Robust Output Regulation for Multi-Dimensional Heat Equation under Boundary Control ⋆

Ren-Xi Zhao [a], Bao-Zhu Guo [b,c,d], Lassi Paunonen [e]

[a] *School of Mathematics, Foshan University, Foshan 528000, China*

[b] *Key Laboratory of System and Control, Academy of Mathematics and Systems Science, Academia Sinica, Beijing 100190, China*

[c] *School of Mathematics and Physics, North China Electric Power University, Beijing 102206, China*

[d] *College of Mathematics, Taiyuan University of Technology, Taiyuan 030024, China*

[e] *Mathematics, Faculty of Information Technology and Communication Sciences, Tampere University, PO. Box 692, 33101 Tampere, Finland*

Abstract

Over the past few years, the internal model principle has been extended to some systems described by one-dimensional partial differential equations (PDEs) from the PDE perspective. However, robustness has remained limited to specific cases, primarily due to the challenges in formulating it within the PDE framework. In this paper, we explore output regulation for a multi-dimensional heat equation under boundary control, where the output space is infinite-dimensional. We not only derive an analytic tracking error feedback control but also demonstrate robustness. This is achieved by leveraging abstract results and PDE design techniques.

Key words: Output tracking, disturbance rejection, multi-dimensional heat equation.

1 Introduction

The output regulation is a pivotal concern in control systems. The well-known internal model principle, initially developed in Francis and Wonham (1976) and Davison (1976) for lumped parameter systems, has evolved to encompass infinite-dimensional systems (Rebarber and Weiss, 2003; Paunonen, 2016, 2017). By the internal model principle, a controller aiming for robust output regulation in a control system must incorporate copies of the exosystem that generates references and disturbances. The number of these copies should be equal to or exceed the dimensionality of the system's output. When applying the abstract Paunonen (2016, 2017) to specific PDEs, it necessitates the verification of numerous abstract conditions, which can be arduous. Conversely, employing a PDE approach for output regulation in certain partial differential equations (PDEs) can yield analytical forms of tracking error feedback control, akin to those achieved for 1-D parabolic systems (Deutscher, 2015; Guo and Meng, 2020), first-order hyperbolic systems (Deutscher, 2017a), wave systems (Feng, Guo, and Wu, 2020a), and Euler-Bernoulli beam systems (Guo and Meng, 2021b). However, the PDE approach encounters difficulties in addressing robustness, which is more straightforward to address through an abstract setup. This challenge arises as it is practically infeasible to formulate all potential system variations within the PDE framework.

In this paper, we examine the output regulation for a multi-dimensional heat equation controlled at its boundary. The uniqueness of this problem arises from the distributed nature of the output, making it inherently infinite-dimensional. According to the internal model principle, the controller must contain infinitely many copies of the exosystem. This aspect distinguishes

⋆ This work was supported partially by the National Natural Science Foundation of China under grant nos. 12131008 and U23B2033, and the Research Council of Finland under grant no. 349002. A special case of this paper was presented in 2023 IFAC held in Yokohama, Japan. Corresponding author: Bao-Zhu Guo.

Email addresses: `zhaorenxi@amss.ac.cn` (Ren-Xi Zhao), `bzguo@iss.ac.cn` (Bao-Zhu Guo), `lassi.paunonen@tuni.fi` (Lassi Paunonen).

our work significantly from previous research on output regulation for 1-D PDEs with finite-dimensional output spaces. By leveraging the abstract theory of the internal model principle, as elaborated in Paunonen (2017), and adopting a PDE-based methodology, we are able to derive not just the analytic expression for the feedback control but also attain robustness. Our exosystem can include nontrivial Jordan blocks. In the case of diagonal exosystems, robust regulation can also be achieved using the low-gain robust controller described in theorem 1.1 of Rebarber and Weiss (2003); see also Humaloja and Paunonen (2019). Additionally, the design of the low-gain robust controller has been extended to accommodate exosystems featuring Jordan blocks, but with a finite-dimensional output space (Hämäläinen and Pohjolainen, 2002).

We proceed as follows. In the next section, Section 2, we precisely define the output regulation problem for a multi-dimensional heat system. In Section 3, we design an error feedback regulator and demonstrate its efficacy in solving the robust output regulation challenge. Section 4 focuses on robustness, integrating our proposed error feedback regulator with abstract findings on the internal model principle. Finally, conclusions are presented in Section 5.

Notations. For a linear operator A from Hilbert space X to Y, we denote the domain, kernel, and range of A by $D(A), \mathcal{N}(A)$ and $\mathcal{R}(A)$ respectively. The space of all bounded linear operators from X to Y is represented by $\mathcal{L}(X,Y)$. If $X=Y$, then $\sigma(A)$ and $\rho(A)$ represent the spectrum and resolvent set of A respectively. The resolvent operator is defined as $R(\lambda,A)=(\lambda-A)^{-1}$. If $A:D(A)\subset X\to X$ generates a C_0-semigroup $T(t)$ on X, we define $X_1=D(A)$ equipped with the graph norm of A. Furthermore, we define X_{-1} as the completion of X with respect to the norm $\|x\|_{-1}:=\|(\lambda_0-A)^{-1}x\|_X$ for a fixed $\lambda_0\in\rho(A)$. Subsequently, A can be extended to be an operator from X to X_{-1} (also denoted by A).

2 Problem statement

The system that we consider is governed by the following multi-dimensional heat equation:

$$\begin{cases} w_t(x,t)=\Delta w(x,t)+F_1(x)^\top p(t),\ x\in\Omega,\ t>0,\\ w(x,t)|_{\Gamma_0}=F_2(x)^\top p(t),\ t\ge 0,\\ \dfrac{\partial w(x,t)}{\partial\nu}|_{\Gamma_1}=u(x,t)+F_3(x)^\top p(t),\ t\ge 0,\\ w(x,0)=w_0(x),\ x\in\Omega,\\ y_o(x,t)=w(x,t)|_{\Gamma_1},\ t\ge 0,\end{cases} \tag{1}$$

where $w_0(x)$ denotes the initial state; The domain $\Omega\subset\mathbb{R}^d (d\ge 2)$ is an open and bounded domain with a smooth C^2-boundary denoted by $\Gamma=\overline{\Gamma_0}\cup\overline{\Gamma_1}$. The open subsets Γ_0 and Γ_1 satisfy the conditions $\Gamma_0\neq\emptyset$, $\Gamma_1\neq\emptyset$, $\Gamma_0\cap\Gamma_1=\emptyset$; The symbol ν refers to the unit normal vector of Γ pointing towards the exterior of Ω; $u(x,t)$ is the control input, while $y_o(x,t)$ represents the performance output signal that needs regulation; $F_1(\cdot)\in L^2(\Omega;\mathbb{C}^n)$ and $F_2(\cdot)\in H^{1/2}(\Gamma_0;\mathbb{C}^n)$ and $F_3(\cdot)\in L^2(\Gamma_1;\mathbb{C}^n)$ are unknown disturbance coefficients affecting both the in-domain and the boundary.

The $p(t)$ represents an unknown external disturbance, which is presumed to satisfy the following differential equation:

$$\begin{cases}\dot p(t)=Gp(t),\\ p(0)=p_0\in\mathbb{C}^n,\end{cases} \tag{2}$$

where G is known, but p_0 remains unknown. We assume that

$$G=\operatorname{diag}\{i\omega_1,i\omega_2,\dots,i\omega_{n-q},G_q\},$$

$$G_q=\begin{bmatrix}0_{(q-1)\times 1} & I_{(q-1)\times(q-1)}\\ 0 & 0_{1\times(q-1)}\end{bmatrix}+i\omega_{n-q+1}I_{q\times q},$$

where the eigenvalues $\{i\omega_k\}_{k=1}^{n-q+1}\subset i\mathbb{R}$ are distinct and assumed to be known.

Remark 2.1 *The results presented in this paper can be straightforwardly extended to the scenario where $G=\operatorname{diag}\{J_1,J_2,\dots,J_n\}$, in which $J_1,\cdots,J_n$ are Jordan blocks corresponding to the eigenvalues $i\omega_1,\cdots,i\omega_n$, respectively. For simplicity and clarity of exposition, we focus on a single Jordan block in this paper, avoiding overly complex mathematics.*

Note that $p(t)$ may physically represent the ambient temperature, which influences the heat flux emanating from the boundaries Γ_0,Γ_1 and the domain Ω. System (1) will be analyzed within the conventional state space $X=L^2(\Omega)$, with the control space designated as $U=H^{-1/2}(\Gamma_1)$ and the output space defined as $Y=H^{1/2}(\Gamma_1)$.

The problem we consider can be stated as follows: Given a reference signal

$$r(x,t)=F_4(x)^\top p(t),$$

where $F_4(\cdot)\in H^{1/2}(\Gamma_1;\mathbb{C}^n)$ may also be unknown, the task is to design a tracking error feedback control for the uncertain system (1). The aim is to mitigate the external disturbance and achieve output tracking as follows:

$$e(\cdot,\cdot)=y_o(\cdot,\cdot)-r(\cdot,\cdot)\in L^2_\alpha(0,\infty;Y),$$

where $L^2_\alpha(0,\infty;Y)=\{f\in L^2(0,\infty;Y)|\int_0^\infty e^{2\alpha t}\|f(\cdot,t)\|_Y^2\,dt<\infty\}$ for $\alpha>0$.

Let $A=\Delta$ be the usual Laplacian with

$$D(A)=\left\{\phi\in H^2(\Omega):\phi|_{\Gamma_0}=0,\ \frac{\partial\phi}{\partial\nu}|_{\Gamma_1}=0\right\}.$$

Then, $-A$ is a positive operator on $L^2(\Omega)$. According to Pazy (1983, Theorem 2.7, Chapter 7), A generates an exponentially stable analytic C_0-semigroup e^{At} on $L^2(\Omega)$. Furthermore, it is widely known (see, e.g.,(Lasiecka and Triggiani, 2000, pp. 668)) that

$$D((-A)^{1/2})=H^1_{\Gamma_0}(\Omega):=\left\{\phi\in H^1(\Omega):\phi|_{\Gamma_0}=0\right\}$$

and $(-A)^{1/2}$ is an isomorphism mapping $H^1_{\Gamma_0}(\Omega)$ onto $L^2(\Omega)$. We consider $L^2(\Omega)$ as the pivot space and then the following compact inclusions of Gelfand triple holds:

$$\begin{aligned} H^1_{\Gamma_0}(\Omega) = D((-A)^{1/2}) \hookrightarrow L^2(\Omega) = L^2(\Omega)' \\ \hookrightarrow D((-A)^{1/2})' = H^{-1}_{\Gamma_0}(\Omega), \end{aligned}$$

where $H^{-1}_{\Gamma_0}(\Omega)$ is the dual space of $H^1_{\Gamma_0}(\Omega)$ with respect to $L^2(\Omega)$. The operator A possesses an extension, denoted as $A \in \mathcal{L}(H^1_{\Gamma_0}(\Omega), H^{-1}_{\Gamma_0}(\Omega))$ and is defined as follows:

$$\langle A\phi, \psi\rangle_{H^{-1}_{\Gamma_0}(\Omega), H^1_{\Gamma_0}(\Omega)} = -\langle (-A)^{1/2}\phi, (-A)^{1/2}\psi\rangle_X,$$

applicable to all $\phi, \psi \in H^1_{\Gamma_0}(\Omega)$. Analogously, A also allows for an extension $A \in \mathcal{L}(L^2(\Omega), [D(A)]')$ signifies the dual space of $D(A)$ relative to $L^2(\Omega)$ ((Tucsnak and Weiss, 2009, Section 2.10)). The Neumann map, denoted as $N \in \mathcal{L}(H^{-1/2}(\Gamma_1), H^1_{\Gamma_0}(\Omega))$ (Lasiecka and Triggiani, 2000, pp. 668), establishes a relationship where $Nu = h$ if and only if

$$\begin{cases} \Delta h = 0 \text{ in } \Omega, \\ h\big|_{\Gamma_0} = 0, \ \dfrac{\partial h}{\partial \nu}\big|_{\Gamma_1} = u. \end{cases}$$

Define the Dirichlet map $N_1 \in \mathcal{L}(L^2(\Gamma_0), H^{1/2}(\Omega))$ (Lions and Magenes, 1972, pp. 188-189), which means $N_1 u = h$ if and only if

$$\begin{cases} \Delta h = 0 \text{ in } \Omega, \\ h\big|_{\Gamma_0} = u, \ \dfrac{\partial h}{\partial \nu}\big|_{\Gamma_1} = 0. \end{cases}$$

Utilizing both the Neumann and Dirichlet maps, system (1) can be abstractly represented in $[D(A)]'$ as

$$\begin{aligned} &\dot{w}(\cdot,t) = Aw(\cdot,t) + \mathbb{B}\left(u(\cdot,t) + F_3(\cdot)^\top p(t)\right) \\ &+\mathbb{B}_1\left(F_2(\cdot)^\top p(t)\right) + F_1(\cdot)^\top p(t), \end{aligned}$$

where $\mathbb{B} \in \mathcal{L}(H^{-1/2}(\Gamma_1), H^{-1}_{\Gamma_0}(\Omega)), \mathbb{B}_1 \in \mathcal{L}(L^2(\Gamma_0), [D(A)]')$ are given by

$$\begin{aligned} &\mathbb{B}u = -ANu, \ \forall u \in H^{-1/2}(\Gamma_1); \\ &\mathbb{B}_1 u = -AN_1 u, \ \forall u \in L^2(\Gamma_0). \end{aligned}$$

We emphasize that that while the operator $\mathbb{B}$ is admissible for the semigroup e^{At} (Byrnes, Gilliam, Shubov, and Weiss, 2002, Theorem 2), $\mathbb{B}_1$ is not (Lasiecka and Triggiani, 2000, Sec. 3.1, pp. 180). This crucial difference implies that system (1) does not qualify as a regular linear system. However, there is an important exception: when $F_2 = 0$, the system is indeed a regular linear system (Byrnes, Gilliam, Shubov, and Weiss, 2002, Theorem 2). This specific system can be represented as follows:

$$\begin{cases} \dot{w}(\cdot,t) = Aw(\cdot,t) + \mathbb{B}u(\cdot,t) + B_d p(t), \\ w(\cdot,0) = w_0 \in X, \\ e(\cdot,t) = C_\Lambda w(\cdot,t) + Du(t) + Fp(t), \end{cases} \tag{3}$$

where $w(\cdot,t) \in L^2(\Omega)$, $u(\cdot,t) \in U = H^{-1/2}(\Gamma_1)$, $p(t) \in U_d = \mathbb{C}^n$, and the output $e(t) \in Y = H^{1/2}(\Gamma_1)$. The operators $F \in \mathcal{L}(U_d, Y)$, $B_d \in \mathcal{L}(U_d, H^{-1}_{\Gamma_0}(\Omega))$, and $C \in \mathcal{L}(H^1_{\Gamma_0}(\Omega), Y)$ are defined as:

$$\begin{cases} F = -F_4^\top(\cdot), \\ B_d = \mathbb{B}F_3^\top(\cdot) + F_1^\top(\cdot), \\ C\phi = \phi\big|_{\Gamma_1}, \ \forall \phi \in H^1_{\Gamma_0}(\Omega). \end{cases}$$

The input and output operators $\mathbb{B}, B_d$, and C are admissible with respect to e^{At}, and $D \in \mathcal{L}(U, Y)$ is the feedthrough operator. The admissibility of these operators implies that for all $t > 0$, $u \in L^2(0,t;U)$, and $p \in L^2(0,t;U_d)$:

$$\int_0^t T(t-s)(\mathbb{B}u(\cdot,s) + B_d p(s))ds \in L^2(\Omega),$$

and there exists a constant $\kappa_t > 0$ such that

$$\int_0^t \|CT(s)\phi\|^2\, ds \leq \kappa_t \|\phi\|^2, \forall \phi \in D(A).$$

We define the Λ-extension C_Λ of C as the limit: $C_\Lambda \phi = \lim_{\lambda\to\infty} \lambda C R(\lambda, A)\phi$, where $D(C_\Lambda)$ consists of those $\phi \in X$ for which this limit exists. In a regular linear system, we have $\mathcal{R}(R(\lambda, A)B) \subset D(C_\Lambda)$ for all $\lambda \in \rho(A)$. In our context, $D = 0$ (Byrnes, Gilliam, Shubov, and Weiss, 2002). When the input function $u(\cdot,\cdot)$ belongs to $L^2_{loc}([0,\infty); H^{-1/2}(\Gamma_1))$, we can immediately conclude that system (3) has a unique solution in $C([0,\infty); L^2(\Omega))$, given by:

$$w(\cdot,t) = e^{At}w(\cdot,0) + \int_0^t e^{A(t-s)}(\mathbb{B}u(\cdot,s) + B_d p(t))\, ds,$$

with a well-defined output:

$$\begin{aligned} e(\cdot,t) &= C_\Lambda e^{At} w(\cdot,0) \\ &+ C_\Lambda \int_0^t e^{A(t-s)}(\mathbb{B}u(\cdot,s) + B_d p(s))\, ds + Fp(t). \end{aligned}$$

3 Output feedback regulator design

In this section, we delve into the output regulation for system (1), accommodating all conceivable unknown coefficients such as $F_1(\cdot) \in L^2(\Omega; \mathbb{C}^n)$,

$F_2(\cdot) \in H^{1/2}(\Gamma_0;\mathbb{C}^n), F_4(\cdot) \in H^{1/2}(\Gamma_1;\mathbb{C}^n)$ and $F_3(\cdot) \in L^2(\Gamma_1;\mathbb{C}^n)$. However, we do not presume $F_2(\cdot)=0$, indicating that (1) is not a regular linear system in general. Firstly, we introduce a transformation tailored for systems (1) and (2):

$$z(x,t) = w(x,t) + g(x)^\top p(t), \tag{4}$$

where $g:\Omega\to\mathbb{C}^n$ fulfills the Sylvester-type equation:

$$\begin{cases} \Delta g(x)^\top = g(x)^\top G + F_1(x)^\top, \\ g(x)^\top|_{\Gamma_0} = -F_2(x)^\top, \\ \dfrac{\partial g(x)^\top}{\partial\nu}|_{\Gamma_1} = -F_3(x)^\top. \end{cases} \tag{5}$$

Lemma 3.1 *The boundary value problem* (5) *has a unique solution* $g\in H^1(\Omega;\mathbb{C}^n)$.

Proof. Denote by e_k the k-th column of the $n\times n$ identity matrix. For $1\le k\le n-q$, we right-multiply (5) by e_k to get:

$$\begin{cases} \Delta g_k(x) = g_k(x) i\omega_k + F_1(x)^\top e_k, \\ g_k(x)|_{\Gamma_0} = -F_2(x)^\top e_k, \\ \dfrac{\partial g_k(x)}{\partial\nu}|_{\Gamma_1} = -F_3(x)^\top e_k, \end{cases} \tag{6}$$

where $g_k(x)=g(x)^\top e_k$. From Lions and Magenes (1972, pp. 188-189), we know that both of the following boundary value problems:

$$\begin{cases} \Delta f_k(x) = 0, \\ f_k(x)|_{\Gamma_0} = -F_2(x)^\top e_k, \ \dfrac{\partial f_k(x)}{\partial\nu}|_{\Gamma_1} = 0, \end{cases}$$

and

$$\begin{cases} \Delta h_k(x) = 0, \\ h_k(x)|_{\Gamma_0} = 0, \ \dfrac{\partial h_k(x)}{\partial\nu}|_{\Gamma_1} = -F_3(x)^\top e_k, \end{cases}$$

have unique solutions in $H^1(\Omega)$. Next, invoking the Fredholm alternative theorem (Evans, 1998, Sec. 6.2), we find that the boundary value problem:

$$\begin{cases} \Delta H_k(x) = i\omega_k H_k(x) + i\omega_k(f_k(x)+h_k(x)) + F_1(x)^\top e_k, \\ H_k(x)|_{\Gamma_0} = 0, \ \dfrac{\partial H_k(x)}{\partial\nu}|_{\Gamma_1} = 0, \end{cases}$$

also has a unique solution in $H^1(\Omega)$. Therefore, $g_k(x) = f_k(x)+h_k(x)+H_k(x)$ is the unique solution to (6). For $k=n-q+1$, we right-multiply (5) by e_k to obtain

$$\begin{cases} (\Delta - i\omega_{n-q+1}) g_k(x) = F_1(x)^\top e_k, \\ g_k(x)|_{\Gamma_0} = -F_2(x)^\top e_k, \\ \dfrac{\partial g_k(x)}{\partial\nu}|_{\Gamma_1} = -F_3(x)^\top e_k, \end{cases} \tag{7}$$

which admits a unique solution in $H^1(\Omega)$ by repeating the same steps as the proof for previous $1\le k\le n-q$. For $n-q+2\le k\le n$, right-multiply (5) by e_k to obtain

$$\begin{cases} (\Delta - i\omega_{n-q+1}) g_k(x) = g_{k-1}(x) + F_1(x)^\top e_k, \\ g_k(x)|_{\Gamma_0} = -F_2(x)^\top e_k, \\ \dfrac{\partial g_k(x)}{\partial\nu}|_{\Gamma_1} = -F_3(x)^\top e_k, \end{cases} \tag{8}$$

which also admits a unique solution in $H^1(\Omega)$ by repeating the same steps as the proof for previous $1\le k\le n-q$. ∎

To facilitate control design, we introduce the extended system $(z(\cdot,\cdot),p(\cdot))$ described by:

$$\begin{cases} z_t(x,t) = \Delta z(x,t), \\ z(x,t)|_{\Gamma_0} = 0, \\ \dfrac{\partial z(x,t)}{\partial\nu}|_{\Gamma_1} = u(x,t), \\ \dot p(t) = Gp(t), \\ e(x,t) = z(x,t)|_{\Gamma_1} - (g(x)|_{\Gamma_1} + F_4(x))^\top p(t). \end{cases}$$

According to the Sobolev trace theorem, $g\in H^{1/2}(\Gamma_1;\mathbb{C}^n)$. The term $(g(\cdot)|_{\Gamma_1}+F_4(\cdot))^\top p(t) \in H^{1/2}(\Gamma_1)$ is expressed as:

$$\begin{aligned} (g(x)|_{\Gamma_1} + F_4(x))^\top p(t) &= \sum_{k=1}^{n-q} A_k(x) e^{i\omega_k t} \\ &+ \sum_{k=1}^{q} B_k(x) e^{i\omega_{n-q+1}t} \frac{t^{k-1}}{(k-1)!}, \end{aligned}$$

where $A_k(\cdot), B_k(\cdot) \in H^{1/2}(\Gamma_1)$ are unknown parameters. For control design purposes, we consider $(g(x)|_{\Gamma_1} + F_4(x))^\top p(t)$ to be generated by the following new exosystem:

$$\begin{cases} d_t(x,t) = Gd(x,t), \ x\in\Gamma_1, t\ge 0, \\ d(x,0) = d_0(x), \ x\in\Gamma_1, \\ (g(x)|_{\Gamma_1} + F_4(x))^\top p(t) = \gamma d(x,t), \ x\in\Gamma_1. \end{cases} \tag{9}$$

Here, $d(x,t) = (d_1(x,t),\cdots,d_n(x,t)) \in \mathbb{C}^n$, and

$$\begin{cases} \gamma = (1,1,\ldots,1,0_{1\times(q-1)}), \\ d_0(x) = (A_1(x),\ldots,A_{n-q}(x),B_1(x),\cdots,B_q(x))^\top. \end{cases} \tag{10}$$

Remark 3.1 *The infinite-extension* (9) *can be viewed as an infinite-dimensional analog of the p-extension proposed in Eq.(22) of Deutscher (2017b) for finite-dimensional systems.*

We then formulate $(z(\cdot,t),d(\cdot,t))$ to be governed by:

$$\begin{cases} z_t(x,t)=\Delta z(x,t),\\ z(x,t)|_{\Gamma_0}=0,\\ \dfrac{\partial z(x,t)}{\partial\nu}|_{\Gamma_1}=u(x,t),\\ d_t(x,t)=Gd(x,t),\\ e(x,t)=z(x,t)|_{\Gamma_1}-\gamma d(x,t).\end{cases} \tag{11}$$

Subsequently, our focus shifts to designing an error feedback control for the transformed system (11), which simplifies the original system (1). The design of the error feedback control for system (11) will involve first designing a feedforward control, followed by an observer-based output feedback control, as detailed in the next two subsections.

3.1 Feedforward control design

We introduce the Dirichlet maps $\Upsilon_k\in\mathcal{L}(H^{1/2}(\Gamma_1), H^1(\Omega)), k=1,2,\cdots,n-q+1$ (Lions and Magenes, 1972, pp. 188-189). These maps are defined such that $\Upsilon_k(r)=h_0$ if and only if

$$\begin{cases}(\Delta-i\omega_k I)h_0=0 \text{ in } \Omega,\\ h_0\big|_{\Gamma_0}=0,\ h_0\big|_{\Gamma_1}=r.\end{cases}$$

For any $r\in H^{1/2}(\Gamma_1)$, we have $h_0\in H^1(\Omega)$ and

$$\|h_0(\cdot)\|_{H^1(\Omega)}\le C_q\|r(\cdot)\|_{H^{1/2}(\Gamma_1)}$$

for some constant $C_q>0$. By Tucsnak and Weiss (2009, Theorem 13.7.6), the Neumann trace operator $S_1\in\mathcal{L}(H^2(\Omega),H^{1/2}(\Gamma_1))$ defined by $S_1\phi=\frac{\partial\phi}{\partial\nu}\big|_{\Gamma_1}$, $\forall\phi\in H^2(\Omega)$ admits a unique extension $S_1\in\mathcal{L}(D(\Delta),H^{-1/2}(\Gamma_1))$,, where $D(\Delta)$ is the Hilbert space $D(\Delta)=\{f\in H^1(\Omega)|\Delta f\in L^2(\Omega)\}$, endowed with the norm

$$\|f\|_{D(\Delta)}=\sqrt{\|f\|^2_{H^1(\Omega)}+\|\Delta f\|^2_{L^2(\Omega)}},\ \forall f\in D(\Delta).$$

If $\Upsilon_k(r)=h_0$ for some $r\in H^{1/2}(\Gamma_1)$ and $k=1,2,\cdots,n-q+1$, then $h_0\in D(\Delta)$ and there exists a constant $C_p>0$ such that

$$\begin{aligned}&\|S_1h_0(\cdot)\|_{H^{-1/2}(\Gamma_1)}\le\|S_1\|\|h_0(\cdot)\|_{D(\Delta)}\\ &\le\|S_1\|\sqrt{\|h_0(\cdot)\|^2_{H^1(\Omega)}+\|i\omega_k h_0(\cdot)\|^2_{L^2(\Omega)}}\\ &\le C_p\|h_0(\cdot)\|_{H^1(\Omega)}\le C_pC_q\|r(\cdot)\|_{H^{1/2}(\Gamma_1)},\end{aligned}$$

showing that $S_1\Upsilon_k\in\mathcal{L}(H^{1/2}(\Gamma_1),H^{-1/2}(\Gamma_1))$. Define an operator $\mathbb{A}_1: D(\mathbb{A}_1)(\subset L^2(\Omega))\to L^2(\Omega)$ by

$$\begin{cases}\mathbb{A}_1\phi=(\Delta-i\omega_{n-q+1})\phi,\ \forall\,\phi\in D(\mathbb{A}_1),\\ D(\mathbb{A}_1)=\left\{\phi\in H^2(\Omega)|\ \phi\big|_{\Gamma}=0\right\}.\end{cases}$$

By Lions and Magenes (1972, pp. 188-189), $\mathbb{A}_1^{-1}\in\mathcal{L}(L^2(\Omega),H^2(\Omega))$.

Lemma 3.2 *Let*

$$v(x,t)=\sum_{k=1}^{q}\mathbb{A}_1^{-(k-1)}\Upsilon_{n-q+1}(d_{n-q+k}(\cdot,t)). \tag{12}$$

Then $v(x,t)$ is a solution of the following equation:

$$\begin{cases}v_t(x,t)=\Delta v(x,t)\ \text{in}\ \Omega,\\ v(x,t)|_{\Gamma_0}=0,\\ v(x,t)|_{\Gamma_1}=d_{n-q+1}(x,t).\end{cases} \tag{13}$$

Proof. We can verify that $v(x,t)$ defied by (12) satisfies every equation of (13). Firstly, by the definition of Υ_{n-q+1} and $\mathbb{A}_1$,

$$\begin{aligned}&v(x,t)|_{\Gamma_0}=\Upsilon_{n-q+1}(d_{n-q+1}(\cdot,t))|_{\Gamma_0}=0,\\ &v(x,t)|_{\Gamma_1}=\Upsilon_{n-q+1}(d_{n-q+1}(\cdot,t))|_{\Gamma_1}=d_{n-q+1}(\cdot,t).\end{aligned}$$

Next, since for each $1\le k\le q-1$, $\left(\frac{\partial}{\partial t}-i\omega_{n-q+1}\right)d_{n-q+k}(x,t)=d_{n-q+k+1}(x,t)$, $\left(\frac{\partial}{\partial t}-i\omega_{n-q+1}\right)d_n(x,t)=0$, recalling of $d_{n-q+k}\in C^\infty([0,\infty);H^{1/2}(\Gamma_1))$, $\forall 1\le k\le q$, we have

$$\left(\frac{\partial}{\partial t}-i\omega_{n-q+1}\right)v(\cdot,t)=\sum_{k=1}^{q-1}\mathbb{A}_1^{-(k-1)}\Upsilon_{n-q+1}(d_{n-q+k+1}(\cdot,t)).$$

Using the fact $(\Delta-i\omega_{n-q+1})\Upsilon_{n-q+1}(d_{n-q+1}(\cdot,t))=0$, it follows that

$$(\Delta-i\omega_{n-q+1})v(\cdot,t)=\sum_{k=2}^{q}\mathbb{A}_1^{-(k-2)}\Upsilon_{n-q+1}(d_{n-q+k}(\cdot,t)).$$

Therefore, $v_t(x,t)=\Delta v(x,t)$ in Ω. ∎

Let

$$\begin{aligned}\varepsilon(\cdot,t)&=z(\cdot,t)-\sum_{k=1}^{n-q}\Upsilon_k(d_k(\cdot,t))\\ &\quad-\sum_{k=1}^{q}\mathbb{A}_1^{-(k-1)}\Upsilon_{n-q+1}(d_{n-q+k}(\cdot,t)).\end{aligned}$$

For $1\le k\le n-q$, given that $d_k(x,t)=A_k(x)e^{i\omega_k t}$, we obtain

$$\frac{d}{dt}\Upsilon_k(d_k(\cdot,t))=i\omega_k\Upsilon_k(d_k(\cdot,t))=\Delta\Upsilon_k(d_k(\cdot,t)).$$

Hence, $\varepsilon(\cdot,\cdot)$ is described by the system:

$$\begin{cases} \varepsilon_t(x,t) = \Delta\varepsilon(x,t), \\ \varepsilon(x,t)|_{\Gamma_0} = 0, \\ \dfrac{\partial\varepsilon(\cdot,t)}{\partial\nu}|_{\Gamma_1} = u(\cdot,t) - Hd(\cdot), \\ e(x,t) = \varepsilon(x,t)|_{\Gamma_1}, \end{cases} \tag{14}$$

where $H \in \mathcal{L}(H^{1/2}(\Gamma_1;\mathbb{C}^n); H^{-1/2}(\Gamma_1))$ is given by

$$\begin{cases} Hd(\cdot) = S_1\Bigg(\displaystyle\sum_{k=1}^{n}\Upsilon_k(d_k(\cdot)) \\ \qquad + \displaystyle\sum_{k=1}^{q}\mathbb{A}_1^{-(k-1)}\Upsilon_{n-q+1}(d_{n-q+k}(\cdot))\Bigg), \\ \forall d(\cdot) = (d_1(\cdot),\cdots,d_n(\cdot))^\top \in H^{1/2}(\Gamma_1;\mathbb{C}^n). \end{cases} \tag{15}$$

To compensate the disturbance, we naturally introduce a feedforward control defined as:

$$u(\cdot,t) = Hd(\cdot,t). \tag{16}$$

This control ensures that $u \in C([0,\infty); H^{-1/2}(\Gamma_1))$. With this control, system (14) transforms into:

$$\begin{cases} \varepsilon_t(x,t) = \Delta\varepsilon(x,t), \\ \varepsilon(x,t)|_{\Gamma_0} = 0, \quad \dfrac{\partial\varepsilon(x,t)}{\partial\nu}|_{\Gamma_1} = 0, \\ e(x,t) = \varepsilon(x,t)|_{\Gamma_1}. \end{cases} \tag{17}$$

For any initial state $\varepsilon(\cdot,0) \in L^2(\Omega)$, this system admits a unique solution $\varepsilon(\cdot,t) = e^{At}\varepsilon(\cdot,0) \in L^2(\Omega)$ that decays exponentially.

3.2 Observer design

An observer for the z-subsystem, as described in (11), can be trivially designed as follows:

$$\begin{cases} \hat{z}_t(x,t) = \Delta\hat{z}(x,t), \\ \hat{z}(x,t)|_{\Gamma_0} = 0, \\ \dfrac{\partial\hat{z}(x,t)}{\partial\nu}|_{\Gamma_1} = u(x,t), \end{cases} \tag{18}$$

From this design, the observer error $\tilde{z}(x,t) = z(x,t) - \hat{z}(x,t)$ evolves according to:

$$\begin{cases} \tilde{z}_t(x,t) = \Delta\tilde{z}(x,t), \\ \tilde{z}(x,t)|_{\Gamma_0} = 0, \\ \dfrac{\partial\tilde{z}(x,t)}{\partial\nu}|_{\Gamma_1} = 0. \end{cases} \tag{19}$$

For any initial state $\tilde{z}(\cdot,0) \in L^2(\Omega)$, system (19) admits a unique solution in $C([0,\infty); L^2(\Omega))$ that decays exponentially. The admissibility of C leads to lemma 3.3 following.

Lemma 3.3 *The solution of system (19) satisfies*

$$\int_0^\infty e^{\beta t}\|\tilde{z}(\cdot,t)\|^2_{H^{1/2}(\Gamma_1)}\,dt \le \tilde{C}\|\tilde{z}(\cdot,0)\|^2_{L^2(\Omega)} \tag{20}$$

for some $\beta > 0$.

In (11), the term $\gamma d(x,t)$ in the error $e(x,t) = z(x,t)|_{\Gamma_1} - \gamma d(x,t)$ is unknown. Since $z(x,t)$ can be estimated by the observer (19), we introduce a known function

$$y_d(x,t) = -e(x,t) + \hat{z}(x,t)|_{\Gamma_1} = \gamma d(x,t) - \tilde{z}(x,t)|_{\Gamma_1}$$

to estimate $\gamma d(x,t)$ and consider the system:

$$\begin{cases} d_t(x,t) = Gd(x,t), \ x \in \Gamma_1, \ t \ge 0, \\ d(x,0) = d_0(x), \ x \in \Gamma_1, \\ y_d(x,t) = \gamma d(x,t) - \tilde{z}(x,,t)|_{\Gamma_1}, \ x \in \Gamma_1. \end{cases} \tag{21}$$

Based on the output $y_d(x,t)$, we design the following observer for (21):

$$\begin{cases} \hat{d}_t(x,t) = G\hat{d}(x,t) + L(y_d(x,t) - \gamma\hat{d}(x,t)), \\ \hat{d}(x,0) = \hat{d}_0(x) \in H^{1/2}(\Gamma_1;\mathbb{C}^n). \end{cases} \tag{22}$$

Here, $L = (l_1,\cdots,l_n)^\top \in \mathbb{C}^n$ is chosen so that $G - L\gamma$ is Hurwitz. Such an L always exists because (G,γ) is observable for γ defined by (10). Defining the observer error as $\tilde{d}(x,t) = d(x,t) - \hat{d}(x,t)$, we obtain:

$$\tilde{d}_t(x,t) = (G - L\gamma)\tilde{d}(x,t) + L\tilde{z}(x,t)|_{\Gamma_1}. \tag{23}$$

Rewriting system (23) as an evolution equation on Y^n gives:

$$\dot{\tilde{d}}(\cdot,t) = (G_1 - L_1\gamma_1)\tilde{d}(\cdot,t) + L_1\tilde{z}(\cdot,t)|_{\Gamma_1}, \tag{24}$$

where

$$G_1 = \mathrm{diag}\left\{i\omega_1 I_Y, i\omega_2 I_Y, \ldots, \begin{bmatrix} 0 & I_{Y^{q-1}} \\ 0 & 0 \end{bmatrix} + i\omega_{n-q+1}I_{Y^q}\right\},$$

and $\gamma_1 = [I_Y, I_Y, \ldots, I_Y, 0, \cdots, 0] \in \mathcal{L}(Y^n, Y), L_1 = [l_1 I_Y, \cdots, l_n I_Y]^\top \in \mathcal{L}(Y, Y^n)$.

Lemma 3.4 *For any initial state $\tilde{d}(\cdot,0) \in H^{1/2}(\Gamma_1;\mathbb{C}^n)$, and $\tilde{z}(\cdot,t)$ from (19), system (23) admits a unique solution $\tilde{d} \in C([0,\infty); H^{1/2}(\Gamma_1;\mathbb{C}^n))$.*

Proof. Since $G_1 - L_1\gamma_1$ is bounded and $G - L\gamma$ is Hurwitz, $G_1 - L_1\gamma_1$ generates an exponentially stable uniformly continuous semigroup $S_g(t)$ with $\|S_g(t)\| \le M_g e^{-\omega_g t}$ for some $M_g, \omega_g > 0$. Because L_1 is bounded, it follows from Tucsnak and Weiss (2009, Proposition 4.2.5) that system (23) admits a unique solution $\tilde{d} \in C([0,\infty); H^{1/2}(\Gamma_1;\mathbb{C}^n))$. ∎

3.3 Observer-based error feedback control

By referring to equation (16), we can naturally design a tracking error feedback control, as follows:

$$u(\cdot,t) = H\hat{d}(\cdot,t), \tag{25}$$

where H is as defined in (15). Since $H \in \mathcal{L}(H^{1/2}(\Gamma_1;\mathbb{C}^n); H^{-1/2}(\Gamma_1))$ and $u \in C([0,\infty); H^{-1/2}(\Gamma_1))$, the closed-loop system arising from (1) under the control (25) can be expressed as:

$$\begin{cases} w_t(x,t) = \Delta w(x,t) + F_1(x)^\top p(t), \\ w(x,t)|_{\Gamma_0} = F_2(x)^\top p(t), \\ \dfrac{\partial w(\cdot,t)}{\partial \nu}|_{\Gamma_1} = H\hat{d}(\cdot,t) + F_3(\cdot)^\top p(t), \\ \hat{z}_t(x,t) = \Delta \hat{z}(x,t), \\ \hat{z}(x,t)|_{\Gamma_0} = 0, \ \dfrac{\partial \hat{z}(\cdot,t)}{\partial \nu}|_{\Gamma_1} = H\hat{d}(\cdot,t), \\ \hat{d}_t(x,t) = G\hat{d}(x,t) + L(-e(x,t) + \hat{z}(x,t)|_{\Gamma_1} \\ \qquad -\gamma \hat{d}(x,t)), \\ e(x,t) = w(x,t)|_{\Gamma_1} - F_4(x)^\top p(t), \end{cases} \tag{26}$$

which is equivalent to:

$$\begin{cases} \varepsilon_t(x,t) = \Delta \varepsilon(x,t), \\ \varepsilon(x,t)|_{\Gamma_0} = 0, \ \dfrac{\partial \varepsilon(\cdot,t)}{\partial \nu}|_{\Gamma_1} = -H\tilde{d}(\cdot,t), \\ \tilde{z}_t(x,t) = \Delta \tilde{z}(x,t), \\ \tilde{z}(x,t)|_{\Gamma_0} = 0, \ \dfrac{\partial \tilde{z}(x,t)}{\partial \nu}|_{\Gamma_1} = 0, \\ \tilde{d}_t(x,t) = (G - L\gamma)\tilde{d}(x,t) + L\tilde{z}(x,t)|_{\Gamma_1}, \\ e(x,t) = \varepsilon(x,t)|_{\Gamma_1}, \end{cases} \tag{27}$$

where $\varepsilon(\cdot,\cdot)$, $\tilde{z}(\cdot,\cdot)$ and $\tilde{d}(\cdot,\cdot)$ are as specified in (14), (19) and (23), respectively. Now, let's consider system (27) within the Hilbert space $\mathcal{H} = (L^2(\Omega))^2 \times H^{1/2}(\Gamma_1;\mathbb{C}^n)$, equipped with the norm:

$$\|(\phi_1,\phi_2,\phi_3)\|_{\mathcal{H}} = \Bigg(\nu^2 \textstyle\int_\Omega |\phi_1(x)|^2 dx + \int_\Omega |\phi_2(x)|^2 dx + \nu\langle \phi_3, P\phi_3\rangle_{H^{1/2}(\Gamma_1;\mathbb{C}^n)}\Bigg)^{1/2}, \quad \forall (\phi_1,\phi_2,\phi_3) \in \mathcal{H}, \tag{28}$$

where $P \in L(Y^n)$ is a positive definite operator satisfying:

$$P(G_1 - L_1\gamma_1) + (G_1 - L_1\gamma_1)^* P = -2I. \tag{29}$$

System (27) can be rewritten as:

$$\frac{d}{dt}\begin{pmatrix} \varepsilon(\cdot,t) \\ \tilde{z}(\cdot,t) \\ \tilde{d}(\cdot,t) \end{pmatrix} = A_E \begin{pmatrix} \varepsilon(\cdot,t) \\ \tilde{z}(\cdot,t) \\ \tilde{d}(\cdot,t) \end{pmatrix},$$

where $A_E : D(A_E) \to (L^2(\Omega))^2 \times H^{1/2}(\Gamma_1;\mathbb{C}^n)$ is defined as:

$$\begin{cases} A_E \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} A\phi_1 - \mathbb{B}H\phi_3 \\ A\phi_2 \\ (G_1 - L_1\gamma_1)\phi_3 + L_1\phi_2|_{\Gamma_1} \end{pmatrix}, \\ D(A_E) = \Big\{(\phi_1,\phi_2,\phi_3)^\top \in H^1_{\Gamma_0}(\Omega) \times (H^2(\Omega)\cap H^1_{\Gamma_0}(\Omega)) \\ \times H^{1/2}(\Gamma_1;\mathbb{C}^n) : A\phi_1 - \mathbb{B}H\phi_3 \in L^2(\Omega), \dfrac{\partial \phi_2}{\partial \nu}|_{\Gamma_1} = 0\Big\}. \end{cases} \tag{30}$$

Lemma 3.5 *The operator A_E defined by (30) generates an exponentially stable C_0-semigroup on $\mathcal{H} = (L^2(\Omega))^2 \times H^{1/2}(\Gamma_1;\mathbb{C}^n)$.*

Proof. For any triplet $(\varphi_1,\varphi_2,\varphi_3) \in \mathcal{H}$, solving the equation

$$A_E \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}$$

leads us to the following system of equations:

$$\begin{cases} \phi_2 = A^{-1}\varphi_2 \in H^2(\Omega)\cap H^1_{\Gamma_0}(\Omega), \\ \phi_3 = (G_1 - L_1\gamma_1)^{-1}(\varphi_3 - L_1\phi_2|_{\Gamma_1}) \in H^{1/2}(\Gamma_1;\mathbb{C}^n), \end{cases}$$

where ϕ_1 satisfies

$$\Delta\phi_1(x) = \varphi_1(x), \ \phi_1(x)|_{\Gamma_0} = 0, \ \frac{\partial \phi_1(x)}{\partial \nu}|_{\Gamma_1} = -H\phi_3(\cdot).$$

Since $\varphi_1 \in L^2(\Omega)$, and $H\phi_3(\cdot) \in H^{-1/2}(\Gamma_1)$, according to Lions and Magenes (1972, pp. 188-189), we deduce that $\phi_1 \in H^1_{\Gamma_0}(\Omega)$, which establishes that $\mathcal{R}(A_E) = \mathcal{H}$. For any triplet $(\phi_1,\phi_2,\phi_3) \in D(A_E)$, and for any $\nu > 0$, we have the following inequality:

$$\begin{aligned} &\mathrm{Re}\langle A_E(\phi_1,\phi_2,\phi_3),(\phi_1,\phi_2,\phi_3)\rangle_{\mathcal{H}} \\ &= \mathrm{Re}\Big(\nu^2\langle A\phi_1,\phi_1\rangle_{H^{-1}_{\Gamma_0}(\Omega),H^1_{\Gamma_0}(\Omega)} + \int_\Omega \Delta\phi_2(x)\overline{\phi_2(x)}dx \\ &+\nu\langle (G_1 - L_1\gamma_1)\phi_3, P\phi_3\rangle_{H^{1/2}(\Gamma_1;\mathbb{C}^n)} + \nu\langle L_1\phi_2|_{\Gamma_1}, P\phi_3\rangle_{H^{1/2}(\Gamma_1;\mathbb{C}^n)} \\ &-\nu^2\langle \mathbb{B}H\phi_3,\phi_1\rangle_{H^{-1}_{\Gamma_0}(\Omega),H^1_{\Gamma_0}(\Omega)}\Big) \\ &= \mathrm{Re}\Big(-\nu^2\langle (-A)^{1/2}\phi_1,(-A)^{1/2}\phi_1\rangle_{L^2(\Omega)} - \int_\Omega |\nabla\phi_2(x)|^2 dx \\ &-\nu\|\phi_3\|^2_{H^{1/2}(\Gamma_1;\mathbb{C}^n)} + \nu\langle L_1 C\phi_2, P\phi_3\rangle_{H^{1/2}(\Gamma_1;\mathbb{C}^n)} \\ &-\nu^2\langle H\phi_3, C\phi_1\rangle_{H^{-1/2}(\Gamma_1),H^{1/2}(\Gamma_1)}\Big) \\ &\le -\nu^2\int_\Omega |\nabla\phi_1(x)|^2 dx - \int_\Omega |\nabla\phi_2(x)|^2 dx \\ &-\nu\|\phi_3\|^2_{H^{1/2}(\Gamma_1;\mathbb{C}^n)} + \nu C_2\|\phi_2\|_{H^1_{\Gamma_0}(\Omega)}\|\phi_3\|_{H^{1/2}(\Gamma_1;\mathbb{C}^n)} \\ &+\nu^2 C_3\|\phi_1\|_{H^1_{\Gamma_0}(\Omega)}\|\phi_3\|_{H^{1/2}(\Gamma_1;\mathbb{C}^n)}, \end{aligned}$$

for some constants $C_2, C_3 > 0$ independent of ν. Considering the symmetric matrix

$$\begin{pmatrix} -\nu^2 & 0 & \frac{1}{2}\nu^2 C_3 \\ 0 & -1 & \frac{1}{2}\nu C_2 \\ \frac{1}{2}\nu^2 C_3 & \frac{1}{2}\nu C_2 & -\nu \end{pmatrix},$$

which is negative definite for sufficiently small $\nu > 0$. By Poincaré's inequality, there exist constants $C_A, C_B > 0$ such that

$$\begin{aligned} &\mathrm{Re}\langle A_E(\phi_1,\phi_2,\phi_3),(\phi_1,\phi_2,\phi_3)\rangle_{\mathcal{H}} \\ &\le -C_A\|(\phi_1,\phi_2,\phi_3)\|^2_{H^1_{\Gamma_0}(\Omega)\times H^1_{\Gamma_0}(\Omega)\times H^{1/2}(\Gamma_1;\mathbb{C}^n)} \\ &\le -C_B\|(\phi_1,\phi_2,\phi_3)\|^2_{\mathcal{H}}. \end{aligned} \tag{31}$$

Together with the Lumer-Philips theorem (Pazy, 1983, Theorem 1.4.3), this implies that A_E generates an exponentially stable C_0-semigroup on $\mathcal{H}$.

Theorem 3.1 *For any unknown coefficients $F_1(\cdot) \in L^2(\Omega;\mathbb{C}^n)$, $F_2(\cdot) \in H^{1/2}(\Gamma_0;\mathbb{C}^n)$, $F_4(\cdot) \in H^{1/2}(\Gamma_1;\mathbb{C}^n)$ and $F_3(\cdot) \in L^2(\Gamma_1;\mathbb{C}^n)$, and for any initial state*

$$(w(\cdot,0),\hat{z}(\cdot,0),\hat{d}(\cdot,0)) \in (L^2(\Omega))^2 \times H^{1/2}(\Gamma_1;\mathbb{C}^n),$$

system (26) admits a unique solution $(w,\hat{z},\hat{d}) \in C([0,\infty);(L^2(\Omega))^2 \times H^{1/2}(\Gamma_1;\mathbb{C}^n))$, and the output tracking of the closed-loop system (26) is guaranteed such that

$$\int_0^\infty e^{\alpha t}\|e(\cdot,t)\|^2_{H^{1/2}(\Gamma_1)}\,dt < \infty \tag{32}$$

for some $\alpha > 0$.

Proof. Since the operator A_E generates an exponentially stable C_0-semigroup on $\mathcal{H} = (L^2(\Omega))^2 \times H^{1/2}(\Gamma_1;\mathbb{C}^n)$, system (27) admits a unique solution in $C([0,\infty);(L^2(\Omega))^2 \times H^{1/2}(\Gamma_1;\mathbb{C}^n))$ satisfying

$$\begin{aligned} &\left\|\varepsilon(\cdot,t),\tilde{z}(\cdot,t),\tilde{d}(\cdot,t)\right\|_{(L^2(\Omega))^2\times H^{1/2}(\Gamma_1;\mathbb{C}^n)} \\ &\le M_1 e^{-\mu_1 t}\left\|\varepsilon(\cdot,0),\tilde{z}(\cdot,0),\tilde{d}(\cdot,0)\right\|_{(L^2(\Omega))^2\times H^{1/2}(\Gamma_1;\mathbb{C}^n)} \end{aligned} \tag{33}$$

for some $M_1, \mu_1 > 0$. Therefore, the transforms

$$\begin{cases} w(\cdot,t) = \varepsilon(\cdot,t) + \sum_{k=1}^{n-q}\Upsilon_k(d_k(\cdot,t)) \\ +\sum_{k=1}^{q}\mathbb{A}_1^{-(k-1)}\Upsilon_{n-q+1}(d_{n-q+k}(\cdot,t)) - g(\cdot)p(t), \\ \hat{z}(\cdot,t) = \varepsilon(\cdot,t) + \sum_{k=1}^{n-q}\Upsilon_k(d_k(\cdot,t)) \\ +\sum_{k=1}^{q}\mathbb{A}_1^{-(k-1)}\Upsilon_{n-q+1}(d_{n-q+k}(\cdot,t)) - \tilde{z}(\cdot,t), \\ \hat{d}(\cdot,t) = d(\cdot,t) - \tilde{d}(\cdot,t), \end{cases}$$

show that $(w,\hat{z},\hat{d})$ is well-defined and bounded in $(L^2(\Omega))^2 \times H^{1/2}(\Gamma_1;\mathbb{C}^n)$ with respect to time. The ε-subsystem now reads

$$\begin{cases} \varepsilon_t(x,t) = \Delta\varepsilon(x,t), \\ \varepsilon(x,t)|_{\Gamma_0} = 0, \\ \dfrac{\partial\varepsilon(x,t)}{\partial\nu}|_{\Gamma_1} = \tilde{u}(x,t), \end{cases} \tag{34}$$

where

$$\tilde{u}(\cdot,t) = -H\tilde{d}(\cdot,t),$$

which satisfies $\tilde{u} \in C([0,\infty);H^{-1/2}(\Gamma_1))$ and

$$\begin{aligned} &\|\tilde{u}(\cdot,t)\|_{H^{-1/2}(\Gamma_1)} \le \|H\|\|\tilde{d}(\cdot,t)\|_{H^{1/2}(\Gamma_1;\mathbb{C}^n)} \\ &\le \|H\|M_1 e^{-\mu_1 t}\left\|\varepsilon(\cdot,0),\tilde{z}(\cdot,0),\tilde{d}(\cdot,0)\right\|_{(L^2(\Omega))^2\times H^{1/2}(\Gamma_1;\mathbb{C}^n)}. \end{aligned} \tag{35}$$

Without loss of generality, we consider only the real solution. Let

$$V_1(t) = \frac{1}{2}\int_\Omega \varepsilon^2(x,t)dx.$$

It follows from (33) that $V_1(t)$ decays exponentially, i.e.,

$$\begin{aligned} &V_1(t) = \frac{1}{2}\|\varepsilon(x,t)\|^2_{L^2(\Omega)} \\ &\le \frac{1}{2}M_1^2 e^{-2\mu_1 t}\left\|\varepsilon(\cdot,0),\tilde{z}(\cdot,0),\tilde{d}(\cdot,0)\right\|^2_{(L^2(\Omega))^2\times H^{1/2}(\Gamma_1;\mathbb{C}^n)}. \end{aligned} \tag{36}$$

Differentiating $V_1(t)$ along the solution of (34) and using Green's formula yields

$$\begin{aligned} \dot{V}_1(t) &= -\int_\Omega |\nabla\varepsilon(x,t)|^2dx + \int_{\Gamma_1}\varepsilon(x,t)\tilde{u}(x,t)dx. \\ &\le -\int_\Omega |\nabla\varepsilon(x,t)|^2dx + \|\varepsilon(\cdot,t)\|_{H^{1/2}(\Gamma_1)}\|\tilde{u}(\cdot,t)\|_{H^{-1/2}(\Gamma_1)} \\ &\le -\|\varepsilon(\cdot,t)\|^2_{H^1_{\Gamma_0}(\Omega)} + m_0\|\varepsilon(\cdot,t)\|_{H^1_{\Gamma_0}(\Omega)}\|\tilde{u}(\cdot,t)\|_{H^{-1/2}(\Gamma_1)} \\ &\le -(1-\delta m_0)\|\varepsilon(\cdot,t)\|^2_{H^1_{\Gamma_0}(\Omega)} + \frac{m_0}{4\delta}\|\tilde{u}(\cdot,t)\|^2_{H^{-1/2}(\Gamma_1)}, \end{aligned} \tag{37}$$

where we used Young's inequality and the Sobolev trace theorem that $\|\varepsilon(\cdot,t)\|_{H^{1/2}(\Gamma_1)} \le m_0\|\varepsilon(\cdot,t)\|_{H^1(\Omega)}$ with m_0 being a positive constant. Let $\delta = \frac{1}{2m_0}$. Then,

$$\dot{V}_1(t) \le -\frac{1}{2}\|\varepsilon(\cdot,t)\|^2_{H^1_{\Gamma_0}(\Omega)} + \frac{m_0}{4\delta}\|\tilde{u}(\cdot,t)\|^2_{H^{-1/2}(\Gamma_1)}. \tag{38}$$

From (38) and (36), we obtain

$$\begin{aligned} &\frac{1}{2}\int_0^\infty e^{\alpha t}\|\varepsilon(\cdot,t)\|^2_{H^1_{\Gamma_0}(\Omega)}dt \\ &\le V_1(0) + \int_0^\infty \alpha e^{\alpha t}V_1(t)dt \\ &+ \int_0^\infty \frac{m_0}{4\delta}e^{\alpha t}\|\tilde{u}(\cdot,t)\|_{H^{-1/2}(\Gamma_1)}dt \\ &\le C_e\left\|\varepsilon(\cdot,0),\tilde{z}(\cdot,0),\tilde{d}(\cdot,0)\right\|^2_{(L^2(\Omega))^2\times H^{1/2}(\Gamma_1;\mathbb{C}^n)} \end{aligned}$$

for any $\alpha < 2\mu_1$ and some $C_e > 0$. This, together with the Sobolev trace theorem, gives

$$\int_0^\infty e^{\alpha t} \|\varepsilon(\cdot,t)\|^2_{H^{1/2}(\Gamma_1)}\, dt < \infty.$$

∎

4 Robustness of the control

In this section, we delve into the robustness of the controller within a simplified context where $F_2 = 0$ and $q = 1$. The assumption $F_2 = 0$ is to make the system be regular. For $q > 1$, there is difficulty to discuss the robustness. Under these conditions, $G = \operatorname{diag}\{i\omega_1, i\omega_2, \ldots, i\omega_n\}$, and $\gamma = (1, 1, \cdots, 1)$. For simplicity, L can be chosen as $L = (1, 1, \cdots, 1)^\top$, ensuring that $G - L\gamma$ satisfies the Hurwitz criterion. We demonstrate that the feedback control introduced in the preceding section:

$$\begin{cases} u(\cdot,t) = H(\hat{d}(\cdot,t)), \\ \hat{z}_t(x,t) = \Delta \hat{z}(x,t), \\ \hat{z}(x,t)|_{\Gamma_0} = 0, \\ \dfrac{\partial \hat{z}(\cdot,t)}{\partial \nu}|_{\Gamma_1} = H(\hat{d}(\cdot,t)), \\ \hat{d}_t(x,t) = G\hat{d}(x,t) + L(-e(x,t) \\ +\hat{z}(x,t)|_{\Gamma_1} - \gamma \hat{d}(x,t)), \end{cases} \tag{39}$$

fulfills the $\mathcal{G}-$ conditions, thereby solving the robust output regulation problem. To facilitate our analysis, we introduce a bounded and invertible coordinate transformation:

$$\begin{pmatrix} \hat{\varepsilon}(\cdot,t) \\ \hat{d}(\cdot,t) \end{pmatrix} = \begin{pmatrix} \hat{z}(\cdot,t) - H_1(\hat{d}(\cdot,t)) \\ \hat{d}(\cdot,t) \end{pmatrix}, \tag{40}$$

where H_1 is defined as

$$\begin{cases} H_1 d(\cdot) = \displaystyle\sum_{k=1}^n \Upsilon_k(d_k(\cdot)), \\ \forall d(\cdot) = (d_1(\cdot), \cdots, d_n(\cdot))^\top \in H^{1/2}(\Gamma_1; \mathbb{C}^n). \end{cases}$$

It's worth noting that this type of general coordinate transformation is analogous to the one presented in (3.19) of Immonen (2007). Evidently, $H_1 \in \mathcal{L}(H^{1/2}(\Gamma_1;\mathbb{C}^n); H^1(\Omega))$. The control (39) can be equivalently expressed as:

$$\begin{cases} u(\cdot,t) = H(\hat{d}(\cdot,t)), \\ \hat{\varepsilon}_t(\cdot,t) = \Delta\hat{\varepsilon}(\cdot,t) + \displaystyle\sum_{k=1}^n \Upsilon_k(-e(\cdot,t) + \hat{\varepsilon}(\cdot,t)|_{\Gamma_1}), \\ \hat{\varepsilon}(x,t)|_{\Gamma_0} = 0, \\ \dfrac{\partial\hat{\varepsilon}(x,t)}{\partial\nu}|_{\Gamma_1} = 0, \\ \hat{d}_t(x,t) = G\hat{d}(x,t) + L(-e(x,t) + \hat{\varepsilon}(x,t)|_{\Gamma_1}), \end{cases} \tag{41}$$

This system can be further abstracted as:

$$\begin{cases} \dot{z}(t) = \mathbb{G}_1 z(t) + \mathbb{G}_2 e(t), \\ u(t) = Kz(t), \end{cases} \tag{42}$$

where $z = (\hat{\varepsilon}, \hat{d})^\top$ and the triple $(\mathbb{G}_1, \mathbb{G}_2, K)$ is given by:

$$\mathbb{G}_1 = \begin{pmatrix} A + H_1 L_1 C_\Lambda & 0 \\ L_1 C_\Lambda & G_1 \end{pmatrix}, \ \mathbb{G}_2 = \begin{pmatrix} -H_1 L_1 \\ -L_1 \end{pmatrix}, \ K = \begin{pmatrix} 0 & H \end{pmatrix}.$$

Note that both $\mathbb{G}_2$ and K are bounded operators. The following lemma establishes key properties of the system operators:

Lemma 4.1 *The operator $\mathbb{G}_1$ generate a C_0-semigroup on $L^2(\Omega) \times H^{1/2}(\Gamma_1;\mathbb{C}^n)$). The triple $(\mathbb{G}_1, \mathbb{G}_2, K)$ satisfies the $\mathcal{G}$-conditions, specifically:*

$$\begin{cases} \mathcal{R}(i\omega_k - \mathbb{G}_1) \cap \mathcal{R}(\mathbb{G}_2) = \{0\}, \ \forall 1 \le k \le n, \\ \mathcal{N}(\mathbb{G}_2) = \{0\}, \end{cases} \tag{43}$$

Proof. According to Weiss (1994, Section 7), $\mathbb{G}_1$ generate an C_0-semigroup on $L^2(\Omega) \times H^{1/2}(\Gamma_1;\mathbb{C}^n)$) due to the following operator representation:

$$\mathbb{G}_1 = \begin{bmatrix} A & 0 \\ 0 & G_1 \end{bmatrix} - \mathbb{G}_2 \begin{bmatrix} C_\Lambda & 0 \end{bmatrix}. \tag{44}$$

The operator L_1 takes the form $L_1 = (I_Y, \cdots, I_Y)^\top \in \mathcal{L}(Y, Y^n)$, implying that $\mathcal{N}(\mathbb{G}_2) = \{0\}$. Now, let $1 \le k \le n$ be arbitrary and assume that

$$\begin{pmatrix} w \\ v \end{pmatrix} = (i\omega_k - \mathbb{G}_1)\begin{pmatrix} h_0 \\ h \end{pmatrix} = \mathbb{G}_2 h_1 \in \mathcal{R}(i\omega_k - \mathbb{G}_1) \cap \mathcal{R}(\mathbb{G}_2)$$

for some $(h_0, h)^\top \in (H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)) \times H^{1/2}(\Gamma_1;\mathbb{C}^n)$, and $h_1 \in H^{1/2}(\Gamma_1)$. Denote $h = (h^1, \cdots, h^n)^\top$. Then,

$$\begin{pmatrix} w \\ v \end{pmatrix} = \begin{pmatrix} (i\omega_k - A - H_1 L_1 C_\Lambda) h_0 \\ (-L_1 C_\Lambda) h_0 + (i\omega_k - G_1) h \end{pmatrix} = \begin{pmatrix} -H_1 L_1 h_1 \\ -L_1 h_1 \end{pmatrix}.$$

The second row above implies that $(i\omega_k - G_1)h = L_1(C_\Lambda h_0 + h_1)$, which specifically means that $0 = (i\omega_k - i\omega_k)h^k = (C_\Lambda h_0 + h_1)$. Hence, the first row above implies that $(i\omega_k - A)h_0 = 0$ which leads to $h_0 = 0$, leading to $h_0 = 0$ since $i\omega_k \in \rho(A)$. Therefore, $h_1 = -C_\Lambda h_0 = 0$ and $(w, v)^\top = \mathbb{G}_2 h_1 = 0$. This concludes that $\mathcal{R}(i\omega_k - \mathbb{G}_1) \cap \mathcal{R}(\mathbb{G}_2) = \{0\}$. Since $1 \le k \le n$ are arbitrary, the $\mathcal{G}$-conditions is satisfied.

∎

We emphasize that in this section, when operators $A, \mathbb{B}, B_d, C$ and D are perturbed to $\tilde{A} : D(\tilde{A}) \subset X \to X$,

$\tilde{\mathbb{B}} \in \mathcal{L}(U, \tilde{X}_{-1}), \tilde{B}_1 \in \mathcal{L}(U_d, \tilde{X}_{-1}), C \in \mathcal{L}(\tilde{X}_1, Y)$, $\tilde{D} \in \mathcal{L}(U, Y)$ respectively, where $\tilde{X}_1$ and $\tilde{X}_{-1}$ are scale spaces linked to $\tilde{A}$, we presume that $(\tilde{A}, \tilde{\mathbb{B}}, \tilde{B}_d, \tilde{C}, \tilde{D})$ constitutes a regular linear system, and F is perturbed in a way that $\tilde{F} \in \mathcal{L}(U_d, Y)$.

Denoting the operators of the closed-loop system formed by the perturbed plant and the controller as $\tilde{C}_e = [\tilde{C}_\Lambda, \tilde{D}K]$, $\tilde{D}_e = \tilde{F}$, and

$$\tilde{A}_e = \begin{bmatrix} \tilde{A} & \tilde{B}K \\ \mathbb{G}_2\tilde{C}_\Lambda & \mathbb{G}_1 + \mathbb{G}_2\tilde{D}K \end{bmatrix}, \; \tilde{B}_e = \begin{bmatrix} \tilde{B}_d \\ \mathbb{G}_2\tilde{F} \end{bmatrix}, \qquad (45)$$

we have the following lemma:

Lemma 4.2 *(Paunonen, 2017, Theorem 2.3) Assume $q = 1$ and $F_2 = 0$. The closed-loop system $(\tilde{A}_e, \tilde{B}_e, \tilde{C}_e, \tilde{D}_e)$ is a regular system.*

The class $\mathcal{O}$ of perturbations encompasses operators $(\tilde{A}, \tilde{\mathbb{B}}, \tilde{B}_d, \tilde{C}, \tilde{D}, \tilde{F})$ satisfying the aforementioned assumptions. Notably, the class $\mathcal{O}$ includes the nominal plant, i.e., $(A, \mathbb{B}, B_d, C, D, F) \in \mathcal{O}$. Given that we have established the exponential stability of the nominal closed-loop system $A_e = \Gamma^{-1}A_E\Gamma$, with Γ being

$$\begin{pmatrix} I & 0 & 0 \\ I & -I & 0 \\ 0 & 0 & I \end{pmatrix},$$

the succeeding theorem is a direct consequence of Paunonen (2017, Theorem 3.8).

Theorem 4.1 *The control (39) is robust with respect to all perturbations for which the perturbed closed-loop system $\tilde{A}_e$ is exponentially stable.*

5 Conclusions

In this paper, we design a robust output regulator for a multi-dimensional heat equation with infinite-dimensional output space by the internal model principle approach. The approach we adopt is again the observer-based feedback control approach. Two transformations are made. The first transformation is for the observer design for which we re-generate the exosystem so that the tracking error can detect both PDE and ODE. The second transformation is for the feedforward control design. The tracking error feedback control is therefore designed with replacement of the state and disturbance by their estimates. The closed-loop system is proved to be conditionally robust when G is a diagonal matrix. For the special case, since the plant is a impedance passive and stable regular linear system, it is possible to use the simple controller design in (see, e.g., Rebarber and Weiss (2003))

$$\begin{cases} u(x,t) = -\gamma\hat{d}(x,t), \\ \dot{\hat{d}}(x,t) = G\hat{d}(x,t) + \gamma^\top e(x,t). \\ \hat{d}(x,0) = \hat{d}_0(x) \in H^{1/2}(\Gamma_1; \mathbb{C}^n). \end{cases}$$

The approach of Rebarber and Weiss (2003) is not working for non-trivial Jordan block case. On the other hand, the low-gain simple control design has been extended to the stable systems and non-trivial Jordan blocks in Hämäläinen and Pohjolainen (2002), which is, however, only limited to finite-dimensional output space. Our approach has potential application for unstable heat equation, which has been shown working well for 1-d case. Another potential problem that we need to consider in the next future work is the non-collocated case which has been solved for 1-d PDEs. Given that the control is observer-based, the finite approximations represent an intriguing area for future investigation.

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Ren-Xi Zhao received his Ph.D. from the Academy of Mathematics and Systems Science, Chinese Academy of Sciences, in 2024. Currently, he serves as an Assistant Professor in the School of Mathematics at Foshan University, China. His research interests encompass the control theory of infinite-dimensional systems.

Fig. 1. Ren-Xi Zhao

Fig. 2. Bao-Zhu Guo

Fig. 3. Lassi Paunonen

Bao-Zhu Guo received his Ph.D. in Applied Mathematics from the Chinese University of Hong Kong in 1991. Since 2000, he has been affiliated with the Academy of Mathematics and Systems Science, Chinese Academy of Sciences, where he holds the position of Research Professor in Mathematical System Theory. Additionally, he is currently associated with the School of Mathematics and Physics at North China Electrical Power University in China. His research pursuits include the control of nonlinear systems, as well as the theory and application of infinite-dimensional systems control.
Lassi Paunonen received his Ph.D. in Mathematics from Tampere University of Technology, Tampere, Finland, in 2011. He currently holds the position of Associate Professor in the Department of Mathematics at Tampere University of Technology. His primary research focus is on the asymptotic behavior of strongly continuous semigroups, operator theory, and the robust control of infinite-dimensional linear systems.