On Excitable Control of Stable Infinite-Dimensional Systems

Tuulia Tokola, Mohamed Fkirine, and Lassi Paunonen

***Abstract*— We study the control of a class of stable infinite-dimensional linear systems. Our main goal is to design a controller that makes the closed-loop system excitable in the sense of the system's sensitivity to external input pulses exceeding a given threshold. We achieve this with a dynamic feedback controller combining positive and negative feedback. As our main results we prove the well-posedness of the infinite-dimensional nonlinear closed-loop system and analyse the closed-loop stability properties for different parameter configurations of the controller. We illustrate the excitability properties in the case of one-dimensional beam and heat equations.**

***Index Terms*— Distributed parameter systems, nonlinear control, excitability, stability, FitzHugh–Nagumo equation.**

I. Introduction

Neuromorphic control is a research area where the control strategies are inspired by the dynamical behaviour of biological systems, especially the functionality of the brain [20]. One of the most characteristic features of the neuromorphic control schemes is *excitability*. This refers to the capability of the system to produce qualitatively different behaviour for external input pulses that are either below or above a given threshold [21]. Excitability can especially be induced by *mixed feedback* controllers which incorporate both negative and positive feedback [20, Sec. III].

In this article we study the neuromorphic control of linear infinite-dimensional systems and partial differential equations (PDEs). More precisely, we investigate a stable linear infinite-dimensional system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0 \in X \tag{1a}$$
$$y(t) = Cx(t) \tag{1b}$$

with a nonlinear mixed feedback controller. Our first main result establishes the existence and uniform boundedness of the closed-loop trajectories. Subsequently we investigate the stability of the closed-loop equilibrium points. Finally, we use numerical simulations to demonstrate the excitable behaviour of the controlled system for an Euler–Bernoulli beam equation and a one-dimensional heat equation.

Neuromorphic control and excitable systems have been investigated actively in the setting of finite-dimensional linear and nonlinear systems [22], [19], [26], [2], [18], [14]. Earlier research on excitable infinite-dimensional systems and PDEs include the study of a spatially invariant reaction–diffusion equation with a static mixed feedback interconnection in [15]. In addition, PDE versions of the FitzHugh–Nagumo equation and other reaction–diffusion equations with non-monotone nonlinearities have been investigated, for instance, in [10], [5], [8]. Our analysis is also related to the study of PDEs with Van der Pol type boundary conditions where the nonlinearity enters through mixed feedback at the boundary. Such boundary feedback laws can lead to self-excited oscillations and complex transient dynamics even when the underlying PDE system is passive. The dynamics of such models have especially been analysed in the case of the one-dimensional wave equation in [4], [11], [9] and the heat equation in [24].

The article is organised as follows. In Section II we state the standing assumptions on our system and the controller. We prove the existence and boundedness of solutions of the closed-loop system in Section III. In Section IV we analyse the locations and stability of the equilibrium points of the closed-loop system in the case of a cubic controller inspired by the FitzHugh–Nagumo model. The excitability properties of the controlled beam and heat equations are demonstrated in Section V.

Notation. If X and Y are Banach spaces and $A : X \to Y$ is a linear operator, we denote by $D(A)$, $\mathcal{N}(A)$ and $\mathcal{R}(A)$ the domain, kernel and range of A, respectively. The space of bounded linear operators from X to Y is denoted by $\mathcal{L}(X, Y)$. If $A : D(A) \subset X \to X$, then $\sigma(A)$ and $\rho(A)$ denote the spectrum and the resolvent set of A, respectively. The inner product on a Hilbert space is denoted by $\langle \cdot, \cdot \rangle$. An operator $A : D(A) \subset X \to X$ on a real Hilbert space X is *dissipative* if $\langle Ax, x \rangle \leq 0$ for all $x \in D(A)$. We use complexifications in the spectral analysis of operators on real Hilbert spaces.

II. Preliminaries and Standing Assumptions

We consider an infinite-dimensional linear system of the form (1) on a real Hilbert space X. We assume that A generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ on X and assume that the system has scalar-valued input and output so that $B \in \mathcal{L}(\mathbb{R}, X)$ and $C \in \mathcal{L}(X, \mathbb{R})$. Throughout the paper we assume that $(T(t))_{t\geq 0}$ is exponentially stable. We denote the transfer function of the system by $P(\lambda) = C(\lambda - A)^{-1}B$ for $\lambda \in \rho(A)$.

We consider nonlinear controllers of the form

$$\dot{v}(t) = f_c(v(t)) + u_c(t), \qquad v(0) = v_0 \in \mathbb{R} \tag{2a}$$
$$y_c(t) = v(t) \tag{2b}$$

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and interconnect the plant and the controller so that $u(t) = y_c(t)$ and $u_c(t) = -\gamma y(t) + I_{\text{app}}(t)$, where $\gamma > 0$ and where I_{app} is an external input to the closed-loop system. The closed-loop structure is similar to [2]. It differs from the mixed feedback structure in [20], where the external input I_{app} is fed into the plant. This illustrates the possibility of achieving excitability with different configurations.

Our main interest is in the cases where f_c is a locally Lipschitz continuous non-monotone function such that $f_c(v) \to \mp\infty$ as $v \to \pm\infty$. In particular, in the neuromorphic controller inspired by the FitzHugh–Nagumo model we have

$$f_c(v) = -\alpha v^3 + \beta v$$

with $\alpha, \beta > 0$. The closed-loop system on $X \times \mathbb{R}$ has the form

$$\dot{x}(t) = Ax(t) + Bv(t) \tag{3a}$$
$$\dot{v}(t) = f_c(v(t)) - \gamma Cx(t) + I_{\text{app}}(t). \tag{3b}$$

III. Existence and Boundedness of Solutions

In our main result below we show that the closed-loop system (3) has well-defined solutions which are uniformly bounded with respect to time. The condition (4) holds especially when f_c is a polynomial with an odd degree at least 3 and with a negative leading coefficient.

Theorem 3.1: Assume that the semigroup generated by A is exponentially stable and that there exists $\eta \in \mathbb{R}$ such that $\langle Ax, x\rangle \leq \eta\|x\|^2$ for all $x \in D(A)$. Let $f_c : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz continuous function such that

$$vf_c(v) \leq -c_0|v|^{\delta} + c_1, \quad \forall v \in \mathbb{R}, \tag{4}$$

for some $c_0 > 0$, $c_1 \geq 0$ and $\delta > 2$. Then for every initial state $(x_0, v_0) \in X \times \mathbb{R}$ and piecewise continuous and uniformly bounded input $I_{\text{app}} : [0, \infty) \to \mathbb{R}$ the closed-loop system (3) has a unique mild solution (x, v) on $[0, \infty)$, and $\sup_{t\geq 0}\big(\|x(t)\| + \|v(t)\|\big) < \infty$. Moreover, if $x_0 \in D(A)$, then (x, v) is a strong solution if I_{app} is locally Lipschitz continuous, and a classical solution if f_c and I_{app} are continuously differentiable.

Proof: The closed-loop system (3) on $X \times \mathbb{R}$ has the form

$$\dot{z}(t) = \mathbb{A}z(t) + F(t, z(t)) \quad z(0) = (x_0, v_0),$$

with

$$\mathbb{A} = \begin{bmatrix} A & B \\ -\gamma C & 0 \end{bmatrix}, \quad F\left(t, \begin{bmatrix} x \\ v \end{bmatrix}\right) = \begin{bmatrix} 0 \\ f_c(v) + I_{\text{app}}(t) \end{bmatrix}.$$

This is a semilinear equation where the function F is piecewise continuous with respect to t and locally Lipschitz continuous with respect to (x, v) (uniformly in t). By [17, Thm. 6.1.4][1] the closed-loop system (3) has a unique mild solution (x, v) on $[0, t_0)$ for some $t_0 > 0$. Moreover, if $x_0 \in D(A)$, then $(x_0, v_0) \in D(\mathbb{A})$, and by [17, Thm. 6.1.5] (x, v) is a strong solution provided that I_{app} is locally Lipschitz continuous, and a classical solution if f_c and I_{app} are continuously differentiable.

We will now show that all solutions are uniformly bounded with respect to t. By [17, Thm. 1.4] this will also imply that the solutions (x, v) exist on the infinite time interval $[0, \infty)$. By [3, Thm. 1], there exists a Fréchet differentiable $V : X \to [0, \infty)$ such that

$$m_1\|x\|^2 \leq V(x) \leq m_2\|x\|^2, \qquad x \in X \tag{5a}$$
$$\|\nabla V(x)\| \leq m_0\sqrt{V(x)}, \qquad x \in X, \tag{5b}$$
$$\langle \nabla V(x), Ax\rangle \leq -\mu V(x), \qquad x \in D(A) \tag{5c}$$

for some constants $m_0, m_1, m_2, \mu > 0$. We define a Lyapunov candidate $V_L(x, v) := V(x) + \frac{1}{2}v^2$. If (x, v) is a strong solution of (3), then for a.e. $t \geq 0$ we have

$$\begin{aligned}\dot{V}_L(x, v) &= \langle \nabla V(x), \dot{x}\rangle + v\dot{v} \\ &= \langle \nabla V(x), Ax + Bv\rangle + v(f_c(v) - \gamma Cx + I_{\text{app}}).\end{aligned}$$

The estimates (5) imply that

$$|\langle \nabla V(x), Bv\rangle| + |\gamma vCx| \leq m_3\sqrt{V(x)}\,|v|,$$

where $m_3 := m_0\|B\| + \frac{\gamma\|C\|}{\sqrt{m_1}}$. Applying Young's inequality $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$ with $a = \sqrt{\mu V(x)}$ and $b = m_3|v|\mu^{-1/2}$ yields

$$|\langle \nabla V(x), Bv\rangle| + |\gamma vCx| \leq \frac{\mu}{2}V(x) + \frac{m_3^2}{2\mu}v^2.$$

If $M > 0$ is such that $|I_{\text{app}}(t)| \leq M$ for all $t \geq 0$, then $|vI_{\text{app}}| \leq \frac{v^2}{2} + \frac{M^2}{2}$. Combining the estimates yields

$$\begin{aligned}&\dot{V}_L(x, v) \\ &\quad\leq -\frac{\mu}{2}V(x) + \frac{m_3^2}{2\mu}v^2 - c_0|v|^{\delta-2} + c_1 + \frac{v^2}{2} + \frac{M^2}{2} \\ &\quad= -\frac{\mu}{2}V_L(x, v) + \left(\frac{m_3^2}{2\mu} + \frac{1}{2} + \frac{\mu}{4}\right)v^2 - c_0|v|^{\delta-2} \\ &\qquad + c_1 + \frac{1}{2}M^2.\end{aligned}$$

Since $\delta > 2$, we can maximise $g(v) := \left(\frac{m_3^2}{2\mu} + \frac{1}{2} + \frac{\mu}{4}\right)v^2 - c_0|v|^{\delta-2}$ with respect to v to note that there exists a $K_0 > 0$ for which $g(v) \leq K_0$ for all v. Thus there exists a constant $a_1 > 0$ such that $\dot{V}_L(x, v) \leq -\frac{\mu}{2}V_L(x, v) + a_1$. Since (x, v) was an arbitrary strong solution of the closed-loop system, we have from [6, Thm. 2.1.2 & Ex. 2.1.3] that the solutions of the closed-loop system are uniformly bounded with respect to $t \geq 0$. ■

IV. Equilibrium Points and Stability

In this section we describe the equilibrium points of the closed-loop system (3) with constant external input and characterise their stability properties.

Lemma 4.1: Assume that I_{app} is a constant function so that $I_{\text{app}}(t) \equiv I_{\text{app}} \in \mathbb{R}$. The equilibrium points of the closed-loop system (3) are of the form $((-A)^{-1}Bv_e, v_e) \in X \times \mathbb{R}$, where v_e is a solution of the scalar equation

$$0 = f_c(v_e) - \gamma P(0)v_e + I_{\text{app}}. \tag{6}$$

[1] The reference [17, Thm. 6.1.4] assumes F to be continuous with respect to t. We obtain existence of mild solutions for piecewise continuous functions by concatenating the trajectories between points of discontinuity.

Proof: The equilibrium points (x_e, v_e) of (3) are the solutions of the system

$$0 = Ax_e + Bv_e, \tag{7a}$$
$$0 = f_c(v_e) - \gamma C x_e + I_{\text{app}}. \tag{7b}$$

Since A generates an exponentially stable semigroup, (7a) is equivalent to $x_e = (-A)^{-1}Bv_e$. Since $P(0) = C(-A)^{-1}B$, (7b) is exactly (6). ■

We will now study the stability properties of the equilibrium points in the case where $f_c(v) = -\alpha v^3 + \beta v$. This nonlinearity is of the same form as the nonlinear part of the FitzHugh–Nagumo model of neural excitability. The following proposition describes the stability properties of the equilibrium points in the case where the external input I_{app} is zero. In particular, it introduces conditions under which the closed-loop system is *bistable* in the sense that it has exactly two locally stable equilibrium points.

Proposition 4.2: Assume that $f_c(v) = -\alpha v^3 + \beta v$, that $I_{\text{app}}(t) \equiv 0$, and that $\beta/\gamma > P(0)$. Then the closed-loop system has three distinct equilibrium points $0 \in X \times \mathbb{R}$ and $(\pm(-A)^{-1}Bv_*, \pm v_*) \in X \times \mathbb{R}$ with $v_* = \sqrt{(\beta - \gamma P(0))/\alpha} > 0$ and the following hold.

- The point $0 \in X \times \mathbb{R}$ is unstable.
- The points $(\pm(-A)^{-1}Bv_*, \pm v_*)$ are locally exponentially stable if $\lambda - 3\gamma P(0) + 2\beta + \gamma P(\lambda) \neq 0$ for all $\lambda \in \mathbb{C}_+$. Local exponential stability holds especially if A is dissipative, $C = B^*$, and $\beta/\gamma > \frac{3}{2}P(0)$.

Proof: By Lemma 4.1, the equilibrium points of the system are determined by the solutions of $-\alpha v^3 + \beta v - \gamma P(0) v = 0$. Since $\beta/\gamma > P(0)$, the polynomial has three distinct real roots, namely $v = 0$ and $v = \pm\sqrt{(\beta - \gamma P(0))/\alpha}$. This implies the claim regarding the locations of the equilibrium points.

To study local stability, we linearise the system around the equilibrium $z_e = (x_e, v_e)$. Since f_c is continuously differentiable, the semigroup generator $\mathbb{A}(z_e) : D(\mathbb{A}(z_e)) \subset X \times \mathbb{R} \to X \times \mathbb{R}$ of the linearised system is defined by $D(\mathbb{A}(z_e)) = D(A) \times \mathbb{R}$ and

$$\mathbb{A}(z_e) = \begin{bmatrix} A & B \\ -\gamma C & f_c'(v_e) \end{bmatrix}, \quad \text{where } f_c'(v_e) = -3\alpha v_e^2 + \beta.$$

Since A generates an exponentially stable semigroup, we have $\sigma(A) \subset \mathbb{C}_-$. Thus for every $\lambda \in \overline{\mathbb{C}_+}$ we have

$$\lambda - \mathbb{A}(z_e) = \begin{bmatrix} I & 0 \\ \gamma C R_\lambda & I \end{bmatrix} \begin{bmatrix} \lambda - A & 0 \\ 0 & S(\lambda) \end{bmatrix} \begin{bmatrix} I & -R_\lambda B \\ 0 & I \end{bmatrix},$$

where $R_\lambda = (\lambda - A)^{-1}$ and $S(\lambda) = \lambda - f_c'(v_e) + \gamma P(\lambda)$. Thus $\lambda \in \overline{\mathbb{C}_+}$ satisfies $\lambda \in \rho(\mathbb{A}(z_e))$ if and only if $S(\lambda) \neq 0$, and in this case $(\lambda - \mathbb{A}(z_e))^{-1}$ is given by

$$\begin{bmatrix} I & R_\lambda B \\ 0 & I \end{bmatrix} \begin{bmatrix} (\lambda - A)^{-1} & 0 \\ 0 & S(\lambda)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -\gamma C R_\lambda & I \end{bmatrix}.$$

Since $\lambda \mapsto (\lambda - A)^{-1}$ is uniformly bounded on $\mathbb{C}_+$, the Gearhart–Prüss–Greiner theorem [7, Thm. V.1.11] implies that the semigroup generated by $\mathbb{A}(z_e)$ is exponentially stable if and only if there exists $c > 0$ such that $|S(\lambda)| \geq c$ for all $\lambda \in \mathbb{C}_+$.

The non-zero equilibria: If $z_e = (\pm(-A)^{-1}Bv_*, \pm v_*)$, then $f_c'(v_e) = -3\alpha v_*^2 + \beta = 3\gamma P(0) - 2\beta$ and $S(\lambda) = \lambda + 2\beta - 3\gamma P(0) + \gamma P(\lambda)$. We first note that if A is dissipative and $C = B^*$, the transfer function of (A, B, B^*) is positive real in the sense that $\operatorname{Re} P(\lambda) \geq 0$ for all $\lambda \in \mathbb{C}_+$. In this case the condition $\beta/\gamma > \frac{3}{2}P(0)$ guarantees that $S(\lambda) \neq 0$ for all $\lambda \in \mathbb{C}_+$. Assume now that $S(\lambda) \neq 0$ for all $\lambda \in \mathbb{C}_+$. Since P is uniformly bounded on $\mathbb{C}_+$, we can define $c_0 = \sup_{\operatorname{Re}\lambda \geq 0} |2\beta - 3\gamma P(0) + \gamma P(\lambda)| < \infty$, and then $|S(\lambda)| \geq c_0$ for all $\lambda \in \mathbb{C}_+$ with $|\lambda| \geq 2c_0$. Since S is continuous and nonzero in $\{\, \lambda \in \mathbb{C}_+ \mid |\lambda| \leq 2c_0 \,\}$, there exists $c > 0$ such that $|S(\lambda)| \geq c$ for all $\lambda \in \mathbb{C}_+$. As stated above, this implies that the semigroup generated by $\mathbb{A}(z_e)$ is exponentially stable. Since f_c is continuously differentiable, the equilibrium point z_e is locally exponentially stable by [25, Prop 4.17].

The equilibrium at zero: If $z_e = 0$, then $f_c'(v_e) = \beta$ and $S(\lambda) = \lambda - \beta + \gamma P(\lambda)$. We have $S(\lambda) \in \mathbb{R}$ for $\lambda \geq 0$ and $\lim_{\lambda\to\infty} P(\lambda) = 0$ implies that $S(\lambda) \to \infty$ as $\lambda \to \infty$. Since $S(0) = -\beta + \gamma P(0) < 0$, there exists $\lambda_0 > 0$ such that $S(\lambda_0) = 0$. The instability of z_e will follow from [25, Prop. 4.18] if we can show that $\sigma(\mathbb{A}(z_e)) \cap \mathbb{C}_+$ consists of a finite number of eigenvalues with finite multiplicities. Since C is a rank one operator, the structure of $\mathbb{A}(z_e)$ and [12, Thm. IV.6.2 & IV.6.5] imply that $\sigma(\mathbb{A}(z_e)) \cap \overline{\mathbb{C}_+}$ only includes isolated spectral points which are eigenvalues with finite multiplicities. Finally, to show that $\sigma(\mathbb{A}(v_e)) \cap \mathbb{C}_+$ is finite, we recall that $\lambda \in \mathbb{C}_+$ satisfies $\lambda \in \sigma(\mathbb{A}_e(z_e))$ if and only if $S(\lambda) = 0$, and all zeros of S in $\mathbb{C}_+$ are contained in the set $\Omega = \{\, \lambda \in \overline{\mathbb{C}_+} \mid |\lambda| \leq 2c_0 \,\}$. Since Ω is compact and P is analytic in Ω, also S is analytic in Ω. We argue by contraposition and assume that $\sigma(\mathbb{A}(v_e)) \cap \mathbb{C}_+$ is infinite. Equivalently, $S(\lambda)$ has infinitely many zeros. As Ω is compact, the zeros have an accumulation point. Then it follows from [1, Cor. 26.3] that $S(\lambda)$ should be identically zero in Ω. Since $|S(2c_0)| \geq c_0 > 0$, this is a contradiction, and therefore the number of zeros of S in Ω must be finite. Together the above properties imply that $\sigma(\mathbb{A}(v_e)) \cap \mathbb{C}_+$ consists of a finite number of eigenvalues with finite multiplicities, and by [25, Prop. 4.18] the equilibrium point z_e is unstable. ■

Finally, to study the excitability of the closed-loop system in Section V, we want to be able to choose a constant external input in such a way that the system has a single stable equilibrium point. The following proposition provides necessary and sufficient conditions for such choices of I_{app} in the case where f_c is a cubic polynomial.

Proposition 4.3: Assume that $f_c(v) = -\alpha v^3 + \beta v$ with $\alpha, \beta > 0$, and that $I_{\text{app}} \neq 0$ is a constant input. If either $\beta/\gamma \leq P(0)$ or alternatively $\beta/\gamma > P(0)$ and

$$|I_{\text{app}}| > \frac{2\sqrt{(\beta - \gamma P(0))^3}}{3\sqrt{3\alpha}}, \tag{8}$$

then the closed-loop system has a single equilibrium point

$((-A)^{-1}Bv_e, v_e)$, where $v_e = v_+^{1/3} + v_-^{1/3}$ and

$$v_\pm = \frac{1}{2\alpha}\left(I_{\text{app}} \pm \sqrt{I_{\text{app}}^2 - \frac{4(\beta-\gamma P(0))^3}{27\alpha}}\right). \qquad (9)$$

The equilibrium (x_e, v_e) is unstable if $3\alpha v_e^2 - \beta + \gamma P(0) < 0$, and locally exponentially stable if $\lambda + 3\alpha v_e^2 - \beta + \gamma P(\lambda) \neq 0$ for all $\lambda \in \overline{\mathbb{C}_+}$. In particular, local exponential stability holds if $(3\alpha v_e^2 - \beta)/\gamma > \sup_{\operatorname{Re}\lambda \geq 0} |\operatorname{Re} P(\lambda)|$.

Proof: By Lemma 4.1, the equilibrium points are determined by the roots of the polynomial

$$p(v_e) = v_e^3 - \frac{\beta - \gamma P(0)}{\alpha} v_e - \frac{I_{\text{app}}}{\alpha}. \qquad (10)$$

The discriminant of p is $\Delta = I_{\text{app}}^2/(4\alpha^2) - (\beta - \gamma P(0))^3/(27\alpha^3)$. If $\beta/\gamma \leq P(0)$, then p is strictly monotone and it has a unique root v_e for every fixed I_{app}. If $\beta/\gamma > P(0)$, then (10) has a single real root if and only if $\Delta > 0$, which is equivalent to (8). In this case the unique real root is given by Cardano's formula $v_e = v_+^{1/3} + v_-^{1/3}$, where $v_\pm$ are as in (9), and the corresponding equilibrium point is $z_e = (x_e, v_e) = ((-A)^{-1}Bv_e, v_e)$.

The local stability of (x_e, v_e) can be analysed using linearisations similarly as in the proof of Proposition 4.2. The arguments in the proof of Proposition 4.2 show that z_e is locally exponentially stable if $S(\lambda) = \lambda + 3\alpha v_e^2 - \beta + \gamma P(\lambda) \neq 0$ for all $\lambda \in \mathbb{C}_+$. This is in particular true if $3\alpha v_e^2 - \beta > \gamma \sup_{\operatorname{Re}\lambda \geq 0} |\operatorname{Re} P(\lambda)|$, since this implies $\operatorname{Re} S(\lambda) > 0$ for all $\lambda \in \mathbb{C}_+$. On the other hand, if $S(0) < 0$, then the fact that $S(\lambda) \to \infty$ as $\lambda \to \infty$ and continuity of S imply there exists $\lambda_0 > 0$ such that $S(\lambda_0) = 0$. This implies the instability of z_e similarly as in the proof of Proposition 4.2. ■

V. Excitability

In this section we study the effect of external input pulses on the closed-loop system consisting of a one-dimensional beam equation or a heat equation and a mixed feedback controller. In this situation we have

$$I_{\text{app}}(t) = I_{\text{base}} + I_{\text{pulse}}(t),$$

where I_{base} determines the constant part of the external input and I_{pulse} is non-zero only on relatively short time intervals. Since I_{base} is constant, the results in Section IV determine the dynamics of the closed-loop system between the pulses in I_{pulse}. The pulses can cause two different types of behaviour: relatively small input pulses lead to small deviations from a stable equilibrium state, whereas input pulses exceeding a *threshold* cause relatively large deviations from the equilibrium [20]. This way, the dynamical behaviour can be divided into *subthreshold* and *suprathreshold* responses [23], which is a typical feature of excitability [20]. We investigate the phenomena described above in the cases of a controlled beam system with a unique and locally stable equilibrium point, and a bistable closed-loop system for a heat equation.

A. Controlled Beam Equation

In this case study we consider the control of a cantilever Euler–Bernoulli beam which is attached to a rotating hub at $\xi = 0$ and free at the other end $\xi = 1$. The model of the system is

$$\begin{aligned}
&\rho w_{tt}(\xi,t) + EI w_{\xi\xi\xi\xi}(\xi,t) + d w_t(\xi,t) = \frac{\rho\xi}{I_h} u(t)\\
&w(0,t) = w_\xi(0,t) = 0, \quad w_{\xi\xi}(1,t) = w_{\xi\xi\xi}(1,t) = 0,\\
&y(t) = w(1,t)
\end{aligned}$$

with initial conditions $w(\cdot,0) = w_0$ and $w_t(\cdot,0) = w_1$ [16, Ex. 4.17, 4.27]. The input $u(t)$ is the torque applied to the hub, $\rho > 0$ is the density of the beam, $EI > 0$ is the flexural rigidity, $d > 0$ is the viscous damping parameter, and $I_h > 0$ is the inertia of the rotating hub. By [13, Ex. 2.36] and straightforward computations the system can be represented in the form (1) on the Hilbert space $X = H_\ell^2(0,1) \times L^2(0,1)$ with $H_\ell^2(0,1) = \{ f \in H^2(0,1) \mid f(0) = f'(0) = 0 \}$, and with inner product

$$\begin{aligned}
&\langle (x_1,x_2),(y_1,y_2)\rangle_X\\
&\quad = EI \int_0^1 x_1''(\xi) y_1''(\xi) d\xi + \rho \int_0^1 x_2(\xi) y_2(\xi) d\xi.
\end{aligned}$$

In particular, the associated operator $A : D(A) \subset X \to X$ is dissipative and generates an exponentially stable semigroup on X, and $B \in \mathcal{L}(\mathbb{R}, X)$ and $C \in \mathcal{L}(X, \mathbb{R})$.

We design our controller and set the base value of the external input I_{base} in such a way that the closed-loop system has a single stable equilibrium point and is capable of exhibiting excitable events. We use the neuromorphic controller (2) inspired by the FitzHugh–Nagumo model, with

$$f_c(v) = -\alpha v^3 + \beta v$$

for suitable $\alpha, \beta > 0$. By Theorem 3.1 the closed-loop system consisting of (1) and (2) has well-defined and uniformly bounded solutions on $[0,\infty)$ for all piecewise continuous and uniformly bounded inputs $I_{\text{app}} : [0,\infty) \to \mathbb{R}$. In the simulations we use the parameters $\rho = 1$, $EI = 2$, $d = 3$, $I_h = 1$, $\alpha = 4$, $\beta = 10$, $\gamma = 8$, and $I_{\text{base}} = 6.09$. We verify numerically that the conditions for the uniqueness and local stability of the equilibrium point in Proposition 4.3 are satisfied for the given parameters.

We demonstrate the excitability of the closed-loop system using numerical simulations where we introduce pulses in I_{pulse}. The beam equation is semi-discretised using the Finite Element Method with cubic basis elements and $N = 10$ elements for both the deflection and the velocity profile. Figures 1 and 2 show the behaviour of the controller state and the deflection of the beam resulting from input pulses of length 0.2 with different amplitudes (the pulses are chosen to be smooth to improve numerical accuracy). The first pulses generate a modest response in the controller state $v(t)$, whereas the latter two pulses result in a more dramatic deviation from the equilibrium state. Such qualitative differences in responses to input pulses of different sizes is a distinctive feature of excitable systems.

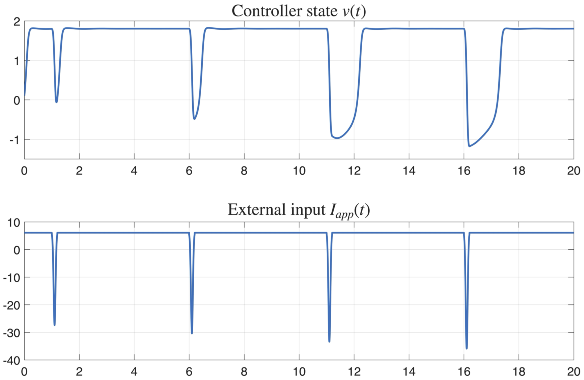

Fig. 1. The controller state for the beam system for input pulses of different sizes. The first two pulses generate modest responses and the latter pulses result in strong responses.

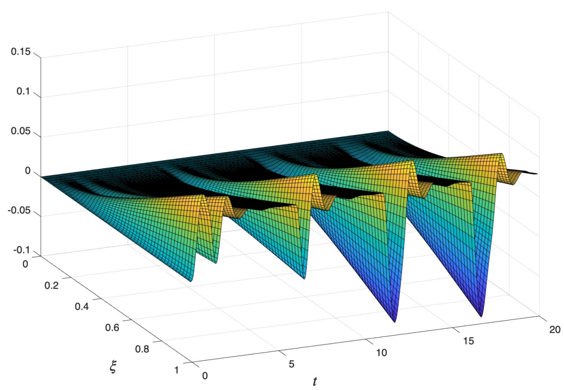

Fig. 2. The deflection of the beam for the input pulses in Figure 1.

B. Heat Equation

We consider a controlled one-dimensional heat equation

$$\begin{aligned} w_t(\xi,t) &= c w_{\xi\xi}(\xi,t) + b(\xi)u(t) \\ w_\xi(0,t) &= w(1,t) = 0, \\ y(t) &= \int_0^1 b(\xi) w(\xi,t) d\xi \end{aligned}$$

with initial condition $w(\cdot,0) = w_0$. Here $c > 0$ is the (constant) conductivity of heat and $b = \chi_{[0,1/10]}$ is the input and output profile function. By [13, Ex. 2.36] and straightforward computations the system can be recast as a linear system of the form (1) on $X = L^2(\Omega)$. In particular, the system is exponentially stable, A is dissipative, $B \in \mathcal{L}(X,\mathbb{R})$, and $C = B^*$.

Our aim is to design a controller so that the closed-loop system has two stable equilibrium points for the nominal external input I_{base}, and the system can be steered from one of these equilibria to the other using pulses in I_{pulse}. To this end, we consider the neuromorphic controller (2) inspired by the FitzHugh–Nagumo model, with

$$f_c(v) = -\alpha v^3 + \beta v$$

for suitable $\alpha, \beta > 0$. By Theorem 3.1 the closed-loop system consisting of (1) and (2) has well-defined and uniformly bounded solutions on $[0,\infty)$ for all piecewise continuous and uniformly bounded inputs $I_{\text{app}} : [0,\infty) \to \mathbb{R}$. We have from Proposition 4.2 that if the controller parameters satisfy $\beta/\gamma > 3P(0)/2$, then the closed-loop system has two locally stable equilibrium points at $(\pm(-A)^{-1}Bv_*, \pm v_*) \in X \times \mathbb{R}$ with $v_* = \sqrt{(\beta - \gamma P(0))/\alpha} > 0$ (as well as an unstable equilibrium point at 0).

We demonstrate the effect of pulses in I_{pulse} using numerical simulations. For this purpose the heat equation is semi-discretised using the Finite Difference method with $N = 100$ points. In the simulations we use the parameters $c = 1.5$, $\alpha = 4$, $\beta = 10$, $\gamma = 8$, and $I_{\text{base}} = 0$. These parameters satisfy $\beta/\gamma > 3P(0)/2$, leading to two locally stable equilibrium points with $v_* \approx 1.03$.

Figure 3 shows the behaviour of the controller state resulting from input pulses of length 0.05 with different amplitudes. These input pulses do not exceed a threshold, and because of this the closed-loop state returns to the same equilibrium point.

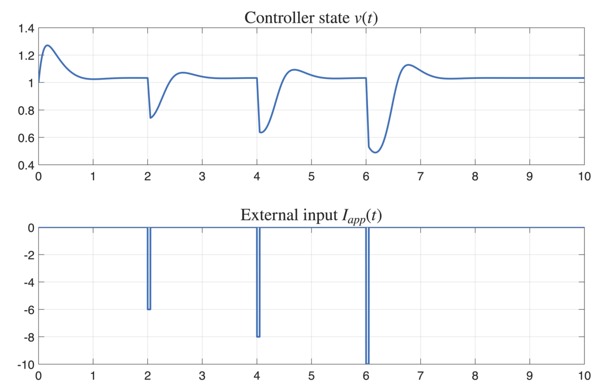

Fig. 3. The controller state for the heat system with subthreshold input pulses which do not trigger a change of the equilibrium state.

On the other hand, Figure 4 shows the behaviour of the controller state resulting from input pulses of length 0.05 with slightly higher amplitudes. Each pulse triggers an event where the closed-loop state is steered into the other stable equilibrium point. Figure 5 depicts the corresponding state of the heat system.

VI. Conclusions

In this paper we have investigated the well-posedness and excitability of infinite-dimensional linear systems with nonlinear non-monotone controllers. In the setting of our study, excitability refers to the capacity of the system to exhibit qualitatively different behaviour in response to external input pulses of different sizes. The more precise meaning of this property is nevertheless case-dependent. In the case of our beam equation with a single equilibrium point in Section V-A, the external inputs either led to a relatively direct return to the equilibrium, or alternatively caused a large deviation in the trajectory. In our controlled heat system in Section V-B

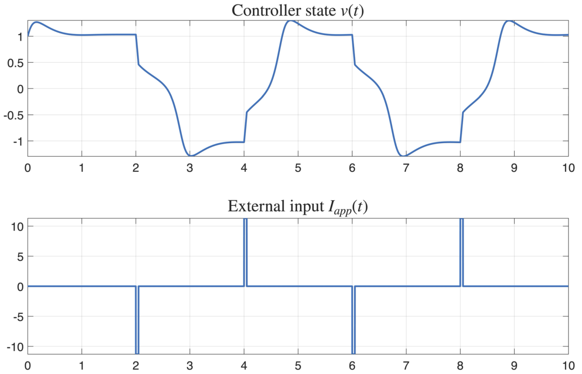

Fig. 4. The controller state for the heat system with suprathreshold input pulses which trigger changes of the equilibrium state.

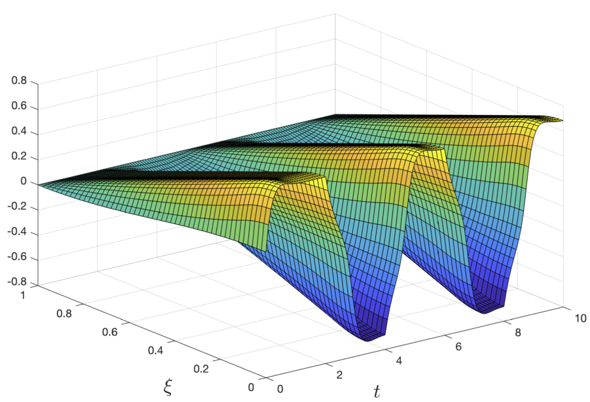

Fig. 5. The controlled heat profile with the suprathreshold input pulses.

the external input pulses either led to the trajectory to return to the same stable equilibrium point, or converge to the other.

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