \mathbb{R}$, and it is covered in the homework problems. In this case the proof is a fairly direct consequence of the Arzela–Ascoli Theorem presented below (the proof of Theorem 3.1.7 can be found for example in Renardy–Rogers, or Rudin – "Real and Complex Analysis"). For stating this result, we need the concept of "*equicontinuity*" of a sequence, which means that the parameters $\varepsilon > 0$ and $\delta > 0$ in the definition of continuity can be chosen to be independent of $n \in \mathbb{N}$ (and $\xi_0 \in \Omega$).

Definition 3.1.6. Let $\Omega \subset \mathbb{R}^n$. A sequence $(f_n)_{n\in\mathbb{N}} \subset C(\Omega)$ is called *equicontinuous* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\xi_0, \xi \in \Omega$ and for all $n \in \mathbb{N}$ we have

$$|f_n(\xi_0) - f_n(\xi)| < \varepsilon \qquad \text{whenever} \qquad \|\xi_0 - \xi\| < \delta.$$

Note that in particular, every function f_n in an equicontinuous sequence is uniformly continuous.

Theorem 3.1.7 (Arzela–Ascoli Theorem)**.** *Assume $\Omega \subset \mathbb{R}^n$ is open and bounded. Let $(f_n)_{n\in\mathbb{N}} \subset C(\overline{\Omega})$ be a bounded and equicontinuous sequence. Then there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ which converges in $C(\overline{\Omega})$.*

3.2 Spectral Representation of A

Using the results in the previous subsection we can prove the following theorem characterising the spectral properties of the elliptic operator A. In particular, the result establishes the *eigenfunction expansion* (or *spectral representation*) for the operator A.

Theorem 3.2.1. *Assume $\Omega \subset \mathbb{R}^n$ is open and bounded with smooth boundary $\partial\Omega$, $\beta \in C^1(\overline{\Omega};\mathbb{R}^{n\times n})$ and there exists $\beta_0 > 0$ such that the matrix $\beta(\xi) - \beta_0 I$ is positive semidefinite for every $\xi \in \overline{\Omega}$. The elliptic operator $A = -\nabla\cdot(\beta\nabla f)$ on $X = L^2(\Omega)$ with domain $\mathcal{D}(A) = H^2(\Omega)\cap H_0^1(\Omega)$ has compact resolvents. In particular, there exist $\{\lambda_k\}_{k\in\mathbb{N}} \subset (0,\infty)$ satisfying $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ and an orthonormal basis $\{\phi_k\}_{k\in\mathbb{N}}$ of X such that*

$$Af = \sum_{k=1}^{\infty} \lambda_k \langle f, \phi_k\rangle_X \phi_k, \tag{3.4a}$$

$$\mathcal{D}(A) = \Big\{ f \in X \;\Big|\; \sum_{k=1}^{\infty} |\lambda_k|^2 |\langle f, \phi_k\rangle_X|^2 < \infty \Big\}. \tag{3.4b}$$

In addition A, has the following properties.

(a) $\sigma(A) = \sigma_p(A) = \{\lambda_k\}_{k\in\mathbb{N}} \subset (0,\infty)$ *(in particular, $\sigma_c(A) = \varnothing$ and $\sigma_r(A) = \varnothing$).*

(b) *The set $\{\lambda_k\}_{k\in\mathbb{N}}$ of eigenvalues has no finite accumulation points.*

(c) *The* multiplicity *of each eigenvalue λ_k is finite, i.e., for any $k \in \mathbb{N}$ there exist at most finite number of $n \in \mathbb{N}$ such that $\lambda_n = \lambda_k$.*

Part (a) of Theorem 3.2.1 is a direct consequence of the spectral representation of A and part (b) of the theorem. Indeed, the eigenfunction expansion implies that there exists an operator $S \in \mathcal{L}(\ell^2, X)$ with $S^{-1} \in \mathcal{L}(X, \ell^2)$ such that the operator $D : \mathcal{D}(D) \subset \ell^2(\mathbb{C}) \to \ell^2(\mathbb{C})$ defined so that $\mathcal{D}(D) = S^{-1}(\mathcal{D}(A))$ and $D = S^{-1}AS$ is an infinite diagonal matrix $D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots)$. In this situation we say that A is *boundedly similar* to the operator D, or *diagonalisable*. A bit similarly as in the case of matrices, the bounded similarity transform preserves spectral properties. Using the conclusions of the first homework problems and the fact that the parts of spectra of D and A coincide, we have that

- $\sigma_p(A) = \sigma_p(D) = \{\lambda_k\}_{k\in\mathbb{N}}$.
- $\sigma_c(A) = \sigma_c(D) = \{\text{acc. points of } \{\lambda_k\}_{k\in\mathbb{N}}\} = \varnothing$ by part (b).
- $\sigma_r(A) = \sigma_r(D) = \varnothing$.

The rest of the parts of Theorem 3.2.1 are proved in the next exercise problem.

Exercise 3.2.1. Prove that under the assumptions of Theorem 3.2.1 the operator A has compact resolvents. In addition, prove parts (b) and (c) of the theorem.

Hint: *In order to prove the first part, simply use the results presented in this and the previous chapter to show that the operator A^{-1} is compact (as an operator $A^{-1} : X \to X$).*

Parts (b) *and* (c) *follow from the basic spectral properties of compact operators (note that the eigenvalues of the operator* $A^{-1} \in \mathcal{L}(X)$ *are* $\lambda_k^{-1} \in (0, \infty)$ *for* $k \in \mathbb{N}$*).* ◇

Exercise 3.2.2. In the case where $\Omega = (0, \ell) \subset \mathbb{R}$ and $\beta > 0$ is constant, use Theorem 3.2.1 express the elliptic operator A in the form

$$(Af)(\xi) = \sum_{k=1}^{\infty} \lambda_k \phi_k(\xi) \int_0^{\ell} f(t)\overline{\phi_k(t)}dt,$$

$$\mathcal{D}(A) = \left\{ f \in X \mid \sum_{k=1}^{\infty} |\lambda_k|^2 \left| \int_0^{\ell} f(t)\overline{\phi_k(t)}dt \right|^2 < \infty \right\}.$$

with correct expressions for λ_n and ϕ_n. Write your formulas in the simplest possible forms. **Hint:** *λ_n and ϕ_n are eigenvalues and eigenfunctions of A, which were computed in the homework problems. You need to scale the eigenfunctions in such a way that $\|\phi_n\| = 1$ (you can use Matlab/Mathematica/WolframAlpha for computing the associated trigonometric integrals if you like).* ◇

The eigenfunction expansion of A can also be used in solving the elliptic equation

$$-\nabla \cdot \big(\beta \nabla f\big) = g, \qquad \text{on} \quad \Omega, \tag{3.5a}$$

$$f = 0, \qquad \text{on} \quad \partial\Omega. \tag{3.5b}$$

Indeed, as we saw in the homework problems on Week 2, the inverse of the operator A in (3.4) is given by

$$A^{-1}h = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \langle h, \phi_k \rangle_X \phi_k, \qquad h \in X.$$

Thus the knowledge of the eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$ and the orthonormal eigenfunctions $\{\phi_k\}_{k\in\mathbb{N}}$ also gives us the strong solution $f \in H^2(\Omega) \cap H_0^1(\Omega)$ of the elliptic equation (3.5) in the form

$$f(\xi) = (A^{-1}g)(\xi) = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \langle g, \phi_k \rangle_{L^2} \phi_k(\xi) \tag{3.6}$$

for any function $g \in L^2(\Omega)$. In particular, the scalar values $\lambda_k^{-1}\langle g, \phi_k \rangle_{L^2}$ are the coordinates of the solution f in the orthonormal basis $\{\phi_k\}_{k\in\mathbb{N}}$ of $L^2(\Omega)$.

The importance of the solution (3.6) of the elliptic equation (3.5) may often be purely theoretical (of course this value is not to be taken lightly!), since the formula involves summation over an infinite number of terms and the requirement of full knowledge of the eigenvalues λ_k and eigenfunctions ϕ_k is typically very unrealistic, unless the considered case is in some way simple geometrically. For example, it is possible to compute the eigenvalues and eigenfunctions in the case where $\beta(\xi) \equiv \beta_0 I_{n\times n}$ (in which case the operator reduces to a scalar multiple of the Laplacian $A = -\beta_0 \Delta$), and the domain has a particular geometry, such as one of the following:

- $n = 1$ and $\Omega = (a, b) \subset \mathbb{R}$
- Ω has rectangular shape (a rectangle in the case $n = 2$, and a "rectangular cuboid" if $n \geq 3$)
- Ω is a disk ($n = 2$) or a sphere ($n \geq 3$).

In other cases, the eigenvalues and eigenfunctions can be solved using computational methods.

3.3 Approximation of The Solutions of $Af = g$

In this section we briefly discuss two methods for approximating the solution f of the abstract linear equation $Af = g$, or more precisely the solution of the elliptic differential equation.

3.3.1 Approximation Using Eigenfunctions

The property that A has compact resolvents opens up a possibility of approximating the solution $g = A^{-1}f$ of the elliptic problem using truncations of the infinite series (3.6). This leads to so-called *modal approximation* (the term arises from the eigenfunctions ϕ_k being "*modes*" of the partial differential equation). This kind of an approximation obviously requires that at least a finite number of eigenvalues λ_k and eigenfunctions ϕ_k are known.

Modal approximation is based on the fundamental property of compact operators, which states that if (and only if!) $K \in \mathcal{L}(X)$ is compact, then there exists a sequence of operators $(K_N)_{N\in\mathbb{N}} \subset \mathcal{L}(X)$ each with $\dim(\mathcal{R}(K_N)) < \infty$ such that

$$\|K_N - K\|_{\mathcal{L}(X)} \to 0 \qquad \text{as} \quad N \to \infty.$$

The operators K_N with the property $\dim(\mathcal{R}(K_N)) < \infty$ are said to have *finite rank*, and correspond to matrices (i.e. operators on finite-dimensional spaces).

Theorem 3.3.1. *Assume $\Omega \subset \mathbb{R}^n$ is open and bounded with smooth boundary $\partial\Omega$, $\beta \in C^1(\overline{\Omega}; \mathbb{R}^{n\times n})$ and there exists $\beta_0 > 0$ such that the matrix $\beta(\xi) - \beta_0 I$ is positive semidefinite for every $\xi \in \overline{\Omega}$. If $\{\lambda_k\}_{k\in\mathbb{N}}$ and $\{\phi_k\}_{k\in\mathbb{N}}$ are as in Theorem* 3.2.1*, then for every $g \in L^2(\Omega)$ and $N \in \mathbb{N}$ we have that the function*

$$f_N(\xi) = \sum_{k=1}^{N} \frac{1}{\lambda_k} \langle g, \phi_k \rangle_{L^2} \phi_k(\xi), \qquad \xi \in \Omega$$

satisfies

$$\|f_N - f\|_{L^2} \leq \frac{1}{\lambda_{N+1}} \|g\|_{L^2}, \tag{3.7}$$

where f is the strong solution of the elliptic equation (3.5)*. In particular, $\|f_N - f\|_{L^2} \to 0$ as $N \to \infty$.*

The last conclusion of the theorem follows immediately from the estimate (3.7), since we assumed that $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ and $\lambda_k \to \infty$ as $k \to \infty$.

Exercise 3.3.1. In this problem we will prove Theorem 3.3.1 in parts.

(a) For an $N \in \mathbb{N}$, construct an operator $K_N \in \mathcal{L}(X)$ such that the function f_N defined in Theorem 3.3.1 satisfies $f_N = K_N g$, and show that K_N has finite rank.

(b) Show that $\|K_N - A^{-1}\|_{\mathcal{L}(X)} \to 0$ as $N \to \infty$, and in particular $\|K_N - A^{-1}\|_{\mathcal{L}(X)} \leq \lambda_{N+1}^{-1}$.

(c) Based on (a) and (b), prove Theorem 3.3.1.

Hint: *The proofs of the parts are quite simple and straightforward. In the required norm estimates you can use the property that $\{\phi_k\}_k$ is an orthonormal basis of X.* ◇

Exercise 3.3.2. Consider the differential equation

$$3f''(\xi) = g(\xi), \qquad \xi \in (0,2)$$
$$f(0) = f(2) = 0.$$

In the case of two functions $g \in L^2(0,2)$:

$$\text{(a)} \qquad g(\xi) = \begin{cases} 0 & 0 \leq \xi \leq 1 \\ 1 & 1 < \xi \leq 2 \end{cases}$$

$$\text{(b)} \qquad g(\xi) = \frac{1}{4} + \xi(\xi-2)^2.$$

Construct an approximate solution $\tilde{f} \in L^2(0,2)$ for the of the above differential equation such that $\|\tilde{f} - f\|_{L^2} \leq 0.01$ where f denotes the exact solution of the equation. Plot these approximate solutions $\tilde{f}$.

Hint: *Use Theorem* 3.3.1 *and Exercise* 3.2.2. *You can again use Matlab or WolframAlpha in computing the required integrals (these are quite doable by hand as well, but practicing integration is not the point of this exercise).* ◇

3.3.2 The Galerkin Method*

As we saw above, the modal approximation requires knowledge of the eigenfunctions of the operator $A : \mathcal{D}(A) \subset X \to X$. In this section we give an overview of the so-called *Galerkin method* which is much more flexible in the sense that it only requires a set of basis elements $\{\psi_k\}_{k=1}^N \subset X$ which can be used to approximate functions in the space X. Especially in the case of partial differential equations where X is a function space, the elements $\{\psi_k\}_{k=1}^N$ can be chosen to be for example piecewise linear functions, polynomials, or locally supported "hat functions". Instead of being a specific method, the Galerkin method is much more of a "general approach" to *finding the best approximation of the solutions of $Af = g$ in the finite-dimensional space $V_N = \operatorname{span}\{\psi_k\}_{k=1}^N$*, and the different choices of basis elements lead to a variety of different concrete approximation schemes. For example, the Finite Element Method corresponds to the choice of $\{\psi_k\}_{k=1}^N$ as "hat functions", and different families of polynomials (Legendre, Chebyshev, ...) lead to "spectral-Galerkin" methods.

The more precise idea in the Galerkin method is that the weak form of the elliptic equation

$$B[h,f] = \langle h, g\rangle_{L^2}, \qquad \forall h \in V$$

(with $V = H_0^1(\Omega)$ in our case) is relaxed to the form

$$B[h,f_N] = \langle h, g\rangle_{L^2}, \qquad \forall h \in V_N \tag{3.8}$$

where $V_N \subset V$ is a finite-dimensional subspace of the Hilbert space V such that $V_N = \operatorname{span}\{\psi_1, \ldots, \psi_N\}$. At the same time, we seek a solution f_N from the same finite-dimensional space, i.e., $f_N \in V_N$. This way, the equation (3.8) is a finite-dimensional linear equation, which can be easily reformulated as a matrix equation. Indeed, since $f_N \in V_N = \operatorname{span}\{\psi_1, \ldots, \psi_N\}$ by assumption, f_N has the form

$$f_N = \sum_{k=1}^N \overline{a_k}\psi_k, \qquad (a_k)_{k=1}^N \in \mathbb{C}^N \tag{3.9}$$

(the complex conjugates $\overline{a_k}$ are taken for the sake of notational convenience in the following formulas). If we choose $h = \psi_j$ for $j = 1, \dots, N$ in (3.8), then we see that

$$\begin{aligned}
& B[h, f_N] = \langle h, g\rangle_{L^2}, \qquad \forall h \in V_N \\
\Leftrightarrow \qquad & B[\psi_j, f_N] = \langle \psi_j, g\rangle_{L^2}, \qquad \forall j \in \{1, \dots, N\} \\
\Leftrightarrow \qquad & \sum_{k=1}^{N} a_k B[\psi_j, \psi_k] = \langle \psi_j, g\rangle_{L^2}, \qquad \forall j \in \{1, \dots, N\} \\
\Leftrightarrow \qquad & [B[\psi_j, \psi_1], \dots, B[\psi_j, \psi_N]] \begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix} = \langle \psi_j, g\rangle_{L^2}, \qquad \forall j \in \{1, \dots, N\}.
\end{aligned}$$

If we denote $\boldsymbol{a} = [a_1, \dots, a_N]^T \in \mathbb{C}^N$, then the last form is equivalent to the matrix equation

$$A_N \boldsymbol{a} = g_N, \qquad \text{where} \quad A_N = (B[\psi_j, \psi_k])_{jk} \in \mathbb{C}^{N\times N}, \quad g_N = (\langle \psi_j, g\rangle_{L^2})_{j=1}^N \in \mathbb{C}^N.$$

Due to (3.9) the vector $\boldsymbol{a} \in \mathbb{C}^n$ determines the solution f_N of the finite-dimensional problem (3.8).

The following result demonstrates how functional analysis is used to derive a general error estimate for the approximation error $\|f - f_N\|$ in terms of the general error which results from approximation of elements in V with those in the finite-dimensional space V_N. In the result, the existence and uniqueness of the solution of the finite-dimensional problem (3.8) follow from the Lax–Milgram Theorem. Note that due to the above computations, this also shows that under the assumptions the matrix equation $A_N \boldsymbol{a} = g_N$ has a unique solution for all g_N, i.e., the matrix A_N is non-singular.

Theorem 3.3.2. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and assume $\beta \in C(\overline{\Omega}; \mathbb{R}^{n\times n})$. Further assume that the $\{\psi_k\}_{k=1}^N \subset H_0^1(\Omega)$ are linearly independent. Then for every $g \in L^2(\Omega)$ there exists a unique $f_N \in V_N = \operatorname{span}\{\psi_1, \dots, \psi_N\}$ such that* (3.8) *holds.*

Moreover, if $c_0, c_1 > 0$ are such that

$$\begin{aligned}
|B[h_1, h_2]| &\le c_1 \|h_1\|_{H^1} \|h_2\|_{H^1}, \qquad && h_1, h_2 \in H_0^1(\Omega) \\
B[h, h] &\ge c_0 \|h\|_{H^1}^2 && h \in H_0^1(\Omega)
\end{aligned}$$

and $f \in H_0^1(\Omega)$ is the unique weak solution of the elliptic equation in Theorem 2.4.3*, then*

$$\|f_N - f\|_{H^1} \le \frac{c_1}{c_0} \inf_{h \in V_N} \|h - f\|_{H^1}.$$

The estimate for the error $\|f - f_N\|_{H^1}$ shows that the solution f_N of (3.8) is *the best approximation of the true solution f in the space V_N up to the constant c_1/c_0*. This part of the theorem is known as *Céa's Lemma*.

Proof of Theorem 3.3.2. Since the form $B[\cdot, \cdot]$ is bounded and coercive on $H_0^1(\Omega)$, it is also bounded and coercive on V_N (with the same norm $\|\cdot\|_{H^1}$). Because of this, the Lax–Milgram Theorem shows that the equation (3.8) has a unique solution $f_N \in V_N$.

To derive the estimate for the norm $\|f_N - f\|_{H^1}$, we first note that since f is a weak solution of the elliptic problem and f_N satisfies (3.8), for every $h \in V_N \subset H_0^1(\Omega)$ we have

$$B[f, h] = \langle h, g\rangle_{L^2} = B[f_N, h],$$

which implies $B[f-f_N,h]=0$ for all $h\in V_N$. Now the boundedness and coercivity of $B[\cdot,\cdot]$ imply that for an arbitrary $h\in V_N$ we have

$$\begin{aligned}\|f-f_N\|_{H^1}^2 &\le \frac{1}{c_0}B[f-f_N,f-f_N]=\frac{1}{c_0}\big(B[f-f_N,f-h]+B[f-f_N,\overbrace{h-f_N}^{\in V_N}]\big)\\ &=\frac{1}{c_0}B[f-f_N,f-h]\le\frac{c_1}{c_0}\|f-f_N\|_{H^1}\|f-h\|_{H^1}.\end{aligned}$$

This implies $\|f-f_N\|_{H^1}\le\frac{c_1}{c_0}\|f-h\|_{H^1}$, and taking an infimum over all $h\in V_N$ completes the proof. □

In the case where the form $B[\cdot,\cdot]$ satisfies $B[f,g]=\overline{B[g,f]}$ for all $f,g\in H_0^1(\Omega)$, the error estimate in Theorem 3.3.2 can be improved in the following way (note that in boundedness and coercivity we always have $c_1\ge c_0$).

Lemma 3.3.3. *Let the assumptions of Theorem* 3.3.2 *be satisfied. If in addition $B[f,g]=\overline{B[g,f]}$ for all $f,g\in H_0^1(\Omega)$, then*

$$\|f_N-f\|_{H^1}\le\sqrt{\frac{c_1}{c_0}}\inf_{h\in V_N}\|h-f\|_{H^1}.$$

Proof. In the homework problems for Week 4 we proved that if $B[f,g]=\overline{B[g,f]}$ for all $f,g\in H_0^1(\Omega)$, then the mapping $(f,g)\mapsto B[f,g]$ defines an inner product on $H_0^1(\Omega)$ and

$$c_0\|h\|_{H^1}^2\le\|h\|_B^2\le c_1\|h\|_{H^1}^2,$$

where $\|h\|_B:=\sqrt{B[h,h]}$. Similarly as in the proof of Theorem 3.3.2, the property $B[f-f_N,h]=0$ for all $h\in V_N$ implies that for every fixed $h\in V_N$ we have

$$\|f-f_N\|_B=B[f-f_N,f-f_N]=B[f-f_N,f-h]\overset{C-S}{\le}\|f-f_N\|_B\|f-h\|_B.$$

Thus $\|f-f_N\|_B\le\|f-h\|_B$. Using this estimate and the equivalence of the norms $\|\cdot\|_B$ and $\|\cdot\|_{H^1}$ shows that

$$\|f-f_N\|_{H^1}\le\frac{1}{\sqrt{c_0}}\|f-f_N\|_B\le\frac{1}{\sqrt{c_0}}\|f-h\|_B\le\sqrt{\frac{c_1}{c_0}}\|f-h\|_{H^1}.$$

Taking an infimum over $h\in V_N$ leads to the estimate in the claim. □

4. The Fourier Transform

In this section we take a look at the *Fourier transform* of functions defined on the full n-dimensional space $\mathbb{R}^n$. The Fourier transform is an important tool in the analysis of differential operators due to the fact that differentiation of a function $f : \mathbb{R}^n \to \mathbb{C}$ can be alternatively represented as a multiplication operation of the Fourier transform $\mathcal{F}f$ of f.

4.1 The Fourier Transform on $L^2(\mathbb{R}^n)$

We begin by defining the Fourier transform of a function $f \in L^1(\mathbb{R}^n)$.

Definition 4.1.1. The *Fourier transform* of a function $f \in L^1(\mathbb{R}^n)$ is a function $\mathcal{F}f : \mathbb{R}^n \to \mathbb{C}$ defined as

$$(\mathcal{F}f)(z) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-iz\cdot\xi} f(\xi) d\xi, \qquad z \in \mathbb{R}^n \tag{4.1}$$

(where $z\cdot\xi$ denotes the inner product on $\mathbb{R}^n$, i.e., $z\cdot\xi = z^T\xi$). The *inverse Fourier transform* of a function $g \in L^1(\mathbb{R}^n)$ is a function $\mathcal{F}^{-1}g : \mathbb{R}^n \to \mathbb{C}$ defined as

$$(\mathcal{F}^{-1}g)(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{iz\cdot\xi} g(z) dz, \qquad \xi \in \mathbb{R}^n.$$

The Fourier transform is clearly a linear, i.e., $\mathcal{F}(\alpha f_1 + \beta f_2) = \alpha\mathcal{F}f_1 + \beta\mathcal{F}f_2$ for all $f_1, f_2 \in L^1(\mathbb{R}^n)$ and $\alpha, \beta \in \mathbb{C}$.

Exercise 4.1.1. Show that the Fourier transform $\mathcal{F}f$ is a well-defined and continuous function for any $f \in L^1(\mathbb{R}^n)$, and that

$$\|\mathcal{F}f\|_\infty \le \frac{1}{(2\pi)^{n/2}} \|f\|_{L^1}.$$

Finally, show that $(\mathcal{F}f)(z) \to 0$ as $\|z\| \to \infty$ (which concludes that $\mathcal{F}f \in C_0(\mathbb{R}^n) := \{ g \in C(\mathbb{R}^n) \mid |g(z)| \to 0 \text{ as } \|z\| \to \infty \}$).

Hint: *Your first task is to show that the value $(\mathcal{F}f)(z)$ is well-defined for all $z \in \mathbb{R}^n$. Showing continuity is a bit more challenging: For a $z \in \mathbb{R}^n$ you can take a sequence $(z_k)_{k\in\mathbb{N}} \subset \mathbb{R}^n$ such that $z_k \to z$ as $k \to \infty$ and use the Lebesgue's Dominated Convergence Theorem in analysing the difference $|(\mathcal{F}f)(z_k) - (\mathcal{F}f)(z)|$. In the final part you can apply the "Riemann–Lebesgue Lemma".* ◇

As its name suggest, the purpose of the "inverse Fourier transform" is to reverse the effect of the Fourier transform. However, we cannot immediately prove a property that these two transforms are inverses. Indeed, for a function $f \in L^1(\mathbb{R}^n)$ the transformed function $(\mathcal{F}f)$ does not necessarily belong to $L^1(\mathbb{R}^n)$ (in the previous example we saw that $(\mathcal{F}f) \in C_0(\mathbb{R}^n)$), and because of this we cannot directly apply the inverse transform to $(\mathcal{F}f)$.

Moreover, in order to investigate the elliptic operator, we would like the Fourier transform to be defined for functions in the space $L^2(\mathbb{R}^n)$. Since we are considering functions on the full Euclidean space $\mathbb{R}^n$, the space $L^2(\mathbb{R}^n)$ is not a subset of $L^1(\mathbb{R}^n)$, and in particular the integral in (4.1) is not well-defined as a Lebesgue integral for every function $f \in L^2(\mathbb{R}^n)$. However, by first considering the Fourier transform for functions in the intersection $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ (which is dense in $L^2(\mathbb{R}^n)$) we will see that the transform $\mathcal{F}$ can be *extended* to the space $L^2(\mathbb{R}^n)$ in such a way that $\mathcal{F}f$ is given by the formula (4.1) for all $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

Theorem 4.1.2. *For every $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ we have $\mathcal{F}f \in L^2(\mathbb{R}^n)$ and*

$$\|\mathcal{F}f\|_{L^2} = \|f\|_{L^2}. \tag{4.2}$$

The Fourier transform has a unique extension (also denoted by $\mathcal{F}$) to an unitary operator $\mathcal{F} \in \mathcal{L}(L^2(\mathbb{R}^n))$, meaning that $\mathcal{F}$ is bijective and

$$\langle \mathcal{F}f, \mathcal{F}g \rangle_{L^2} = \langle f, g \rangle_{L^2}, \qquad f, g \in L^2(\mathbb{R}^n).$$

The property that $\mathcal{F}$ is *unitary* means that $\mathcal{F}$ is bijective and *isometric* in the sense that (4.2) holds for all $f \in L^2(\mathbb{R}^n)$. The property that Fourier transform preserves the L^2-norm is known as the *Plancherel's Theorem*. Another important thing to note is that the Fourier transform determines $\mathcal{F}f$ as an L^2-function, which means that the value $(\mathcal{F}f)(z)$ is defined for *almost all* $z \in \mathbb{R}^n$. However, in Exercise 4.1.1 we saw that for any $f \in L^1(\mathbb{R}^n)$ the function $\mathcal{F}f$ is a continuous function with well-defined values $(\mathcal{F}f)(z)$ for all $z \in \mathbb{R}^n$. This difference is precisely due to the extension of the Fourier transform from $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.

Proof of Theorem 4.1.2. The identity (4.2) means that

$$\int_{\mathbb{R}^n} |(\mathcal{F}f)(z)|^2 dz = \int_{\mathbb{R}^n} |f(\xi)|^2 d\xi, \qquad \forall f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n).$$

We do not present a proof of this property of the Fourier transform here, a complete proof can be found for example in Evans 2010 (Section 4.3.1) or Rudin 1987 (Theorem 9.13).

The space $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$. By (4.2) the mapping $\mathcal{F} : L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is bounded, and thus $\mathcal{F}$ has an extension to a bounded operator $\mathcal{F} \in \mathcal{L}(L^2(\mathbb{R}^n))$. This extension is a very standard construction in linear operator theory, but since its proof was omitted on the Introduction to Functional Analysis, we sketch it briefly here: For a function $f \in L^2(\mathbb{R}^n)$ we define $\mathcal{F}f$ by taking a sequence $(f_k)_{k\in\mathbb{N}} \subset L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ such that $\|f_k - f\|_{L^2} \to 0$ as $k \to \infty$. By (4.2) we have

$$\|\mathcal{F}f_k - \mathcal{F}f_j\|_{L^2} = \|\mathcal{F}(f_k - f_j)\|_{L^2} = \|f_k - f_j\|_{L^2} \to 0 \qquad \text{as } k, j \to \infty.$$

Thus $(\mathcal{F}f_k)_{k\in\mathbb{N}} \subset L^2(\mathbb{R}^n)$ is Cauchy, and there exists $h \in L^2(\mathbb{R}^n)$ such that $\|\mathcal{F}f_k - h\|_{L^2} \to 0$ as $k \to \infty$. If we define $\mathcal{F}f = h$, we can show that $\mathcal{F}$ is indeed a linear operator and the value $\mathcal{F}f$ does not depend on the sequence $(f_k)_{k\in\mathbb{N}} \subset L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

The property (4.2) (together with the fact that $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$) also imply that $\|\mathcal{F}f\|_{L^2} = \|f\|_{L^2}$ holds for all $f \in L^2(\mathbb{R}^n)$. Thus for all $f \in L^2(\mathbb{R}^n)$ we have

$$\langle \mathcal{F}^*\mathcal{F}f, f\rangle_{L^2} = \langle \mathcal{F}f, \mathcal{F}f\rangle_{L^2} = \|\mathcal{F}f\|_{L^2}^2 = \|f\|_{L^2}^2 = \langle f, f\rangle_{L^2},$$

and since $\mathcal{F}^*\mathcal{F}$ is self-adjoint, we have that $\mathcal{F}^*\mathcal{F} = I$. This means that the Fourier transform $\mathcal{F}$ is an *isometry* on $L^2(\mathbb{R}^n)$.

Finally, to show that $\mathcal{F}$ is unitary, we need to further show that $\mathcal{F}$ is bijective. Injectivity of $\mathcal{F}$ follows directly from (4.2), and this same identity implies that the range of $\mathcal{F} \in \mathcal{L}(L^2(\mathbb{R}^n))$ is closed (exactly as in Exercise 1.3.5(c)). Because of this, we only need to show that $\mathcal{R}(\mathcal{F})$ is dense in $L^2(\mathbb{R}^n)$. Since $\mathcal{R}(\mathcal{F})^\perp = \mathcal{N}(\mathcal{F}^*)$, we can equivalently show that $\mathcal{N}(\mathcal{F}^*) = \{0\}$. To compute $\mathcal{F}^*$, let $f, g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ be arbitrary. Fubini's theorem implies that

$$\begin{aligned}\langle \mathcal{F}f, g\rangle_{L^2} &= \int_{\mathbb{R}^n} \left(\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-iz\cdot\xi} f(\xi) d\xi \right) \overline{g(z)} dz \\ &= \int_{\mathbb{R}^n} f(\xi) \overline{\left(\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{iz\cdot\xi} g(\xi) dz \right)} d\xi = \langle f, \mathcal{F}^{-1}g\rangle_{L^2},\end{aligned}$$

where $\mathcal{F}^{-1}g$ is the inverse Fourier transform of g. Thus $\mathcal{F}^*g = \mathcal{F}^{-1}g$ for all $g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ (note that at this point the notation $\mathcal{F}^{-1}$ is still formal, we have not yet showed that $\mathcal{F}^{-1}$ in Definition 4.1.1 is an inverse of $\mathcal{F}$). We can now note that by definition $(\mathcal{F}^{-1}g)(z) = (\mathcal{F}g)(-z)$ for all $g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and for all $z \in \mathbb{R}^n$. Since this minus sign does not change the L^2-norm of the function, the identity (4.2) implies that for all $g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ we also have

$$\|\mathcal{F}^*g\|_{L^2} = \|\mathcal{F}^{-1}g\|_{L^2} = \|\mathcal{F}g\|_{L^2} = \|g\|_{L^2}.$$

Since $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, we in particular have that $\mathcal{F}^*$ is injective. Thus $\mathcal{R}(\mathcal{F})^\perp = \mathcal{N}(\mathcal{F}^*) = \{0\}$ and $\mathcal{F}$ has dense range. This completes the proof that $\mathcal{F}$ is unitary.

The property $\langle \mathcal{F}f, \mathcal{F}g\rangle_{L^2} = \langle f, g\rangle_{L^2}$ for $f, g \in L^2(\mathbb{R}^n)$ follows by considering the identity (4.2) for functions $f = h \pm ig$ with $h, g \in L^2(\mathbb{R}^n)$. □

The proof of Theorem 4.1.2 shows that the notation "$\mathcal{F}^{-1}g$" for the inverse Fourier transform is justified. Indeed, the inverse of the operator $\mathcal{F} \in \mathcal{L}(L^2(\mathbb{R}^n))$ is the unique extension of the inverse Fourier transform acting on $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Since $(\mathcal{F}^{-1}f)(z) = (\mathcal{F}f)(-z)$ for all $f \in L^1(\mathbb{R}^n)$ and $z \in \mathbb{R}^n$, this extension of $\mathcal{F}^{-1}$ from $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ can be constructed analogously to the corresponding extension of $\mathcal{F}$.

In our study of differential operators, one of the most important properties of the Fourier transform is that it can be used to express the differentiation of a function in an extremely simple way. We are now considering functions on the unbounded domain $\Omega = \mathbb{R}^n$. A useful result regarding Sobolev spaces is that $C_c^\infty(\mathbb{R}^n)$ is dense in $H^k(\mathbb{R}^n)$ for all $k \in \mathbb{N}$ (in this case the support $\operatorname{supp} f$ of $f \in C_c^\infty(\mathbb{R}^n)$ is compact as a subset of $\mathbb{R}^n$, i.e., it is a closed and bounded set).

Theorem 4.1.3. *Let $f \in H^k(\mathbb{R}^n)$ for some $k \in \mathbb{N}$ and let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi-index with $|\alpha| \leq k$. Then*

$$[\mathcal{F}(D^\alpha f)](z) = (iz)^\alpha (\mathcal{F}f)(z), \qquad \text{for almost every } z \in \mathbb{R}^n, \tag{4.3}$$

where we denote $(iz)^\alpha = (iz_1)^{\alpha_1}(iz_2)^{\alpha_2} \cdots (iz_n)^{\alpha_n}$ for $z = (z_1, \ldots, z_n) \in \mathbb{R}^n$ (and "i" is the complex unit).

Exercise 4.1.2. Show that the formula (4.3) holds for all $f \in C_c^\infty(\mathbb{R}^n)$.

Hint: *You can first derive a formula for $[\mathcal{F}(\partial_{\xi_j} f)](z)$ using integration by parts and then apply it repeatedly to prove* (4.3). ◇

Proof of Theorem 4.1.3. Let $f \in H^k(\mathbb{R}^n)$ be arbitrary and let $|\alpha| \leq k$. Since $C_c^\infty(\mathbb{R}^n)$ is dense in $H^k(\mathbb{R}^n)$, we can choose a sequence $(f_j)_{j \in \mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ such that $\|f_j - f\|_{H^k} \to 0$ as $j \to \infty$. If we define $g_j = D^\alpha f_j \in L^2(\mathbb{R})$, then Exercise 4.1.2 shows that $(\mathcal{F}g_j)(z) = (iz)^\alpha (\mathcal{F}f_j)(z)$ almost everywhere. Plancherel's theorem also implies

$$\|\mathcal{F}g_j - \mathcal{F}g_l\|_{L^2} = \|\mathcal{F}(g_j - g_l)\|_{L^2} = \|g_j - g_l\|_{L^2} = \|D^\alpha f_j - D^\alpha f_l\|_{L^2} \leq \|f_j - f_l\|_{H^k} \stackrel{j,l \to \infty}{\longrightarrow} 0.$$

Thus $(\mathcal{F}g_j)_{j \in \mathbb{N}} \subset L^2(\mathbb{R}^n)$ is a Cauchy sequence, and since $L^2(\mathbb{R}^n)$ is Hilbert, there exists $h \in L^2(\mathbb{R}^n)$ such that $\|\mathcal{F}g_j - h\|_{L^2} \to 0$ as $j \to \infty$. Moreover, since $\mathcal{F} \in \mathcal{L}(L^2(\mathbb{R}^n))$ and since $\|g_j - D^\alpha f\|_{L^2} \to 0$ as $j \to \infty$, we have that

$$\mathcal{F}(D^\alpha f) = \lim_{j \to \infty} \mathcal{F}g_j = h.$$

Thus the proof is complete once we show that $h(z) = (iz)^\alpha (\mathcal{F}f)(z)$ for almost all $z \in \mathbb{R}^n$ (this will also imply that $z \mapsto (iz)^\alpha (\mathcal{F}f)(z) \in L^2(\mathbb{R}^n)$).

Denote $m_\alpha(z) = (iz)^\alpha$. We first note that for any $z \in \mathbb{R}^n$ the value $m_\alpha(z) = (iz)^\alpha = i^{|\alpha|} z^\alpha$ is real if $|\alpha|$ is even, and imaginary if $|\alpha|$ is odd. Because of this, it is always possible to choose $\gamma \in \mathbb{C}$ with $\gamma \neq 0$ such that $|\gamma + m_\alpha(z)| \geq |\gamma| > 0$ for all $z \in \mathbb{R}^n$ (simply by choosing γ to be imaginary if $|\alpha|$ is even, and real if $|\alpha|$ is odd). We will show that $\mathcal{F}f = (\gamma + m_\alpha)^{-1}(h + \gamma \mathcal{F}f)$. This will establish the desired property due to the following equivalences

$$\begin{aligned} \mathcal{F}f = \frac{1}{\gamma + m_\alpha}(h + \gamma \mathcal{F}f) \quad &\Leftrightarrow \quad (\mathcal{F}f)(z) = \frac{h(z) + \gamma(\mathcal{F}f)(z)}{\gamma + (iz)^\alpha} \quad \text{a.e.} \\ &\Leftrightarrow \quad (iz)^\alpha (\mathcal{F}f)(z) = h(z) \quad \text{a.e.} \end{aligned}$$

Using the above convergence properties, $\|(\gamma + m_\alpha)^{-1}\|_\infty = \sup_{z \in \mathbb{R}^n} |\gamma + m_\alpha(z)|^{-1} \leq |\gamma|^{-1}$, and $\mathcal{F}g_j = m_\alpha \mathcal{F} f_j$, we have that

$$\begin{aligned}
\|\mathcal{F}f - \frac{1}{\gamma + m_\alpha}(h + \gamma \mathcal{F}f)\|_{L^2} &= \|\mathcal{F}f - \mathcal{F}f_j + \frac{\gamma + m_\alpha}{\gamma + m_\alpha}\mathcal{F}f_j - \frac{1}{\gamma + m_\alpha}(h + \gamma \mathcal{F}f)\|_{L^2} \\
&\le \|\mathcal{F}f - \mathcal{F}f_j\|_{L^2} + \|\frac{1}{\gamma + m_\alpha}\left[\gamma(\mathcal{F}f_j - \mathcal{F}f) + m_\alpha \mathcal{F}f_j - h\right]\|_{L^2} \\
&\le \|\mathcal{F}f - \mathcal{F}f_j\|_{L^2} + \frac{1}{|\gamma|}\|\gamma(\mathcal{F}f_j - \mathcal{F}f) + m_\alpha \mathcal{F}f_j - h\|_{L^2} \\
&\le 2\|\mathcal{F}f - \mathcal{F}f_j\|_{L^2} + \frac{1}{|\gamma|}\|m_\alpha \mathcal{F}f_j - h\|_{L^2} \\
&= 2\|f - f_j\|_{L^2} + \frac{1}{|\gamma|}\|\mathcal{F}g_j - h\|_{L^2} \to 0
\end{aligned}$$

as $j \to \infty$. This implies that $\mathcal{F}f = (\gamma + m_\alpha)(h + \gamma \mathcal{F}f)$ in the sense of L^2-functions, and as argued above, we have that $h(z) = m_\alpha(z)(\mathcal{F}f)(z)$ for almost all $z \in \mathbb{R}^n$, and particular $m_\alpha \mathcal{F}f \in L^2(\mathbb{R}^n)$. This completes the proof. □

Theorem 4.1.4. *Let $k \in \mathbb{N}$. The Sobolev space $H^k(\mathbb{R}^n)$ can be expressed in the form*

$$H^k(\mathbb{R}^n) = \{\, f \in L^2(\mathbb{R}^n) \mid z \mapsto \sqrt{1 + \|z\|^{2k}}(\mathcal{F}f)(z) \in L^2(\mathbb{R}^n) \,\}.$$

Exercise 4.1.3. Prove Theorem 4.1.4.

Hint: *Prove the inclusions "$\subset$" and "$\supset$" separately. The argument in the first inclusion is a fairly straightforward estimate. In the second part your aim is to show that the weak derivatives $D^\alpha f$ exist and are in $L^2(\mathbb{R}^n)$, i.e., $\langle f, D^\alpha \phi \rangle_{L^2} = (-1)^{|\alpha|}\langle g, \phi\rangle_{L^2}$ for some $g \in L^2(\mathbb{R}^n)$ and for all $\phi \in C_c^\infty(\mathbb{R}^n)$. You can do this using Theorem* 4.1.2 *and the function m_α in the proof of Theorem* 4.1.3. *In the proofs you will benefit from the following inequalities*[1] *(which you do not need to prove!) for $a_1, \dots, a_n \ge 0$ and for a multi-index α with $|\alpha| \le k$:*

$$\begin{aligned}
(a_1 + a_2 + \cdots + a_n)^k &\le n^{k-1}(a_1^k + a_2^k + \cdots + a_n^k) \\
a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n} &\le \frac{(\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_n a_n)^{|\alpha|}}{|\alpha|^{|\alpha|}} \le n^k (a_1 + a_2 + \cdots + a_n)^k
\end{aligned}$$

◇

Another way of expressing the conclusion of Theorem 4.1.4 is using *weighted L^2-spaces*.

Definition 4.1.5. Let $w : \mathbb{R}^n \to (0, \infty)$ be a continuous function such that $\inf_{\xi \in \mathbb{R}^n} w(\xi) > 0$. The *weighted Lebesgue space* $L^2_w(\mathbb{R}^n)$ is defined as

$$L^2_w(\mathbb{R}^n) = \{\, f \in L^2(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} |f(\xi)|^2 w(\xi) d\xi < \infty \,\}$$

with norm

$$\|f\|_{L^2_w} = \left(\int_{\mathbb{R}^n} |f(\xi)|^2 w(\xi) d\xi \right)^{1/2}.$$

[1] The first inequality follows from an application of the Hölder inequality, and the second one is an inequality between the "weighted geometric mean" and the "weighted arithmetic mean" of the values $a_1, \dots, a_n$.

With the aid of the weighted L^2-space the conclusion of Theorem 4.1.4 can be written in the form

$$H^k(\mathbb{R}^n) = \{\, f \in L^2(\mathbb{R}^n) \mid \mathcal{F}f \in L^2_{w_k}(\mathbb{R}^n)\,\}, \qquad \text{where} \quad w_k(z) := 1 + \|z\|^{2k}.$$

In fact, the Fourier transform defines an isomorphism (a bounded and boundedly invertible mapping) between $H^k(\mathbb{R}^n)$ and $L^2_{w_k}(\mathbb{R}^n)$. Moreover, the estimates in the solution for Exercise 4.1.3 show that the mapping

$$f \mapsto \|\mathcal{F}f\|_{L^2_{w_k}}, \qquad f \in H^k(\mathbb{R}^n)$$

defines a norm which is equivalent to the usual norm on $H^k(\mathbb{R}^n)$, defined by $\|f\|^2_{H^k} = \sum_{|\alpha|\le k} \|D^\alpha f\|^2_{L^2}$.

The fact that Sobolev spaces can be characterised in terms of the Fourier transforms of the functions belonging to weighted L^2-spaces also opens up the possibility to define Sobolev spaces $H^s(\mathbb{R}^n)$ of *any order* $s \ge 0$ simply by generalising the weight $w_k(z) = 1 + \|z\|^{2k}$ with $k \in \mathbb{N}$ to $w_s(z) = 1 + \|z\|^{2s}$. This way, the condition $\mathcal{F}f \in L^2_{w_s}(\mathbb{R}^n)$ corresponds to f having differentiability properties of "intermediate order" in the sense that if $k < s < k+1$, then f is "smoother" than functions in $H^k(\mathbb{R}^n)$, but not quite as smooth as functions in $H^{k+1}(\mathbb{R}^n)$.

Definition 4.1.6. Let $s \ge 0$. The Sobolev space of order s is defined as

$$H^s(\mathbb{R}^n) = \{\, f \in L^2(\mathbb{R}^n) \mid \mathcal{F}f \in L^2_{w_s}(\mathbb{R}^n)\,\}, \qquad \text{where} \quad w_s(z) := 1 + \|z\|^{2s}.$$

The norm on $H^s(\mathbb{R}^n)$ is defined as

$$\|f\|_{H^s} = \|\mathcal{F}f\|_{L^2_{w_s}}.$$

Due to Theorem 4.1.4 the space $H^s(\mathbb{R}^n)$ coincides with the "classical" Sobolev space $H^k(\mathbb{R}^n)$ in Definition 2.2.3 whenever $s = k \in \mathbb{N}$, and the norms on $H^s(\mathbb{R}^n)$ and $H^k(\mathbb{R}^n)$ are equivalent. Sobolev spaces of non-integer orders are important in the study of *fractional partial differential equations*, and especially the spaces of orders $s = k + 1/2$ appear frequently also in the analysis of partial differential equations, including elliptic equations.

Exercise 4.1.4. Show that for every $0 \le s \le r$ the space $H^r(\mathbb{R}^n)$ is continuously embedded in $H^s(\mathbb{R}^n)$. **Hint:** *You mainly only need an estimate between $w_r(z)$ and $w_s(z)$ for all $z \in \mathbb{R}^n$. Here it is useful to consider the cases $\|z\| \le 1$ and $\|z\| > 1$ separately.* ◇

4.2 The Elliptic Operator and Fourier Multipliers

Theorem 4.1.3 can be used in studying the differential operators and the associated partial differential equations with constant coefficients (which do not depend on the variable $\xi \in \mathbb{R}^n$) and which are defined on the full space $\Omega = \mathbb{R}^n$. In this section we study the class of differential operators $A : \mathcal{D}(A) \subset \mathbb{R}^n \to \mathbb{R}^n$ of the form

$$Af = -\nabla \cdot (\beta \nabla f) + \gamma^T \nabla f + \delta f, \qquad f \in \mathcal{D}(A) = H^2(\mathbb{R}^n) \tag{4.4}$$

with $\beta \in \mathbb{R}^{n\times n}$, $\gamma \in \mathbb{R}^n$, and $\delta \in \mathbb{R}$. In the case where $\beta \in \mathbb{R}^{n\times n}$ is a positive definite matrix, $\gamma = 0$, and $\delta > 0$, the operator is elliptic[2].

The exercise below show that the operator A can be expressed using the Fourier transform as

$$Af = \mathcal{F}^{-1}(m_A \mathcal{F} f), \qquad f \in H^2(\mathbb{R}^n), \tag{4.5}$$

where $m_A : \mathbb{R}^n \to \mathbb{C}$ is defined as $m_A(z) = z^T\beta z + i\gamma^T z + \delta \in \mathbb{C}$.

Exercise 4.2.1. Here we use the Fourier transform for vector-valued $f \in L^2(\mathbb{R}^n; \mathbb{C}^m)$, which are defined using the same formula as in the case $m = 1$. This clearly leads to $\mathcal{F}f = (\mathcal{F}f_1, \dots, \mathcal{F}f_m)^T$ for $f = (f_1, \dots, f_m)^T$.

(a) Derive a formula for $\mathcal{F}(\nabla f)$ for $f \in H^1(\mathbb{R}^n)$.

(b) Derive a formula for $\mathcal{F}(\nabla \cdot f)$ for $f \in H^1(\mathbb{R}^n; \mathbb{C}^n)$.

(c) Show that for $\beta \in \mathbb{R}^{n\times n}$, $\gamma \in \mathbb{R}^n$, and $\delta \in \mathbb{R}$ the operator $Af = -\nabla\cdot(\beta\nabla f) + \gamma^T\nabla f + \delta f$ for $f \in \mathcal{D}(A) = H^2(\mathbb{R}^n)$ has the representation (4.5) with $m_A(z) = z^T\beta z + i\gamma^T z + \delta$.

Hint: *Use the formulas for derivatives $D^\alpha f$ in parts* (a) *and* (b)*. Part* (c) *follows from the preceding parts, but remember to justify why you can apply the formulas in* (a) *and* (b) *(you especially need that $\beta\nabla f \in H^1(\mathbb{R}^n; \mathbb{C}^n)$).* ◇

The representation (4.5) shows that the application of the elliptic operator A alters the Fourier transform of a function f by multiplying $\mathcal{F}f$ pointwise with the function $m_A(\cdot)$. The class of such operators are called *Fourier multipliers*.

Definition 4.2.1. An operator $T : \mathcal{D}(T) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a *Fourier multiplier* if there exists a measurable function $m_T : \mathbb{R}^n \to \mathbb{C}$ such that

$$\mathcal{F}(Tf) = m_T \mathcal{F} f, \qquad f \in \mathcal{D}(T) \tag{4.6}$$

and $\mathcal{D}(T) = \{\, f \in L^2(\mathbb{R}^n) \mid m_T \mathcal{F} f \in L^2(\mathbb{R}^n) \,\}$. The function m_T is the *symbol* of T.

The multiplication in (4.6) is defined in the pointwise sense so that

$$[\mathcal{F}(Tf)](z) = m_T(z)(\mathcal{F}f)(z), \qquad \text{for almost all } z \in \mathbb{R}^n.$$

In the case of the differential operator (4.4) the function $z \mapsto m_A(z) = z^T\beta z + i\gamma^T z + \delta$ is continuous, which in particular implies that m_A is measurable.

Lemma 4.2.2. *If $\beta \in \mathbb{R}^n$ is positive definite, $\gamma \in \mathbb{R}^n$, and $\delta \in \mathbb{R}$, then the operator $A : \mathcal{D}(A) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ in* (4.4) *is a Fourier multiplier with the symbol m_A defined by $m_A(z) = z^T\beta z + i\gamma^T z + \delta$ for $z \in \mathbb{R}^n$.*

Proof. We only prove one part of the claim (for which the argument is probably the most difficult to come up with!), namely that $\{\, f \in L^2(\mathbb{R}^n) \mid m_A \mathcal{F} f \in L^2(\mathbb{R}^n) \,\} \subset \mathcal{D}(A)$. To this

[2] This terminology is slightly different than in Chapters 2 and 3, since we add the term "δf" with $\delta > 0$. The motivation for this addition comes from the fact that we study the operator on the full space $\Omega = \mathbb{R}^n$ — in the absence of boundary conditions the operator in Chapters 2 and 3 (which corresponds to the case $\delta = 0$) is not boundedly invertible.

end, assume that $f \in L^2(\mathbb{R}^n)$ satisfies $m_A \mathcal{F} f \in L^2(\mathbb{R}^n)$. By Theorem 4.1.4 the function satisfies $f \in \mathcal{D}(A) = H^2(\mathbb{R}^n)$ if $\mathcal{F} f \in L^2_{w_2}(\mathbb{R}^n)$ where $w_2(z) = 1 + \|z\|^4$. In order to use the property $m_A \mathcal{F} f \in L^2(\mathbb{R}^n)$ in proving this, we intuitively need to estimate $w_2(z)$ from above by $|m_A(z)|^2$ (times a constant).

Since $\beta \in \mathbb{R}^{n\times n}$ is positive definite, there exists $\beta_0 > 0$ such that $z^T \beta z \geq \beta_0 \|z\|^2$ for all $z \in \mathbb{R}^n$. Before estimating $|m_A(z)|$, we note that for any $\varepsilon > 0$ and for all $a, b \geq 0$ we have

$$2ab = 2(\sqrt{\varepsilon} a)\left(\frac{b}{\sqrt{\varepsilon}}\right) \leq (\sqrt{\varepsilon} a)^2 + \left(\frac{b}{\sqrt{\varepsilon}}\right)^2 = \varepsilon a^2 + \frac{b^2}{\varepsilon} \qquad \Rightarrow \qquad ab \leq \frac{\varepsilon a^2}{2} + \frac{b^2}{2\varepsilon}$$

Using this (extremely convenient!) inequality with $a = \|z\|$, $b = \|\gamma\|$, and $\varepsilon = \beta_0$ we can estimate

$$\begin{aligned}|m_A(z)| &= |z^T \beta z + i\gamma^T z + \delta| \geq |z^T \beta z| - |\gamma^T z + \delta| \geq \beta_0 \|z\|^2 - (\|\gamma\|\|z\| + |\delta|) \\ &\geq \beta_0 \|z\|^2 - \frac{\beta_0}{2}\|z\|^2 - \frac{1}{2\beta_0}\|\gamma\|^2 - |\delta| = \frac{\beta_0}{2}\|z\|^2 - \frac{1}{2\beta_0}\|\gamma\|^2 - |\delta|.\end{aligned}$$

Thus if $\|z\|$ is so large that $\frac{\beta_0}{2}\|z\|^2 \geq 2(\frac{2}{\beta_0}\|\gamma\|^2 + |\delta|)$ and $\|z\| \geq 1$, then

$$\begin{aligned}|m_A(z)| &\geq \frac{\beta_0}{2}\|z\|^2 - \frac{1}{2\beta_0}\|\gamma\|^2 - |\delta| \geq \frac{\beta_0}{4}\|z\|^2 = \frac{\beta_0}{4}\frac{\|z\|^2}{\sqrt{1+\|z\|^4}}\sqrt{w_2(z)} \\ &\geq \frac{\beta_0}{4}\frac{\|z\|^2}{\sqrt{\|z\|^4+\|z\|^4}}\sqrt{w_2(z)} = \frac{\beta_0}{4\sqrt{2}}\sqrt{w_2(z)}.\end{aligned}$$

Thus if we define $R = \max\{1, 2\sqrt{(2/\beta_0^2)\|\gamma\|^2 + (1/\beta_0)|\delta|}\}$, we have that

$$w_2(z) = 1 + \|z\|^4 \leq \begin{cases} \frac{16}{\beta_0^2}|m_A(z)|^2 & \text{if } \|z\| \geq R \\ 1 + R^4 & \text{if } \|x\| < R. \end{cases}$$

Using this estimate we can conclude that

$$\begin{aligned}\int_{\mathbb{R}^n} |(\mathcal{F}f)(z)|^2 w_2(z)dz &= \int_{\|z\|\leq R} |(\mathcal{F}f)(z)|^2 w_2(z)dz + \int_{\|z\|\geq R} |(\mathcal{F}f)(z)|^2 w_2(z)dz \\ &= (1+R^4)\int_{\|z\|\leq R} |(\mathcal{F}f)(z)|^2 dz + \frac{16}{\beta_0^2}\int_{\|z\|\geq R} |(\mathcal{F}f)(z)|^2 |m_A(z)|^2 dz \\ &\leq (1+R^4)\int_{\mathbb{R}^n} |(\mathcal{F}f)(z)|^2 dz + \frac{16}{\beta_0^2}\int_{\mathbb{R}^n} |(\mathcal{F}f)(z)|^2 |m_A(z)|^2 dz \\ &< \infty.\end{aligned}$$

Thus $\mathcal{F} f \in L^2_{w_2}(\mathbb{R}^n)$, and $f \in H^2(\mathbb{R}^n) = \mathcal{D}(A)$ by Theorem 4.1.4.

□

Exercise 4.2.2. Consider the operator $A : \mathcal{D}(A) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ in (4.4).

(a) Complete the proof of Lemma 4.2.2. **Hint:** *The identity* (4.6) *follows fairly directly from Exercise 4.2.1, but you need to complete the proof of the representation of the domain* $\mathcal{D}(A)$ *(by proving the second inclusion). This part involves an estimate of* $|m_A(z)|$ *from above by* $w_2(z)$ *(times a constant), and this estimate is much easier than the one presented above.*

(b) Explain what can go wrong if $\beta \in \mathbb{R}^{n\times n}$ is not positive definite or non-singular, for example if $n = 2$ and $\beta = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$.

◇

Exercise 4.2.3 (Have some fun!)**.** Let $f \in H^8(\mathbb{R}^3)$. Find a way to define the derivative $D^\alpha f$, where $\alpha = (\pi, e^1, \sqrt{2})$. Check that $D^\alpha f \in L^2(\mathbb{R}^3)$. ◇

The application of the Fourier transform can be used to turn a partial differential equation on $\mathbb{R}^n$ to an *algebraic equation* involving the Fourier transforms of the solution and the known functions. Indeed, applying the Fourier transform on both sides of $Af = g$ and using Lemma 4.2.2 implies that for all $z \in \mathbb{R}^n$

$$[\mathcal{F}(Af)](z) = (\mathcal{F}g)(z) \qquad \Leftrightarrow \qquad m_A(z)(\mathcal{F}f)(z) = (\mathcal{F}g)(z).$$

If the parameters $\beta \in \mathbb{R}^{n\times n}$, $\gamma \in \mathbb{R}^n$ and $\delta \in \mathbb{R}$ are such that $m_A(z) = z^T\beta z + i\gamma^T z + \delta$ satisfies $m_A(z) \neq 0$ for all $z \in \mathbb{R}^n$ then this algebraic equation has a solution

$$(\mathcal{F}f)(z) = \frac{(\mathcal{F}g)(z)}{m_A(z)}, \qquad z \in \mathbb{R}^n.$$

However, the real challenge is to compute the solution f of the original differential equation. Finding f is theoretically possible using the inverse Fourier transform, but it is very typical that the resulting integrals cannot be computed analytically.

Exercise 4.2.4. Assume $\beta \in \mathbb{R}^n$ is positive definite, $\gamma \in \mathbb{R}^n$, and $\delta \in \mathbb{R}$, are such that $m_A(z) = z^T\beta z + i\gamma^T z + \delta$ satisfies $m_A(z) \neq 0$ for all $z \in \mathbb{R}^n$. Let $Af = -\nabla\cdot(\beta\nabla f) + \gamma^T\nabla f + \delta f$ with domain $\mathcal{D}(A) = H^2(\mathbb{R}^n)$. For $g \in L^2(\mathbb{R}^n)$, consider the abstract linear equation $Af = g$.

(a) Show that if $Af = g$ has a solution $f \in \mathcal{D}(A)$, then the solution is unique. **Hint:** *Consider two solutions.*

(b) Show that the equation $Af = g$ has a solution $f \in \mathcal{D}(A)$ if and only if $g \in L^2(\mathbb{R}^n)$ is such that

$$z \mapsto \frac{(\mathcal{F}g)(z)}{m_A(z)} \in L^2(\mathbb{R}^n).$$

Show that for such $g \in L^2(\mathbb{R}^n)$ the solution f of $Af = g$ has the form $f = Tg$ where T is a Fourier multiplier. **Hint:** *From the expression $\mathcal{F}f = m_A^{-1}\mathcal{F}g$ and Theorem* 4.1.4 *you immediately get an "easy" condition that $f \in \mathcal{D}(A)$ if and only if $m_A^{-1}\mathcal{F}g \in L^2_{w_2}(\mathbb{R}^n)$. To show that this is further equivalent to the condition in the claim, you can use the estimate between $w_2(z)$ and $m_A(z)$ and the estimates for the integrals in the proof of Lemma* 4.2.2.

(c) Show that if there exists $\varepsilon > 0$ such that $|m_A(z)| \geq \varepsilon > 0$ for all $z \in \mathbb{R}^n$, then $Af = g$ has a unique solution for any $g \in L^2(\mathbb{R}^n)$ and $\|f\|_{L^2} \leq \frac{1}{\varepsilon}\|g\|_{L^2}$. **Hint:** *In the first part you can simply verify that the condition of* (b) *holds for any $g \in L^2(\mathbb{R}^n)$, and the second part follows from the fundamental properties of Fourier transforms.*

◇

As part (c) of the above exercise you also proved that a Fourier multiplier is a bounded operator if its symbol is a bounded function in the sense that $\sup_{z\in\mathbb{R}^n}|m_T(z)| < \infty$, and in this case $\|Tf\|_{L^2} \leq \|m_T\|_\infty \|f\|_{L^2}$ for all $f \in \mathcal{D}(T)$. The boundedness of m_T is also a necessary condition for T to be bounded.

The measurable function m_T defines a *multiplication operator* (see Example 1.3.8) $M_T : \mathcal{D}(M_T) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ such that for all $g \in \mathcal{D}(M_T) := \{ g \in L^2(\mathbb{R}^n) \mid m_T g \in L^2(\mathbb{R}^n) \}$ we have

$$(M_T g)(z) = m_T(z)g(z), \qquad \text{for almost every } z \in \mathbb{R}^n.$$

It follows quite directly from Definition 4.2.1 that the Fourier multiplier $T : \mathcal{D}(T) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is in fact *boundedly similar* to the multiplication operator M_T defined by the symbol m_T in the sense that there exists $S \in \mathcal{L}(L^2(\mathbb{R}^n))$ with $S^{-1} \in \mathcal{L}(L^2(\mathbb{R}^n))$ such that $\mathcal{D}(M_T) = S(\mathcal{D}(T)) = \{ Sf \mid f \in \mathcal{D}(T) \}$ and $Tf = S^{-1}M_T Sf$ for all $f \in \mathcal{D}(T)$. In fact, it follows from (4.6) that $S = \mathcal{F} \in \mathcal{L}(L^2(\mathbb{R}^n))$. By Lemma 4.2.2 this is in particular true for the differential operator A with a positive definite $\beta \in \mathbb{R}^{n\times n}$. The significance of the similarity transformation comes from the fact that the spectral properties of a Fourier multiplier T are completely determined by the multiplication operator M_T, as shown in the following lemma.

Lemma 4.2.3. *Let X and Y be Banach spaces. If $A : \mathcal{D}(A) \subset X \to X$ and $B : \mathcal{D}(B) \subset Y \to Y$ are boundedly similar so that there exists $S \in \mathcal{L}(X, Y)$ with $S^{-1} \in \mathcal{L}(Y, X)$ such that $\mathcal{D}(B) = S(\mathcal{D}(A)) = \{ Sf \mid f \in \mathcal{D}(A) \}$ and $Ax = S^{-1}BSx$ for all $x \in \mathcal{D}(A)$, then the following hold.*

(a) $\sigma_p(A) = \sigma_p(B)$.

(b) $\sigma_c(A) = \sigma_c(B)$.

(c) $\sigma_r(A) = \sigma_r(B)$.

(d) $\rho(A) = \rho(B)$.

Proof. Part (a): $\lambda \in \mathbb{C}$ satisfies $\lambda \in \sigma(A)$ if and only if

$$\begin{aligned}
&& \exists x \in \mathcal{D}(A), x \neq 0 : && (\lambda - A)x = 0 \\
\iff && \exists x \in \mathcal{D}(A), x \neq 0 : && S^{-1}(\lambda - B)Sx = 0 \\
\overset{y\leftrightarrow Sx}{\iff} && \exists y \in \mathcal{D}(B), y \neq 0 : && (\lambda - B)y = 0,
\end{aligned}$$

which is equivalent to $\lambda \in \sigma_p(B)$.

Parts (b)–(d): For any $\lambda \in \mathbb{C}$ we have that

$$\begin{aligned}
\mathcal{R}(\lambda - A) &= \{ z \in X \mid \exists x \in \mathcal{D}(A) : \ z = (\lambda - A)x \} \\
&= \{ z \in X \mid \exists x \in \mathcal{D}(A) : \ z = S^{-1}(\lambda - B)Sx \} \\
&= \{ S^{-1}\tilde{z} \in X \mid \exists y \in \mathcal{D}(B) : \ \tilde{z} = (\lambda - B)y \} \\
&= S^{-1}(R(\lambda - B)).
\end{aligned}$$

Since S^{-1} is boundedly invertible, for any $\lambda \in \mathbb{C} \setminus \sigma_p(A)$ we have

- $\mathcal{R}(\lambda - A) = X$ if and only if $\mathcal{R}(\lambda - B) = Y$ (i.e., $\rho(A) = \rho(B)$).
- $\overline{\mathcal{R}(\lambda - A)} \neq X$ if and only if $\overline{\mathcal{R}(\lambda - B)} \neq Y$ (i.e., $\sigma_r(A) = \sigma_r(B)$).

Finally, part (c) follows from parts (a), (b), and (d). □

The spectra of multiplication operators on $\mathbb{R}^n$ are understood very well. The following theorem summarises these properties. The proof of this result is presented in full, since at this point our main interest is in *using* the properties of multiplication operators to analyse the differential operator A.

Theorem 4.2.4. *Let $M_h : \mathcal{D}(M) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ be a multiplication operator such that $(M_h f)(\xi) = h(\xi)f(\xi)$ for almost all $\xi \in \mathbb{R}^n$ where $h \in C(\mathbb{R}^n; \mathbb{C})$. Then $\sigma_r(M_h) = \varnothing$, and*

$$\sigma(M_h) = \overline{h(\mathbb{R}^n)} := \overline{\{ \mu \in \mathbb{C} \mid \exists \xi \in \mathbb{R}^n : \mu = h(\xi) \}}.$$

Moreover, for any $\lambda \in \sigma(M_h) = \overline{h(\mathbb{R}^n)}$ we have

$$\lambda \in \sigma_p(M_h) \quad \text{if and only if} \quad \{ \xi \in \mathbb{R}^n \mid \lambda = h(\xi) \} \subset \mathbb{R}^n \quad \text{has nonzero measure.}$$

The statement of Theorem 4.2.4 shows that $\lambda \in \mathbb{C}$ is in the continuous spectrum of M_h if either $\lambda \in h(\mathbb{R}^n)$ but the set $\{ \xi \in \mathbb{R}^n \mid \lambda = h(\xi) \}$has measure zero in $\mathbb{R}^n$, or $\lambda \notin h(\mathbb{R}^n)$ but it is an accumulation point of $h(\mathbb{R}^n) \subset \mathbb{C}$.

Proof of Theorem 4.2.4. We will first show that $\sigma(M_h) \subset \overline{h(\mathbb{R}^n)}$. Let $\lambda \notin \overline{h(\mathbb{R}^n)}$ be arbitrary. Since $\mathbb{C} \setminus \overline{h(\mathbb{R}^n)}$ is an open set, there exists $\varepsilon > 0$ such that $|\lambda - h(\xi)| \geq \varepsilon > 0$ for all $\xi \in \mathbb{R}^n$. Thus if $f \in \mathcal{D}(M_h)$ and if $(\lambda - M_h)f = 0$, then $(\lambda - h(\xi))f(\xi) = (\lambda f - M_h f)(\xi) = 0$ for almost all $\xi \in \mathbb{R}^n$, and necessarily $f = 0$ (in the sense of L^2-functions). Thus $\lambda - M_h$ is injective. Moreover, if $g \in L^2(\mathbb{R}^n)$ and we can define a measurable function $f : \mathbb{R}^n \to \mathbb{C}$ such that $f(\xi) = g(\xi)/(\lambda - h(\xi))$ for almost every $\xi \in \mathbb{R}^n$. Our aim is to show that $f \in \mathcal{D}(M_h)$ and $(\lambda - M_h)f = g$. We have

$$\int_{\mathbb{R}^n} |f(\xi)|^2 d\xi = \int_{\mathbb{R}^n} \frac{|g(\xi)|^2}{|\lambda - h(\xi)|^2} d\xi \leq \frac{1}{\varepsilon^2} \int_{\mathbb{R}^n} |g(\xi)|^2 d\xi = \frac{1}{\varepsilon^2} \|g\|_{L^2}^2 < \infty$$

and

$$\begin{aligned}
\int_{\mathbb{R}^n} |h(\xi)f(\xi)|^2 d\xi &= \int_{\mathbb{R}^n} \left| \frac{h(\xi)g(\xi)}{\lambda - h(\xi)} \right|^2 d\xi = \int_{\mathbb{R}^n} \left| \frac{(\lambda - h(\xi))g(\xi) - \lambda g(\xi)}{\lambda - h(\xi)} \right|^2 d\xi \\
&= \int_{\mathbb{R}^n} \left| g(\xi) - \lambda \frac{g(\xi)}{\lambda - h(\xi)} \right|^2 d\xi \leq 2 \int_{\mathbb{R}^n} \left(|g(\xi)|^2 + |\lambda|^2 \frac{|g(\xi)|^2}{|\lambda - h(\xi)|^2} \right) d\xi \\
&\leq 2(1 + \frac{|\lambda|^2}{\varepsilon^2}) \|g\|_{L^2}^2 < \infty.
\end{aligned}$$

Thus $f \in L^2(\mathbb{R}^n)$ and $hf \in L^2(\mathbb{R}^n)$, and therefore $f \in \mathcal{D}(M_h)$ by definition. Moreover,

$$((\lambda - M_h)f)(\xi) = (\lambda - h(\xi)) \frac{g(\xi)}{\lambda - h(\xi)} = g(\xi), \qquad \text{for almost all } \xi \in \mathbb{R}^n$$

shows that $(\lambda - M_h)f = g$ (equality in $L^2(\mathbb{R}^n)$). Thus $\lambda - M_h$ is surjective, and the above estimates also show that $\|f\|_{L^2} \leq \frac{1}{\varepsilon}\|g\|_{L^2}$, which shows that $(\lambda - M_h)^{-1} \in \mathcal{L}(L^2(\mathbb{R}^n))$. Since $\lambda \notin \overline{h(\mathbb{R}^n)}$ was arbitrary, we have that $\sigma(M_h) \subset \overline{h(\mathbb{R}^n)}$.

To describe the point spectrum $\sigma_p(M_h)$ we note that for an arbitrary $\lambda \in \mathbb{C}$ and $f \in \mathcal{D}(M_h)$ we have

$$(\lambda - M_h)f = 0 \qquad \Leftrightarrow \qquad (\lambda - h(\xi))f(\xi) = 0 \quad \text{a.e.} \quad \xi \in \mathbb{R}^n.$$

If the set $\{ \xi \in \mathbb{R}^n \mid \lambda = h(\xi) \} \subset \mathbb{R}^n$ is either empty or has zero measure in $\mathbb{R}^n$, this condition implies that $f(\xi) = 0$ almost everywhere on $\mathbb{R}^n$ and thus $f = 0$. In these cases we have $\lambda \notin \sigma_p(M_h)$. On the other hand, if $\{ \xi \in \mathbb{R}^n \mid \lambda = h(\xi) \} \subset \mathbb{R}^n$ has nonzero measure in $\mathbb{R}^n$, we can choose a nonzero function $f \in \mathcal{D}(M_h)$ which has nonzero values $f(\xi) \neq 0$ only inside $\{ \xi \in \mathbb{R}^n \mid \lambda = h(\xi) \} \subset \mathbb{R}^n$. In which case we have $(\lambda - M_h)f = 0$ and thus $\lambda \in \sigma_p(M_h)$. This concludes that $\lambda \in \sigma_p(M_h)$ if and only if the set $\{ \xi \in \mathbb{R}^n \mid \lambda = h(\xi) \} \subset \mathbb{R}^n$ has positive measure.

Assume now that $\{ \xi \in \mathbb{R}^n \mid \lambda = h(\xi) \} \subset \mathbb{R}^n$ is either empty or has zero measure. Then as shown above, $\lambda - M_h$ is injective. We will first show that $\mathcal{R}(\lambda - M_h)$ is dense in $L^2(\mathbb{R}^n)$, which will show that $\sigma_r(M_h) = \varnothing$. The property $\overline{\mathcal{R}(\lambda - M_h)} = L^2(\mathbb{R}^n)$ is equivalent to $\mathcal{R}(\lambda - M_h)^\perp = \{0\}$. If $g \in \mathcal{R}(\lambda - M_h)^\perp$ is arbitrary, then

$$
\begin{aligned}
& \langle (\lambda - M_h)f, g \rangle_{L^2} = 0 \qquad \forall f \in \mathcal{D}(M_h) \\
\Rightarrow \quad & \int_{\mathbb{R}^n} (\lambda - h(\xi)) f(\xi) \overline{g(\xi)} d\xi = 0 \qquad \forall f \in \mathcal{D}(M_h) \\
\Rightarrow \quad & \int_{\mathbb{R}^n} f(\xi) \overline{\overline{(\lambda - h(\xi))} g(\xi)} d\xi = 0 \qquad \forall f \in \mathcal{D}(M_h) \\
\Rightarrow \quad & \overline{(\lambda - h(\xi))} g(\xi) = 0 \qquad \text{a.e.} \quad \xi \in \mathbb{R}^n
\end{aligned}
$$

(here we have used the fact that $\mathcal{D}(M_h)$ is dense in $L^2(\mathbb{R}^n)$ in the last implication). But by assumption we had that $\lambda - h(\xi) \neq 0$ for almost every $\xi \in \mathbb{R}^n$, and thus necessarily $g = 0$ (as an L^2-function). Since $g \in \mathcal{R}(\lambda - M_h)^\perp$ was arbitrary, we have that $\mathcal{R}(\lambda - M_h)$ is dense in $L^2(\mathbb{R}^n)$.

In the final part we will show that $h(\mathbb{R}^n) \subset \sigma(M_h)$, which due to the closedness of the spectrum also imply that $\overline{h(\mathbb{R}^n)} \subset \sigma(M_h)$. To this end, let $\lambda \in h(\mathbb{R}^n)$. Our aim is to construct a sequence $(f_k)_{k\in\mathbb{N}} \in \mathcal{D}(M_h)$ such that $\|f_k\|_{L^2} = 1$ for all $k \in \mathbb{N}$ and

$$
\|(\lambda - M_h)f_k\| \to 0, \qquad \text{as} \quad k \to \infty.
$$

This will in particular show that the operator $\lambda - M_h$ cannot have a bounded inverse, and thus necessarily $\lambda \in \sigma(M_h)$. Since we assumed $\lambda \in h(\mathbb{R}^n)$, there exists $\xi_0 \in \mathbb{R}^n$ such that $\lambda = h(\xi_0)$. We will choose f_k to be functions which are only nonzero very close to ξ_0 (where the continuous function $\xi \mapsto \lambda - h(\xi)$ has small values). Since $h(\cdot)$ is continuous, for any $k \in \mathbb{N}$ we can choose a radius r_k such that $|\lambda - h(\xi)| \leq 1/k$ for all $\xi \in B(\xi_0, r_k) := \{ \xi \in \mathbb{R}^n \mid |\xi - \xi_0| < r_k \}$. We can define piecewise constant functions $f_k \in L^2(\mathbb{R}^n)$ by

$$
f_k(\xi) = \begin{cases} c_k & \xi \in B(\xi_0, r_k) \\ 0 & \text{otherwise,} \end{cases}
$$

where the constant $c_k > 0$ is given by $c_k = \left(\int_{|\xi| \leq r_k} 1 d\xi \right)^{-1/2}$. We then have $\|f_k\|_{L^2} = 1$ as required, and

$$
\|(\lambda - M_h)f_k\|_{L^2}^2 = \int_{\mathbb{R}^n} |\lambda - h(\xi)|^2 |f_k(\xi)|^2 d\xi \leq \frac{1}{k^2} \int_{|\xi| \leq r_k} |f_k(\xi)|^2 d\xi = \frac{c_k^2}{k^2} \int_{|\xi| \leq r_k} 1 d\xi = \frac{1}{k^2} \to 0
$$

as $k \to \infty$. Thus $\lambda - M_h$ does not have a bounded inverse, and consequently $\lambda \in \sigma(M_h)$. □

The next exercise combines the results presented in this section to describe the spectrum of the differential operator A in terms of the parameters $\beta \in \mathbb{R}^n$, $\gamma \in \mathbb{R}^n$, and $\delta \in \mathbb{R}$.

Exercise 4.2.5. Assume $\beta \in \mathbb{R}^{n\times n}$ is positive definite, $\gamma \in \mathbb{R}^n$, and $\delta \in \mathbb{R}$, and consider the operator $Af = -\nabla \cdot (\beta \nabla f) + \gamma^T \nabla f + \delta f$ with domain $\mathcal{D}(A) = H^2(\mathbb{R}^n)$.

(a) Characterise the spectrum of A in terms of the function m_A.

(b) Provide a complete description of the spectrum of A in the case $n = 1$. In particular, you should answer the following questions:

- What type of spectrum does A have? (point spectrum, continuous spectrum, residual spectrum, no spectrum at all?)
- What is the shape of the spectrum $\sigma(A)$ in the complex plane, and how does this shape depend on the parameters β, γ, and δ?

Hint: *Your first task is to combine the above theorems to show that you can characterise the spectrum of A in terms of the function m_A. Note that the equation $\lambda = h(\xi)$ is a quadratic polynomial, and this will allow you to analyse the set $\{\, \xi \in \mathbb{R}^n \mid \lambda = h(\xi) \,\}$ in Theorem* 4.2.4. *Finally, in describing the shape of the spectrum $\sigma(A)$, it is useful to write $\lambda = a + ib$ with $a, b \in \mathbb{R}$ in the equation $\lambda = h(\xi)$.* ◇

The answer to the above exercise shows that in this section the spectrum of the also the elliptic operator $Af = -\nabla \cdot (\beta \nabla f) + \delta f$ is very different from the case in Section 3.2, where we saw that if Ω is open, bounded and has smooth boundary, then its spectrum *only* contains eigenvalues! The reason for this difference is that in this section we consider the differential operators on the full space $\Omega = \mathbb{R}^n$, which is an unbounded domain.

4.3 Embedding Theorems for Sobolev Spaces

In this final section of the chapter we use the Fourier transform to prove some results on embeddings of Sobolev spaces. In order to study functions in the Sobolev spaces $H^k(\Omega)$ with a domain $\Omega \subset \mathbb{R}^n$, we first need to know how these the function f can be *extended* to a function on the full space $\mathbb{R}^n$ (since the Fourier transform is only defined on functions on $\mathbb{R}^n$). Any function $f : \Omega \subset \mathbb{R}^n \to \mathbb{C}$ can be trivially extended to the full space $\mathbb{R}^n$ by defining $f(\xi) = 0$ for all $\xi \in \mathbb{R}^n \setminus \Omega$, but this will in general destroy weak differentiability of the function near the boundary $\partial\Omega$ of Ω. In the study of functions in Sobolev spaces we would especially like the extensions to be done in such a way that the extended functions have the same differentiability properties as the original ones. More precisely, we require that the extension of f belongs to $H^k(\mathbb{R}^n)$ if $f \in H^k(\Omega)$.

Definition 4.3.1. Let $k \in \mathbb{N}$. A domain $\Omega \subset \mathbb{R}^n$ is said to have the *k-extension property* if there exists

$$E \in \mathcal{L}(H^k(\Omega), H^k(\mathbb{R}^n))$$

such that for every $f \in H^k(\Omega)$ we have $(Ef)(\xi) = f(\xi)$ for almost all $\xi \in \Omega$.

The behaviour of the extension of the function near the boundary $\partial\Omega$ plays a key role in preserving the differentiability, and because of this the existence of the extension operator E

in Definition 4.3.1 requires some assumptions on $\partial\Omega$. On this course we focus on bounded domains $\Omega \subset \mathbb{R}^n$ with bounded and smooth boundary, and such Ω possess the k-extension property for every $k \in \mathbb{N}$. Similar results do also hold under much weaker conditions on the boundary $\partial\Omega$, but some assumptions are always necessary.

Theorem 4.3.2. *Let $k \in \mathbb{N}$. If $\Omega \subset \mathbb{R}^n$ is bounded and has smooth and bounded boundary $\partial\Omega$, then Ω has the k-extension property. The extension operator $E \in \mathcal{L}(H^k(\Omega), H^k(\mathbb{R}^n))$ can be chosen in such a way that $\operatorname{supp} Ef \subset V$ for some compact set $V \subset \mathbb{R}^n$ and for all $f \in H^k(\Omega)$.*

Proof. See Theorem 6.88 in Renardy & Rogers, 1993. □

With the aid of the extension property, we can use the Fourier transform to analyse properties of functions defined on domains $\Omega \subset \mathbb{R}^n$. Our first result shows that for orders $k > n/2$ the functions $f \in H^k(\Omega)$ are continuous and bounded functions.

Theorem 4.3.3 (Sobolev Embedding Theorem)**.** *Let $n \in \mathbb{N}$ and $k > n/2$. The space $H^k(\mathbb{R}^n)$ is continuously embedded in*

$$C_b(\mathbb{R}^n) := \{ f \in C(\mathbb{R}^n) \mid \exists M > 0 : |f(\xi)| \le M \ \forall \xi \in \mathbb{R}^n \}.$$

Moreover, if $\Omega \subset \mathbb{R}^n$ has the k-extension property, then $H^k(\Omega)$ is continuously embedded in $C_b(\overline{\Omega})$.

The norm on $C_b(\mathbb{R}^n)$ is $\|f\|_\infty = \sup_{\xi\in\mathbb{R}^n}|f(\xi)|$, and $C_b(\mathbb{R}^n)$ is a Banach space.

Exercise 4.3.1. Prove Theorem 4.3.3.

Hint: *Begin with the case $H^k(\mathbb{R}^n)$. A completely analogous argument as in Exercise* 4.1.1 *shows that if $\mathcal{F}f \in L^1(\mathbb{R}^n)$, then f is continuous and $\|f\|_\infty \le (2\pi)^{-n/2}\|\mathcal{F}f\|_{L^1}$ (you do not need to prove this again). To complete the estimate required for the embedding you can write $\mathcal{F}f = w_k^{1/2}\mathcal{F}f \cdot w_k^{-1/2}$ with $w_k(z) = 1 + \|z\|^{2k}$ and use the Hölder inequality for the norm $\|\mathcal{F}f\|_{L^1}$. You can directly (= without proving it) use the knowledge that*

$$\int_{\mathbb{R}^n} \frac{dz}{1+\|z\|^q} < \infty, \qquad \textit{if} \quad q > n.$$

The second claim can be proved using the extension operator $E \in \mathcal{L}(H^k(\Omega), H^k(\mathbb{R}^n))$ and the first part. ◇

In particular Theorem 4.3.3 tells us that if $k > n/2$, then the functions in $H^k(\mathbb{R}^n)$ are continuous and bounded. This condition on the order of the Sobolev space is in fact necessary, and especially for $k = n/2$ the functions $f \in H^k(\mathbb{R}^n)$ do not need to be continuous (see Problem 6.48 of Renardy & Rogers, 1993).

We complete this section by proving a version of the Rellich–Kondrachov Theorem.

Theorem 4.3.4. *Assume $\Omega \subset \mathbb{R}^n$ is bounded and has bounded and smooth boundary $\partial\Omega$. Then $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$.*

A more general version of the theorem shows that $H^k(\Omega)$ is compactly embedded in $H^{k-1}(\Omega)$ for every $k \in \mathbb{N}$. Moreover, the assumptions on the domain Ω can also be weakened. The proof again utilises the Fourier transform through extension of the functions

f defined on Ω to functions Ef defined on the full space. We divide the proof of Theorem 4.3.4 into parts, some of which are completed as exercise problems. In the first exercise we study a way of "smoothing out" a function by truncating its Fourier transform using a suitable Fourier multiplier.

Exercise 4.3.2. For $R > 0$ define a Fourier multiplier $T_R \in \mathcal{L}(L^2(\mathbb{R}^n))$ with the symbol $\chi_{B(0,R)} : \mathbb{R}^n \to [0,1]$ defined by

$$\chi_{B(0,R)}(z) = \begin{cases} 1 & \|z\| \leq R \\ 0 & \|z\| > R \end{cases}$$

(the characteristic function of the ball $B(0,R) \subset \mathbb{R}^n$). Show that the operator T_R has the following properties:

(a) $T_R f \in C^\infty(\mathbb{R}^n)$ for every $f \in L^2(\mathbb{R}^n)$ (that is, the operator T_R makes the function f infinitely smooth).

Hint: *Note first that by definition the function $\mathcal{F}(T_R f)$ has compact support (it is zero outside a bounded set). You can then use Theorems* 4.1.4 *and* 4.3.3 *to show that all derivatives of $T_R f$ are continuous.*

(b) Show that for every $R > 0$ and $f \in H^1(\mathbb{R}^n)$ we have

$$\|T_R f - f\|_{L^2} \leq \frac{1}{R}\|f\|_{H^1}.$$

(This means that for a large $R > 0$ the function $T_R f$ can be used to approximate the function $f \in H^1(\mathbb{R}^n)$ very accurately in the L^2-norm.

Hint: *Write the norm on the left-hand side in terms of the Fourier transforms $\mathcal{F}(T_R f)$ and $\mathcal{F}f$ and recall that $\|f\|_{H^1} = \|\mathcal{F}f\|_{L^2_{w_1}} = \|w_1^{1/2}\mathcal{F}f\|_{L^2}$, where $w_1(z) = 1 + \|z\|^2$. In your estimates you can write $1 = w_1(z)^{1/2}w_1(z)^{-1/2}$.*

(c) Show that for every $R > 0$ there exist constants $M_R^1, M_R^2 > 0$ (which may depend on $R > 0$) such that for all $f \in H^1(\mathbb{R}^n)$ we have

$$\|T_R f\|_\infty \leq M_R^1\|f\|_{H^1} \qquad \text{and} \qquad \|\nabla T_R f\|_\infty \leq M_R^2\|f\|_{H^1}.$$

Hint: *If $g \in L^2(\mathbb{R}^n)$ is such that $\mathcal{F}g \in L^1(\mathbb{R}^n)$, then an analogous argument as in Exercise* 4.1.1 *shows that $\|g\|_\infty \leq (2\pi)^{-n/2}\|\mathcal{F}g\|_{L^1}$. You can easily show that $\mathcal{F}(T_R f) \in L^1(\mathbb{R}^n)$ and $\mathcal{F}(\nabla T_R f) \in L^1(\mathbb{R}^n)$ (see Exercise* 4.2.1*). In the estimating the L^1-norms you can again write $1 = w_1^{1/2}(z)w_1^{-1/2}(z)$. You do not need to find explicit values for $M_R^1, M_R^2 > 0$.*

◇

In the proof of Theorem 4.3.4 we will also use the following lemma on equicontinuity of sequences of continuously differentiable functions. Note that the conditions of the lemma are more restrictive compared to those in the case $n = 1$, which we studied in the Homework problems for Week 5. Indeed, in the homework problems the sequence $(g_k)_{k\in\mathbb{N}} \subset H^1(a,b)$ was shown to be equicontinuous whenever $\sup_{k\in\mathbb{N}}\|g_k\|_{H^1} < \infty$.

Lemma 4.3.5. *Let $\Omega \subset \mathbb{R}^n$ be an open and convex set. If $(g_k)_{k\in\mathbb{N}} \subset C^1(\overline{\Omega})$ is such that $\sup_{k\in\mathbb{N}}\|\nabla g_k\|_\infty < \infty$, then $(g_k)_{k\in\mathbb{N}}$ is equicontinuous.*

Proof. Let $(g_k)_{k\in\mathbb{N}} \subset C^1(\overline{\Omega})$ be such that $\sup_{k\in\mathbb{N}}\|\nabla g_k\|_\infty \leq M < \infty$ for some constant $M > 0$. Let $k \in \mathbb{N}$ and $\xi_0, \xi_1 \in \overline{\Omega}$ be arbitrary. Since $\overline{\Omega}$ is convex, the line segment between $\xi_0 \in \mathbb{R}^n$ and $\xi_1 \in \mathbb{R}^n$ belongs to $\overline{\Omega}$

Define a function $h_k : [0,1] \to \mathbb{C}$ by $h_k(t) = g_k(\xi_0 + t(\xi_1 - \xi_0))$. Due to the chain rule of differentiation[3] we have that

$$h_k'(t) = (\xi_1 - \xi_0)^T \nabla g_k(\xi_0 + t(\xi_1 - \xi_0))$$

and thus $h \in C^1([0,1])$. The fundamental theorem of calculus implies that

$$\begin{aligned} |g_k(\xi_0) - g_k(\xi_1)| = |h_k(0) - h_k(1)| = |\int_0^1 h_k'(t)dt| &\leq \int_0^1 |(\xi_1 - \xi_0)^T \nabla g_k(\xi_0 + t(\xi_1 - \xi_0))|dt \\ &\leq \int_0^1 \|\xi_1 - \xi_0\|\|\nabla g_k(\xi_0 + t(\xi_1 - \xi_0))\|dt \leq \|\xi_1 - \xi_0\|\|\nabla g_k\|_\infty \int_0^1 1dt \\ &\leq M\|\xi_1 - \xi_0\|. \end{aligned}$$

Thus if $\varepsilon > 0$, choosing $\delta = \varepsilon/M > 0$ we have that $|g_k(\xi_0) - g_k(\xi_1)| < \varepsilon$ whenever $\|\xi_0 - \xi_1\| < \delta$. Since $\delta > 0$ can be chosen to be independent of $k \in \mathbb{N}$, the sequence $(g_k)_{k\in\mathbb{N}}$ is equicontinuous. □

Proof of Theorem 4.3.4. Our aim is to show that the identity map $J : H^1(\Omega) \to L^2(\Omega)$ is compact. To this end, let $(f_k)_{k\in\mathbb{N}} \subset H^1(\Omega)$ be a bounded sequence, i.e., $\sup_{k\in\mathbb{N}}\|f_k\|_{H^1} \leq M < \infty$ for some constant $M > 0$. We will show that $(f_k)_{k\in\mathbb{N}}$ has a subsequence which converges in $L^2(\Omega)$. Since Ω has the 1-extension property, there exists an extension operator $E \in \mathcal{L}(H^1(\Omega), H^1(\mathbb{R}^n))$, and E can be chosen in such a way that $\operatorname{supp} f \subset V$ for some compact set $V \subset \mathbb{R}^n$ (obviously $\Omega \subset V$) and for all $f \in H^1(\Omega)$.

In the next stage of the proof we approximate $Ef_k \in H^1(\mathbb{R}^n)$ with "smoothed out" functions defined in Exercise 4.3.2. To this end, for $R > 0$ we define the Fourier multiplier $T_R \in \mathcal{L}(L^2(\mathbb{R}^n))$ with symbol $\chi_{B(0,R)}(z)$, and consider the sequence $(T_R E f_k)_{k\in\mathbb{N}}$. We have from Exercise 4.3.2 that $(T_R E f_k)_{k\in\mathbb{N}} \subset C^\infty(\mathbb{R}^n)$ (by part (a)),

$$\|Ef_k - T_R E f_k\|_{L^2} \leq \frac{1}{R}\|Ef_k\|_{H^1} \leq \frac{1}{R}\|E\|\|f_k\|_{H^1} \leq \frac{M\|E\|}{R} \to 0 \qquad \text{as} \quad R \to \infty, \tag{4.7}$$

(by part (b)), and finally part (c) implies

$$\|T_R E f_k\|_\infty \leq M_R^1 \|Ef_k\|_{H^1} \leq M_R^1 M\|E\| < \infty,$$
$$\|\nabla T_R E f_k\|_\infty \leq M_R^2 \|Ef_k\|_{H^1} \leq M_R^2 M\|E\| < \infty.$$

Thus if we take any bounded, open and convex set $U \subset \mathbb{R}^n$ such that $V \subset U$ (for example a ball $B(0,r)$ with a sufficiently large radius $r > 0$), we can consider the restrictions of the

[3]The same formula also holds for weak derivatives, but unfortunately we do not have the time to prove this. Moreover, in this proof it would actually not be enough to assume $g_k \in H^1(\Omega)$ since we really need that the derivatives of g_k are continuous.

functions $T_R E f_k$ to the set $\overline{U}$. For a fixed $R > 0$ we have from above that $(T_R E f_k)_{k\in\mathbb{N}} \subset C^\infty(\overline{U})$, $\sup_{k\in\mathbb{N}}\|T_R E f_k\|_\infty < \infty$, and $\sup_{k\in\mathbb{N}}\|\nabla T_R E f_k\|_\infty < \infty$. Thus $(T_R E f_k)_{k\in\mathbb{N}} \subset C(\overline{U})$ is a bounded sequence, and by Lemma 4.3.5 it is also equicontinuous. The Arzela–Ascoli Theorem in Theorem 3.1.7 therefore implies that $(T_R E f_k)_{k\in\mathbb{N}}$ has a subsequence which converges in $C(\overline{U})$, i.e.,

$$\|T_R E f_{k_j} - g^R\|_\infty \to 0, \qquad \text{as} \quad j \to \infty$$

for some $g^R \in C(\overline{U})$. But since $\Omega \subset \overline{U}$ and since Ω is bounded, we have that also $g^R \in L^2(\Omega)$ and

$$\|T_R E f_{k_j} - g^R\|_{L^2}^2 = \int_\Omega |T_R E f_{k_j}(\xi) - g^R(\xi)|^2 d\xi \leq \|T_R E f_{k_j} - g^R\|_\infty^2 \int_\Omega 1 d\xi \to 0$$

as $j \to \infty$. Thus the subsequence $(T_R E f_{k_j})_{j\in\mathbb{N}}$ convergences also in $L^2(\Omega)$.

Because by (4.7) any sequence $(E f_{k_j})_{j\in\mathbb{N}}$ can be approximated with $(T_R E f_{k_j})_{j\in\mathbb{N}}$ where $R > 0$ is sufficiently large, the convergence of the subsequence $(T_R E f_{k_j})_{j\in\mathbb{N}}$ is already getting *close* to what we are trying to prove. However, the problem is that the choice of this subsequence depends on the value of $R > 0$. In order to complete the proof, we need to find a subsequence of $(T_R E f_k)_{k\in\mathbb{N}}$ which converges in $L^2(\Omega)$ for *all* values of $R > 0$. This can be achieved using a so-called "*diagonal argument*". We begin by choosing a sequence $(R_l)_{l\in\mathbb{N}} \subset \mathbb{R}$ such that $0 < R_1 < R_2 < \cdots$ such that $R_l \to \infty$ as $l \to \infty$ (one suitable concrete sequence would simply be $R_l = l$ for all $l \in \mathbb{N}$).

Step 1: The arguments above show that there exists a subsequence $(f_{k_{1j}})_{j\in\mathbb{N}}$ of $(f_k)_{k\in\mathbb{N}}$ such that $\|T_{R_1} E f_{k_{1j}} - g^1\|_{L^2} \to 0$ as $j \to \infty$ for some $g^1 \in L^2(\Omega)$.

Step 2: Since also $(T_{R_2} E f_{k_{1j}})_{j\in\mathbb{N}} \subset C(\overline{U})$ is bounded and equicontinuous (because it is a subsequence of $(T_{R_2} E f_k)_{k\in\mathbb{N}}$), the above arguments show that there exists a subsequence $(f_{k_{2j}})_{j\in\mathbb{N}}$ of $(f_{k_{1j}})_{j\in\mathbb{N}}$ such that $\|T_{R_2} E f_{k_{2j}} - g^2\|_{L^2} \to 0$ as $j \to \infty$ for some $g^2 \in L^2(\Omega)$.

Step 3: Since also $(T_{R_3} E f_{k_{2j}})_{j\in\mathbb{N}} \subset C(\overline{U})$ is bounded and equicontinuous (because it is a subsequence of $(T_{R_3} E f_k)_{k\in\mathbb{N}}$), there exists a subsequence $(f_{k_{3j}})_{j\in\mathbb{N}}$ of $(f_{k_{2j}})_{j\in\mathbb{N}}$ such that $\|T_{R_3} E f_{k_{3j}} - g^3\|_{L^2} \to 0$ as $j \to \infty$ for some $g^3 \in L^2(\Omega)$.

This process can be continued indefinitely to obtain an infinite number of sequences of the form $(f_{k_{lj}})_{j\in\mathbb{N}}$ such that $T_{R_l} E f_{k_{lj}} \to g^l$ in $L^2(\Omega)$ as $j \to \infty$ for every fixed $l \in \mathbb{N}$. We will now consider a "diagonal" sequence with indices $l = j$, i.e., the sequence $(f_{k_{jj}})_{j\in\mathbb{N}}$. Due to the above construction, this is a subsequence of $(f_k)_{k\in\mathbb{N}}$. Moreover, we can now show that $(T_{R_l} E f_{k_{jj}})_{j\in\mathbb{N}}$ converges for every $l \in \mathbb{N}$. To this end, let $l \in \mathbb{N}$ be arbitrary. By construction, for every $j \geq l$ the elements $f_{k_{jj}}$ are also members of the sequence $(f_{k_{lj}})_{j\in\mathbb{N}}$. But since $T_{R_l} E f_{k_{lj}} \to g^l$ in $L^2(\Omega)$ as $j \to \infty$, and since a subsequence of a convergent sequence converges to the same limit, we also have $T_{R_l} E f_{k_{jj}} \to g^l$ in $L^2(\Omega)$ as $j \to \infty$.

In the final part of the proof we will show that the estimate (4.7) together with the convergence of $(T_{R_l} E f_{k_{jj}})_{j\in\mathbb{N}}$ for every $l \in \mathbb{N}$ imply that the sequence $(f_{k_{jj}})_{j\in\mathbb{N}}$ converges in $L^2(\Omega)$. This is completed in the following exercise.

Exercise 4.3.3. Complete the proof of Theorem 4.3.4 by showing that $(f_{k_{jj}})_{j\in\mathbb{N}}$ converges in $L^2(\Omega)$.

Hint: *You can do this by showing that* $(f_{k_{jj}})_{j\in\mathbb{N}}$ *is a Cauchy sequence. In doing so, you need to show that by choosing* $l \in \mathbb{N}$ *to be sufficiently large, you can use* (4.7) *to approximate any* *element* $f_{k_{jj}}$ *with* $T_{R_l}Ef_{k_{jj}}$ *with arbitrary accuracy in the norm on* $L^2(\Omega)$. *Here you of course need the property that* $Ef_{k_{jj}}$ *and* $f_{k_{jj}}$ *agree on* Ω. *As shown above, the sequence* $(T_{R_l}Ef_{k_{jj}})_{j\in\mathbb{N}}$ *converges in* $L^2(\Omega)$ *for any fixed* $l \in \mathbb{N}$. ◇

□

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