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Stabilization to trajectories for parabolic equations

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Abstract Both internal and boundary feedback exponential stabilization to trajectories for semilinear parabolic equations in a given bounded domain are addressed. The values of the controls are linear combinations of a finite number of actuators which are supported in a small region. A condition on the family of actuators is given which guarantees the local stabilizability of the control system. It is shown that a linearization based Riccati feedback stabilizing controller can be constructed. The results of numerical simulations are presented and discussed.

Keywords feedback stabilization to trajectories · semilinear parabolic equations

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1 Introduction

We consider controlled parabolic equations, for time $t \geq 0$, in a C^∞-smooth domain $\Omega \subset \mathbb{R}^d$ located locally on one side of its boundary $\Gamma = \partial\Omega$, with d a positive integer. We will consider both the case of internal controls

$$\tfrac{\partial}{\partial t} y - \nu \Delta y + f(y, \nabla y) + f_0 = \sum_{i=1}^{M} u_i \Phi_i; \qquad y|_\Gamma = g; \tag{1}$$

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and the case of boundary controls

$$\tfrac{\partial}{\partial t}y - \nu\Delta y + f(y, \nabla y) + f_0 = 0; \qquad\qquad y|_\Gamma = g + \sum_{i=1}^{M} u_i\Psi_i. \tag{2}$$

In the variables $(t, x, \bar{x}) \in (0, +\infty) \times \Omega \times \Gamma$, the unknown in the equation is the function $y = y(t, x) \in \mathbb{R}$. The diffusion coefficient $\nu > 0$ is a positive constant; the functions $g = g(t, \bar{x}) \in \mathbb{R}$, $f\colon \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, and $f_0(t, x) \in \mathbb{R}$ are fixed.

The functions $\Phi_i = \Phi_i(x)$, respectively $\Psi_i = \Psi_i(\bar{x})$, will play the role of actuators, M is a positive integer, and $u = u(t) \in \mathbb{R}^M$ is a (control) vector function at our disposal.

The problem we address here is the *local* exponential stabilization to trajectories for systems (1) and (2). That is, given a positive constant $\lambda > 0$ and a solution $\hat{y}(t) = \hat{y}(t, \cdot)$ of the (uncontrolled) system with $u = 0$, we want to find a control function u such that the solution $y(t) \coloneqq y(t, \cdot)$ of the system, supplemented with the initial condition

$$y(0) \coloneqq y(0, x) = y_0(x),$$

is defined on $[0, +\infty)$ and approaches $\hat{y}(t)$ exponentially with rate $\frac{\lambda}{2}$, provided $y(0) - \hat{y}(0)$ is *small enough*. In other words, for a suitable Banach space X and positive constants C and ϵ, we want to have that

$$|y(t) - \hat{y}(t)|_X^2 \le C\mathrm{e}^{-\lambda t}\,|y(0) - \hat{y}(0)|_X^2\,, \quad \text{provided} \quad |y(0) - \hat{y}(0)|_X < \epsilon, \tag{3}$$

with ϵ *small enough*. Notice that, the constants C and ϵ may depend on λ, but neither on $\hat{y}(0)$ nor on $y(0)$.

We are particularly interested in actuators which are supported in a small domain: $\mathrm{supp}(\Phi_i) \subseteq \overline{\omega} \subseteq \overline{\Omega}$, $\mathrm{supp}(\Psi_i) \subseteq \overline{\Gamma_\mathrm{c}} \subseteq \Gamma$, where ω and Γ_c are given open subsets of Ω and Γ.

Internal and boundary actuators are taken from $L^2(\Omega)$ and $H^{\frac{3}{2}}(\Gamma)$, respectively. The linear span of the actuators will be denoted, respectively, by

$$\mathcal{S}_\Phi \coloneqq \mathrm{span}\{\Phi_i \mid i \in \{1, 2, \dots, M\}\} \subset L^2(\Omega),$$
$$\mathcal{S}_\Psi \coloneqq \mathrm{span}\{\Psi_i \mid i \in \{1, 2, \dots, M\}\} \subset H^{\frac{3}{2}}(\Gamma).$$

and the orthogonal projections onto the span of the actuators by P_M (using the same notation for both internal and boundary cases will lead to no ambiguity)

$$P_M\colon L^2(\Omega) \mapsto \mathcal{S}_\Phi, \qquad P_M\colon L^2(\Gamma) \mapsto \mathcal{S}_\Psi.$$

In order to state the main results let us denote the indicator operators:

$$1_\omega\colon L^2(\Omega) \mapsto L^2(\Omega), \quad \begin{cases} 1_\omega f(x) = f(x), & \text{if } x \in \omega, \\ 1_\omega f(x) = 0, & \text{if } x \in \Omega \setminus \overline{\omega}, \end{cases}$$
$$1_{\Gamma_\mathrm{c}}\colon L^2(\Gamma) \mapsto L^2(\Gamma), \quad \begin{cases} 1_{\Gamma_\mathrm{c}} g(\bar{x}) = g(\bar{x}), & \text{if } \bar{x} \in \Gamma_\mathrm{c}, \\ 1_{\Gamma_\mathrm{c}} g(\bar{x}) = 0, & \text{if } \bar{x} \in \Gamma \setminus \overline{\Gamma_\mathrm{c}}, \end{cases}$$

Without lack of generality, in either case we suppose that the families of actuators $\{\Phi_i \mid i \in \{1, 2, \dots, M\}\}$ and $\{\Psi_i \mid i \in \{1, 2, \dots, M\}\}$ are linearly independent. So, we consider the bijections $B_\Phi \colon \mathbb{R}^M \to \mathcal{S}_\Phi$ and $B_\Psi \colon \mathbb{R}^M \to \mathcal{S}_\Psi$,

$$B_\Phi u := \sum_{i=1}^{M} u_i \Phi_i, \qquad B_\Psi u := \sum_{i=1}^{M} u_i \Psi_i.$$

We will prove the stabilization results under a general condition on the pair $(\hat{y}, f)$, say $(\hat{y}, f) \in \mathfrak{C}$ for a suitable class $\mathfrak{C}$ to be precised hereafter.

The space of continuous linear mappings from a Banach space X into a Banach space Y will be denoted $\mathcal{L}(X, Y)$. When $X = Y$ we write simply $\mathcal{L}(X) := \mathcal{L}(X, X)$.

The usual Lebesgue and Sobolev spaces $L^p(\Omega)^m = L^p(\Omega, \mathbb{R}^m)$, with $p \in [1, +\infty]$, and $H^s(\Omega)^m = H^s(\Omega, \mathbb{R}^m)$, $s \geq 0$, will be denoted by simply $L^p(\Omega)$ and $H^s(\Omega)$, respectively, whenever there is no ambiguity concerning the superscript $m \in \mathbb{N}_0$. Sometimes to shorten the formulas we will write simply L^p and H^s, if there is no ambiguity concerning the domain Ω. Same notation for the spaces $L^p(\Gamma, \mathbb{R}^m) = L^p(\Gamma) = L^p$, and $H^s(\Gamma, \mathbb{R}^m) = H^s(\Gamma) = H^s$.

1.1 Review on the general procedure and main tools.

To derive the local Riccati-based feedback stabilization results we follow the following sequence of steps:

1. linearize the system around the reference trajectory.
2. find an appropriate set of actuators and construct a *globally* stabilizing open-loop control for the linear system.
3. use the dynamical programming principle and Karush-Kuhn-Tucker Theorem to find a time-dependent feedback control operator.
4. observe that the feedback operator satisfies a differential Riccati equation.
5. use a fixed point argument to prove that the same feedback operator also *locally* stabilizes the full nonlinear system.

The most difficult step is to construct an open-loop stabilizing control for the linearized system by means of a finite number of actuators supported in small regions. This has been done in previous works in the case of internal actuators, and a condition for stabilizability has been given in terms of the orthogonal projection $1 - P_M$. The main novelty of this paper is the construction of such stabilizing control with boundary actuators for parabolic equations, with a corresponding stabilizability condition in terms of the orthogonal projection $1 - P_M$. The second novelty is the consideration of a general class of nonlinearities f, for both internal and boundary cases. The third novelty is the presentation of numerical simulations for the boundary feedback control, confirming the theoretical results.

Below we give further comments on the general steps above.

Linearization of the system around the reference trajectory. We present the computations here because they will be useful to write down the conditions we ask for the targeted trajectory $\hat{y}$ and for the nonlinear function f in an easier way. Namely, as $(\hat{y}, f) \in \mathfrak{C}$, with $\mathfrak{C}$ defined in (8), hereafter.

We want the solution $y(t)$ to go to the reference trajectory $\hat{y}(t)$ exponentially. By direct computations, we find that $z \coloneqq y - \hat{y}$ solves

$$\tfrac{\partial}{\partial t} z - \nu \Delta z + f(y, \nabla y) - f(\hat{y}, \nabla \hat{y}) = \iota_{\rm ib} B_\Phi u, \qquad z|_\Gamma = (1 - \iota_{\rm ib}) B_\Psi u, \tag{4}$$

with $\iota_{\rm ib} = 1$ for (1) and $\iota_{\rm ib} = 0$ for (2).

Writing $(\xi^1, \xi^2) \in \mathbb{R} \times \mathbb{R}^d$ we denote $\partial_1 f \coloneqq \frac{\partial f}{\partial_{\xi^1}}$ and $\partial_2 f \coloneqq \frac{\partial f}{\partial_{\xi^2}}$. Formally,

$$f(y, \nabla y) - f(\hat{y}, \nabla \hat{y}) =: \left[\partial_1 f|_{(\hat{y}, \nabla \hat{y})} \; \partial_2 f|_{(\hat{y}, \nabla \hat{y})}\right] \begin{bmatrix} z \\ \nabla z \end{bmatrix} + F_{\hat{y}}(z) = \widehat{L} z - \widehat{\mathcal{N}}(z). \tag{5a}$$

with $\widehat{L} z \coloneqq \hat{a} z + \nabla \cdot (\hat{b} z)$ and

$$\hat{a} \coloneqq \partial_1 f|_{(\hat{y}, \nabla \hat{y})} - \nabla \cdot \partial_2 f|_{(\hat{y}, \nabla \hat{y})}, \quad \hat{b} \coloneqq \partial_2 f|_{(\hat{y}, \nabla \hat{y})}, \quad \text{and} \quad \widehat{\mathcal{N}}(z) = -F_{\hat{y}}(z), \tag{5b}$$

where the remainder $\widehat{\mathcal{N}}(\cdot) : \mathbb{R} \to \mathbb{R}$ either vanishes or is a nonlinear function.

We will be able to prove the local stabilization to the trajectory $\hat{y}$ provided the triple $(\hat{a}, \hat{b}, \widehat{\mathcal{N}})$, defined by $(\hat{y}, f)$, satisfies

$$\hat{a} \in L^\infty(\mathbb{R}_0, L^d(\Omega, \mathbb{R})), \tag{6a}$$

$$\hat{b} \in L^\infty_w(\mathbb{R}_0, L^\infty(\Omega, \mathbb{R}^d)), \qquad \nabla \cdot \hat{b} \in L^\infty_w(\mathbb{R}_0, L^r(\Omega, \mathbb{R})), \tag{6b}$$

with $r = 2$ if $d \in \{1, 2, 3\}$, $r = \infty$ if $d \geq 4$, and for a suitable constant $\widehat{C} > 0$,

$$\begin{aligned} |\widehat{\mathcal{N}}(z) - \widehat{\mathcal{N}}(\tilde{z})|^2_{L^2} \le \widehat{C} |z - \tilde{z}|^2_{H^1} &(1 + |z|^{\varepsilon_1}_{H^1} + |\tilde{z}|^{\varepsilon_2}_{H^1}) \left(|z|^2_{H^2} + |\tilde{z}|^2_{H^2}\right) \\ &+ \widehat{C} |z - \tilde{z}|^2_{H^2} \left(|z|^{\varepsilon_3}_{H^1} + |\tilde{z}|^{\varepsilon_4}_{H^1}\right), \end{aligned} \tag{7a}$$

and

$$\begin{aligned} &\left(\widehat{\mathcal{N}}(z) - \widehat{\mathcal{N}}(\tilde{z}), z - \tilde{z}\right)_{L^2} \\ &\le \widehat{C} (1 + |z|^{\varepsilon_5}_{H^1} + |\tilde{z}|^{\varepsilon_6}_{H^1})^{\frac{1}{2}} (1 + |z|^2_{H^2} + |\tilde{z}|^2_{H^2})^{\frac{1}{2}} \, |z - \tilde{z}|_{H^1} \, |z - \tilde{z}|_{L^2} \\ &\quad + \widehat{C} (1 + |z|^{\varepsilon_5}_{H^1} + |\tilde{z}|^{\varepsilon_6}_{H^1}) (1 + |z|^2_{H^2} + |\tilde{z}|^2_{H^2}) \, |z - \tilde{z}|^2_{L^2} \end{aligned} \tag{7b}$$

with $\{\varepsilon_1, \varepsilon_2\} \in [0, +\infty)$ and $\{\varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6\} \in [2, +\infty)$.

That is, for our results to hold, *the property asked for the pair* $(\hat{y}, f)$ is that it belongs to the class $\mathfrak{C}$ defined as follows

$$\mathfrak{C} \coloneqq \{(\hat{y}, f) \mid (\hat{a}, \hat{b}, \widehat{\mathcal{N}}) \text{ is defined by (5) and satisfies (6) and (7)}\}. \tag{8}$$

We see that our goal (3) is to find the control u, in system (4), such that

$$|z(t)|^2_X \le C \mathrm{e}^{-\lambda t} \, |z(0)|^2_X, \quad \text{provided} \quad |z(0)|_X < \epsilon.$$

for suitable positive constants $C = C_\lambda$ and $\epsilon = \epsilon_\lambda$.

Construction of a *globally* stabilizing control for the linear system. We consider system (4) without the nonlinearity $\widehat{\mathcal{N}}$,

$$\tfrac{\partial}{\partial t}z - \nu\Delta z + \hat{a}z + \nabla\cdot(\hat{b}z) = \iota_{\mathrm{ib}}B_{\Phi}u, \qquad z|_{\Gamma} = (1-\iota_{\mathrm{ib}})B_{\Psi}u.$$

Observe that $z(t)$ goes exponentially to zero with rate $\frac{\lambda}{2}$ if, and only if, $\mathrm{e}^{\frac{\lambda}{2}t}z(t)$ remains bounded. So, we consider the shifted system solved by $\mathrm{e}^{\frac{\lambda}{2}t}z(t)$:

$$\tfrac{\partial}{\partial t}z - \nu\Delta z + (\hat{a}+\tfrac{\lambda}{2})z + \nabla\cdot(\hat{b}z) = \iota_{\mathrm{ib}}B_{\Phi}\widetilde{u}, \qquad z|_{\Gamma} = (1-\iota_{\mathrm{ib}})B_{\Psi}\widetilde{u}.$$

• For *internal* actuators, $\iota_{\mathrm{ib}} = 1$, we can write the system as

$$\tfrac{\partial}{\partial t}z - \nu\Delta z + (\hat{a}+\tfrac{\lambda}{2})z + \nabla\cdot(\hat{b}z) = P_M\eta, \qquad z|_{\Gamma} = 0, \tag{9}$$

where now we look for a stabilizing control $\eta \in L^2((0,+\infty), L^2(\Omega))$, taking its values in $L^2(\Omega)$ and such that $z \in L^2((0,+\infty), H^1(\Omega))$. The key tool used to find η is the *null controllability* of the system

$$\tfrac{\partial}{\partial t}z - \nu\Delta z + (\hat{a}+\tfrac{\lambda}{2})z + \nabla\cdot(\hat{b}z) = 1_{\omega}\eta, \qquad z|_{\Gamma} = 0, \tag{10}$$

at final time $t = s_1$, for $t \in (s_0, s_1)$, $0 < s_0 < s_1 < +\infty$. Notice that we cannot guarantee that $\{1_{\omega}\eta(t) \in L^2(\omega) \mid t \in (s_0,s_1)\}$ is a subset of a finite-dimensional space, that is, $1_{\omega}\eta$ is an infinite-dimensional control, in general. To find suitable actuators and a finite-dimensional control, suitable truncated observability inequalities can be used as in [13] (together with the smoothing property of the parabolic system). Another option is to use a suitable boundedness/smallness condition on the operator $1 - P_M$ as in [17, 29]. The latter approach led to some estimates on the number M of actuators that allow us to stabilize the system, for example for piecewise constant actuators. The main idea is to construct a control recursively in each interval $J^i := (iT_*, (i+1)T_*)$, $i \in \mathbb{N}$ such that $|z((i+1)T_*)|_H \le \rho\,|z(iT_*)|_H$, where $\rho < 1$. Then, we just take the concatenation of such controls. The time-length T_* is at our disposal, but it will be chosen to somehow minimize "the" cost of null controllability. The control in the interval J^i is constructed as follows: firstly from the null controllability of (10) we can take a control η driving (10) from $z(iT_*)$, at time $t = iT_*$, to $z((i+1)T_*) = 0$, at time $t = iT_* + T_*$, such that $|\eta|^2_{L^2(J^i,L^2(\Omega))} \le C_{\mathrm{nc}}(T_*)\,|z(iT_*)|^2_{L^2(\Omega)}$, secondly we observe that the difference d between the solutions of (10) and (9) satisfies

$$\tfrac{\partial}{\partial t}d - \nu\Delta d + (\hat{a}+\tfrac{\lambda}{2})d + \nabla\cdot(\hat{b}d) = 1_{\omega}(1-P_M)1_{\omega}\eta, \qquad z|_{\Gamma} = 0,$$

and $|d(s)|^2_{L^2} \le \widehat{\Upsilon}(T_*)\,|1_{\omega}(1-P_M)1_{\omega}|^2_{\mathcal{L}(L^2(\Omega),H^{-1}(\Omega))}\,|z(iT_*|^2_{L^2(\Omega)}$, where for suitable constants C_1 and C_2, $\widehat{\Upsilon}(T_*) = C_2\mathrm{e}^{C_1T_*}C_{\mathrm{nc}}(T_*)$. Therefore, if we have that $|1_{\omega}(1-P_M)1_{\omega}|^2_{\mathcal{L}(L^2(\Omega),H^{-1}(\Omega))} \le \rho(\widehat{\Upsilon}(T_*))^{-1}$, with $0 < \rho < 1$, then the solution of system (9), issued from $z(iT_*)$ at time $t = iT_*$, satisfies $|z((i+1)T_*)|^2_{L^2(\Omega)} = |d((i+1)T_*)|^2_{L^2(\Omega)} \le \rho\,|z(iT_*)|^2_{L^2(\Omega)}$.

The constants $\Upsilon(T_*)$ and M may be taken the same in each interval J_i, $i \in \mathbb{N}$, due to the conditions (6). For further details we refer to [17, 29, 30].

• For *boundary* actuators, $\iota_{\rm ib} = 0$, we will find suitable actuators and construct a stabilizing control for the system

$$\tfrac{\partial}{\partial t} z - \nu\Delta z + (\hat{a} + \tfrac{\lambda}{2})z + \nabla \cdot (\hat{b}z) = 0, \qquad z|_\Gamma = B_\Psi u, \tag{11}$$

so that $z \in L^2((0,+\infty), H^1(\Omega))$. Again the smoothing property and the *null controllability* of the system

$$\tfrac{\partial}{\partial t} z - \nu\Delta z + (\hat{a} + \tfrac{\lambda}{2})z + \nabla \cdot (\hat{b}z) = 0, \qquad z|_\Gamma = 1_{\Gamma_{\rm c}}\zeta \tag{12}$$

at final time $t = s_1$, for $t \in (s_0, s_1)$, $0 < s_0 < s_1 < +\infty$, will play a key role, where at each time $\zeta(t)$ takes values in a suitable *infinite-dimensional* subspace of $H^{\frac{3}{2}}(\Gamma)$.

The open-loop stabilizing control will again be constructed recursively in the time intervals $J^i = (iT_*, (i+1)T_*)$, for a suitable T_*.

The constructed control u in each interval $J^i = (iT_*, (i+1)T_*)$ will belong to $H^1(J^i, \mathbb{R}^M)$, which implies that $\varkappa := \frac{\partial}{\partial t} u - \frac{\lambda}{2} u + \varsigma u \in L^2(J^i, \mathbb{R}^M)$. That is, we have the control dynamics

$$\tfrac{\partial}{\partial t} u = -\varsigma u + \tfrac{\lambda}{2} u + \varkappa, \qquad \varkappa \in L^2(J^i, \mathbb{R}^M).$$

The advantage of having such a dynamical control is that we will be able us to rewrite (11) in a canonical extended form, where the control operator is bounded. Indeed, for each actuator $\Psi_i \in H^{\frac{3}{2}}(\Gamma)$, we will take the extension $\widehat{\Psi}_i \in H^2(\Omega)$, which solves the elliptic system

$$-\nu\Delta\widetilde{\Psi}_i + \varsigma\widetilde{\Psi}_i = 0, \qquad \widetilde{\Psi}_i|_\Gamma = \Psi_i,$$

and set the bijection $B_{\widetilde{\Psi}} \colon \mathbb{R}^M \to \mathcal{S}_{\widetilde{\Psi}}$, with $\mathcal{S}_{\widetilde{\Psi}} := \operatorname{span}\{\widetilde{\Psi}_i \mid i \in \{1, \dots, M\}\}$,

$$B_{\widetilde{\Psi}}\kappa := \sum_{i=1}^{M} \kappa_i \widetilde{\Psi}_i.$$

Now, we can consider the extended system (cf. [5, 42]) for the new variables $(v, \kappa) = (z - B_{\widetilde{\Psi}} u, u) \in H^1_0(\Omega) \times \mathbb{R}^M$:

$$\tfrac{\partial}{\partial t} \begin{bmatrix} v \\ \kappa \end{bmatrix} + \begin{bmatrix} -\nu\Delta + \widehat{L} - \frac{\lambda}{2} & \widehat{L}_\varsigma B_{\widetilde{\Psi}} \\ 0 & \varsigma - \frac{\lambda}{2} \end{bmatrix} \begin{bmatrix} v \\ \kappa \end{bmatrix} = \begin{bmatrix} -B_{\widetilde{\Psi}} \\ 1 \end{bmatrix} \varkappa, \tag{13}$$

with $\widehat{L}w := \hat{a}w + \nabla\cdot(\hat{b}w)$ and $\widehat{L}_\varsigma w := \widehat{L}w - 2\varsigma w$. Our (new) control function is $\varkappa$. In particular, the control operator is bounded, $\begin{bmatrix} -B_{\widetilde{\Psi}} \\ 1 \end{bmatrix} \in \mathcal{L}(\mathbb{R}^M, L^2(\Omega) \times \mathbb{R}^M)$.

Remark 1.1 Usually the variable u stands for control. This is why we renamed $\kappa := u$ to underline that in the extended system κ is not a control (it is part of the state). Of course we could simply take $\varsigma = 0$, however, taking $\varsigma \geq 0$ will not bring additional difficulties and, as we observed in numerical simulations, the value of ς may play an important role [35, section 9.5].

Deriving the desired time regularity for the control. Proceeding as in the internal case we may look for the control in (11) in the form $P_M\zeta$ where ζ drives (11) to zero at final time. As we will see this standard procedure will give us a control $u \in H^{\frac{3}{4}}(J^i, \mathbb{R}^M)$. To obtain the desired extra regularity $u \in H^1([0, +\infty), \mathbb{R}^M)$, we will use an extra suitable projection in $L^2(J^i)$ with range contained in $H^1_0(J^i) \overset{\mathrm{d}}{\hookrightarrow} L^2(J^i)$, together with a suitable density argument. To prove that (after concatenation) we will $u \in H^1([0, +\infty), \mathbb{R}^M)$ we will use an uniform property on i (cf. Lemma 2.8).

Remark 1.2 In [42], the analogous result concerning the existence of an open-loop boundary stabilizing control for the linearized Navier–Stokes equations is proven, using the corresponding null controllability result [40] and suitable boundary observability inequalities [41]. The procedure we follow here is different, instead of deriving the appropriate truncated observability inequalities for parabolic equations, we give a condition for stabilizability depending on the operator $1 - P_M$. Then we find a set of actuators satisfying the condition. Further, such condition allow us to derive estimates on the number M of actuators that will allow us to stabilize the system (for actuators taken from a suitable class of functions).

Finding a feedback rule. Once we have the existence of a control stabilizing an evolutionary linear system $\dot{v} = -\mathbb{A}(t)v + \mathbb{B}u$ in a Hilbert space $\mathbf{H}$, with bounded control operator $\mathbb{B} \in \mathcal{L}(\mathbb{R}^M, \mathbf{H})$, that is,

$$|v|^2_{L^2(\mathbb{R}_0, \mathbf{H})} + |u|^2_{L^2(\mathbb{R}_0, \mathbb{R}^M)} \le C\, |v(0)|^2_{\mathbf{H}}\,,$$

we can look for the optimal control minimizing a suitable linear quadratic cost. Then, through the dynamical programming principle and the Karush-Kuhn-Tucker Theorem, we will conclude that the optimal control is given in feedback form $u = -\mathbb{B}\Pi v$ where $(\Pi(s)w, w)_{\mathbf{H}}$ is the optimal "cost to go" with initial condition $v(s) = w$. For more details see [13, 30].

In both internal and boundary cases we can write our linearized system as

$$\dot{v} = -\mathbb{A}_\lambda(t)v + \mathbb{B}u \tag{14}$$

where, in the internal case

$$\mathbf{H} = L^2(\Omega), \quad \mathbb{A}_\lambda = -\nu\Delta + \widehat{L} - \tfrac{\lambda}{2}, \quad \text{and} \quad \mathbb{B} = P_M,$$

and, in the boundary case

$$\mathbf{H} = L^2(\Omega) \times \mathbb{R}^M, \quad \mathbb{A}_\lambda = \begin{bmatrix} -\nu\Delta + \widehat{L} - \frac{\lambda}{2} & \widehat{L}_\varsigma B_{\widetilde{\Psi}} \\ 0 & \varsigma - \frac{\lambda}{2} \end{bmatrix}, \quad \text{and} \quad \mathbb{B} = \begin{bmatrix} -B_{\widetilde{\Psi}} \\ 1 \end{bmatrix}.$$

Differential Riccati equation. Computation of the feedback rule. In case our linear quadratic running cost reads $\int_s^{+\infty} |\mathcal{M}v(\tau)|^2_{\mathbf{H}} + |u(\tau)|^2_{\mathbb{R}^M}\, \mathrm{d}\tau$, then the symmetric bounded feedback operator $\Pi = \Pi^* \colon \mathbf{H} \to \mathbf{H}$ will satisfy the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Pi - \Pi\mathbb{A}_\lambda - \mathbb{A}_\lambda^*\Pi - \Pi\mathbb{B}\mathbb{B}^*\Pi + \mathcal{M}^*\mathcal{M} = 0, \tag{15}$$

For simplicity, let us denote

$$\mathbf{V} = H_0^1(\Omega) \quad \text{and} \quad \mathbf{D}(\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$$

in the internal case, and

$$\mathbf{V} = H_0^1(\Omega) \times \mathbb{R}^M \quad \text{and} \quad \mathbf{D}(\Delta) = (H^2(\Omega) \cap H_0^1(\Omega)) \times \mathbb{R}^M$$

in the boundary case.

Formally, equality (15) may be understood in the weak sense

$$\begin{aligned} &((\tfrac{\mathrm{d}}{\mathrm{d}t}\Pi)z, w)_{L^2(\Omega)} - (\Pi\mathbb{A}_\lambda z, w)_{L^2(\Omega)} - (\mathbb{A}_\lambda^*\Pi z, w)_{L^2(\Omega)} \\ =\ &(\Pi\mathbb{B}\mathbb{B}^*\Pi z, w)_{L^2(\Omega)} - (\mathcal{M}^*\mathcal{M}z, w)_{L^2(\Omega)}, \quad \text{for all} \quad (z, w) \in \mathbf{D}(\Delta)^2. \end{aligned}$$

see [13]. More precisely, we say that Π satisfies (15), in the time interval $(0, T)$, $T > 0$, if for all $z \in \mathbf{D}(\Delta)$, see [20, 30],

$$\Pi(t)z = \mathcal{U}_{(t,s)}^*\Pi(s)\mathcal{U}_{(t,s)}z + \int_s^t \mathcal{U}_{(r,s)}^*\left(\Pi(r)\mathbb{B}\mathbb{B}^*\Pi(r) - \mathcal{M}^*\mathcal{M}\right)\mathcal{U}_{(r,s)}\,\mathrm{d}r, \tag{16}$$

where $\mathcal{U}_{(t,s)}^* w$, with $0 < s < t < T$, stands for the solution of

$$\dot{y} = -\mathbb{A}_\lambda y, \quad y(s) = w, \quad \text{with} \quad w \in \mathbf{H}.$$

Remark 1.3 Notice that once $\Pi(t)z$ is defined by (16) for all $z \in \mathbf{D}(\Delta)$, then by a density argument it can be extended for all $z \in \mathbf{H}$. Notice also, in particular, that (16) makes sense because $\mathbb{B} \colon \mathbf{H} \to \mathbf{H}$ and and $-\mathbb{A}_\lambda \colon \mathbf{D}(\Delta) \to \mathbf{H}$ are bounded. The boundedness of $-\mathbb{A}_\lambda \colon \mathbf{D}(\Delta) \to \mathbf{H}$ will follow from (6) (see (24) hereafter).

Notice that, $\Pi = \Pi_\lambda$ has been obtained prior to its dynamics. Therefore, at this point the existence of a solution for the Riccati equation is known. The uniqueness can be guaranteed in the class of families $P \in L^\infty(\mathbb{R}_0, \mathcal{L}(\mathbf{H}))$ such that $P(t)$ is self-adjoint and positive definite for all $t > 0$, and the family $\{P(t) \mid t > 0\}$ is continuous in the weak operator topology. See [30]. For further references concerning the Riccati equations we refer to [19, 47].

Stabilization of the nonlinear system. We consider the unshifted system (14) with the feedback control, and perturbed with the nonlinearity,

$$\tfrac{\partial}{\partial t}v + \mathbb{A}_0 v + \mathbb{B}\mathbb{B}^*\Pi_\lambda v = \mathcal{N}v, \qquad v(0) = v_0 \in \mathbf{H}. \tag{17}$$

where $\mathcal{N} = \widehat{\mathcal{N}}$ in the internal case, and $\mathcal{N} = \begin{bmatrix} \widehat{\mathcal{N}} \\ 0 \end{bmatrix}$ in the boundary case. At this point, the linear system $\frac{\partial}{\partial t}v - \mathbb{A}_0 v + \mathbb{B}\mathbb{B}^* \Pi_\lambda v = 0$ is known to be globally exponentially stable, with rate $\frac{\lambda}{2}$. That is, $|v(t)|_{\mathbf{H}}^2 \le C\mathrm{e}^{-\lambda t}\,|v(0)|_{\mathbf{H}}^2$.

The space $\mathbf{H}$ is considered as a pivot space, $\mathbf{H} = \mathbf{H}'$. Since the Cauchy problem for the nonlinear system is in general not well-posed for so called weak solutions $v \in \{L^2_{\mathrm{loc}}((0,+\infty),\mathbf{V}) \mid \frac{\partial}{\partial t}v \in L^2_{\mathrm{loc}}(\mathbb{R}_0,\mathbf{V}')\}$, we need to guarantee firstly that the solution for the linear system is strong, $v \in \{L^2_{\mathrm{loc}}(\mathbb{R}_0,\mathbf{D}(\Delta) \mid \frac{\partial}{\partial t}v \in L^2_{\mathrm{loc}}(\mathbb{R}_0,\mathbf{H})\}$. This is possible form conditions (6), and from the compatibility condition $v(0) \in \mathbf{V}$.

Secondly, we will conclude the *local* exponential stability of the nonlinear system, with the same rate $\frac{\lambda}{2}$, by following a standard fixed point argument. We look for the fixed point in a subset

$$\mathcal{Z}^\lambda_\varrho := \left\{ v \in L^2_{\mathrm{loc}}(\mathbb{R}_0,\mathbf{H}) \;\middle|\; \sup_{r\ge 0} \left| \mathrm{e}^{\frac{\lambda}{2}\cdot} v \right|_{W((r,r+1),\mathbf{D}(\Delta),\mathbf{H})} \le \varrho |v(0)|^2_{\mathbf{V}} \right\},$$

for an appropriate $\varrho > 0$. In particular, we need the strong solutions of the linearized closed-loop system to go exponentially to zero in the $H^1(\Omega)$-norm, $|v(t)|^2_{\mathbf{V}} \le C_1\mathrm{e}^{-\lambda t}\,|v(0)|^2_{\mathbf{V}}$, which will follow from the smoothing property for parabolic equations, $|v(s+1)|^2_{\mathbf{V}} \le C_2\,|v(s)|^2_{\mathbf{H}}$ (with C_2 independent of $s \ge 0$, due to (6)). Then from standard estimates on the linear parabolic systems it also follow that, indeed the strong solutions v are in $L^2_{\mathrm{loc}}(\mathbb{R}_0,\mathbf{D}(\Delta))$ and $|v|_{W((r,r+1),\mathbf{D}(\Delta),\mathbf{H})} \le C_3\,|v(r)|^2_{\mathbf{V}} \le C_4\mathrm{e}^{-\lambda r}\,|v(0)|^2_{\mathbf{V}}$.

The fixed point argument is based on the mapping $\bar{v} \mapsto v$ where v solves

$$\frac{\partial}{\partial t}v - \mathbb{A}_0 v + \mathbb{B}\mathbb{B}^* \Pi_\lambda v = \mathcal{N}\bar{v}, \qquad v(0) = v_0 \in \mathbf{V}.$$

See (17). Such mapping will be a contraction provided $|v_0|_{\mathbf{V}}$ is small enough.

Though, the fixed point argument above is standard, we would like to mention that we will consider a general class of nonlinearities.

Remark 1.4 We look for, and find, the fixed point in $\mathcal{Z}^\lambda_\varrho$. Using this set we will be able to use some results in previous works (e.g., as in Step 1 in the proof of Theorem 3.1). It *seems* possible that by looking for the fixed point in a subset of the more classical space $\mathrm{e}^{\frac{\hat{\lambda}}{2}\cdot} W(\mathbb{R}_0,\mathbf{D}(\Delta),\mathbf{H})$ (with $\hat{\lambda} < \lambda$) we would be able to find it as well. However, the details should be checked.

1.2 The main results

Here we state the main results of the paper. Recall the class $\mathfrak{C}$ defined in (8).

Main Theorem 1.1 (Internal case) *Let $\hat{y}$ solve system* (1) *with $u = 0$, and let $(\hat{y}, f) \in \mathfrak{C}$. Then for any given $\lambda > 0$, there is a constant $\widehat{\Upsilon} > 0$ such that: if*

$$|1_\omega(1 - P_M)1_\omega|^2_{\mathcal{L}(L^2(\Omega),H^{-1}(\Omega))} < \widehat{\Upsilon}^{-1},$$

then there exists a family of linear operators $\{\varPi_\lambda(t) \in \mathcal{L}(L^2(\varOmega)) \mid t > 0\}$ *such that the following properties hold true:*

(i) *the mapping* $t \mapsto \varPi_\lambda(t)$ *is continuous in the weak operator topology;*
(ii) *there exists* $\epsilon > 0$ *such that: if*

$$y_0 - \hat{y}_0 \in H_0^1(\varOmega) \quad \textit{and} \quad |y_0 - \hat{y}_0|_{H^1(\varOmega)} < \epsilon,$$

then the solution y *of the system*

$$\tfrac{\partial}{\partial t} y - \nu \varDelta y + f(y, \nabla y) + f_0 = -P_M \varPi_\lambda (y - \hat{y}), \quad y|_\varGamma = g, \quad y(0) = y_0,$$

exists, and is unique, in the affine space

$$\hat{y} + L^2_{\mathrm{loc}}(\mathbb{R}_0, H^2(\varOmega)) \cap C([0, +\infty), H_0^1(\varOmega)),$$

and satisfies

$$|y(t) - \hat{y}(t)|^2_{H^1(\varOmega)} \le C\mathrm{e}^{-\lambda t} |y_0 - \hat{y}_0|^2_{H^1(\varOmega)}, \quad \textit{for all} \quad t \ge 0,$$

for a suitable constant C *independent of* $(\epsilon, y_0 - \hat{y}_0)$.

Now we introduce the space

$$\mathcal{H}(\varGamma_\mathrm{c}) \coloneqq \left\{ \gamma \in \mathcal{L}(H^{\frac{3}{2}}(\varGamma), H^{\frac{1}{2}}(\varGamma)) \cap \mathcal{L}(L^2(\varGamma), H^{-\frac{1}{2}}(\varGamma)) \mid \operatorname{supp} \gamma \subseteq \overline{\varGamma_\mathrm{c}} \right\}. \tag{18}$$

Main Theorem 1.2 (Boundary case) *Let* $\hat{y}$ *solve system* (2) *with* $u = 0$, *and let* $(\hat{y}, f) \in \mathfrak{C}$. *Then for any given* $\lambda > 0$, *there is a constant* $\widehat{\varUpsilon} > 0$ *such that: if*

$$|1 - P_M|^2_{\mathcal{H}(\varGamma_\mathrm{c})} < \widehat{\varUpsilon}^{-1},$$

then there exists a family of linear operators $\{\varPi_\lambda(t) \in \mathcal{L}(L^2(\varOmega) \times \mathbb{R}^M) \mid t > 0\}$ *such that the following properties hold true:*

(i) *the mapping* $t \mapsto \varPi_\lambda(t)$ *is continuous in the weak operator topology;*
(ii) *there exists* $\epsilon > 0$ *such that: if*

$$y_0 - \hat{y}_0 \in H^1(\varOmega), \quad (y_0 - \hat{y}_0)|_\varGamma \in \mathcal{S}_\varPsi, \quad \textit{and} \quad |y_0 - \hat{y}_0|_{H^1(\varOmega)} < \epsilon,$$

then the solution y *of the system*

$$\begin{aligned} &\tfrac{\partial}{\partial t} y - \nu \varDelta y + f(y, \nabla y) + f_0 = 0, \quad y|_\varGamma = g + B_\varPsi \kappa, \quad y(0) = y_0,\\ &\tfrac{\partial}{\partial t} \kappa + \varsigma \kappa = \begin{bmatrix} B^*_{\widetilde{\varPsi}} & -1 \end{bmatrix} \varPi_\lambda \begin{bmatrix} y - \hat{y} - B_{\widetilde{\varPsi}} \kappa \\ \kappa \end{bmatrix}, \quad \kappa(0) = (B_\varPsi)^{-1} \left((y_0 - \hat{y}_0)|_\varGamma \right) \end{aligned}$$

exists, and is unique, in the affine space

$$\hat{y} + L^2_{\mathrm{loc}}(\mathbb{R}_0, H^2(\varOmega)) \bigcap C([0, +\infty), H^1(\varOmega)),$$

and satisfies

$$|y(t) - \hat{y}(t)|^2_{H^1(\varOmega)} \le C\mathrm{e}^{-\lambda t} |y_0 - \hat{y}_0|^2_{H^1(\varOmega)}, \quad \textit{for all} \quad t \ge 0,$$

for a suitable constant C *independent of* $(\epsilon, y_0 - \hat{y}_0)$.

Remark 1.5 In the Main Theorems 1.1 and 1.2 we have the conditions $(y_0 - \hat{y}_0)|_\Gamma = 0$ and $(y_0 - \hat{y}_0)|_\Gamma \in \mathcal{S}_\Psi$, respectively. These are compatibility conditions which are necessary to have strong solutions in $W(\mathbb{R}_0, H^2(\Omega), L^2(\Omega))$ for the system (4) solved by $y - \hat{y}$. In fact, to have strong solutions $v \in W(\mathbb{R}_0, \mathbf{D}(\Delta), \mathbf{H})$ for the linearized system (14), we will need that $v(0) \in \mathbf{V}$. Strong solutions will guarantee the existence and uniqueness of the solution for the nonlinear systems.

In terms of the difference to the target $y - \hat{y}$, in the internal case the feedback control is given in linear form: $y - \hat{y} \mapsto K_{\mathrm{int}}(y - \hat{y})$ with $K_{\mathrm{int}} = P_M \Pi_\lambda \in \mathcal{L}(L^2(\Omega), \mathcal{S}_\Phi)$. In the boundary case the control function $u = \kappa$ is given in dynamic form. In terms of the difference to the target $y - \hat{y}$, the boundary feedback is given in *integral* linear form

$$y|_\Gamma = g + \mathrm{e}^{-\varsigma \cdot}(y_0 - \hat{y}_0)|_\Gamma + B_\Psi \left(\int_0^\cdot \mathrm{e}^{-\varsigma(\cdot - \tau)} K_{\mathrm{bdry}}(y(\tau) - \hat{y}(\tau)) \,\mathrm{d}\tau \right)$$

with $K_{\mathrm{bdry}} = [B^*_{\widetilde{\Psi}} \; -1] \Pi_\lambda \begin{bmatrix} 1 - ((B_\Psi)^{-1} \circ (\cdot|_\Gamma)) \\ (B_\Psi)^{-1} \circ (\cdot|_\Gamma) \end{bmatrix} \in \mathcal{L}(H^1_0(\Omega) + \mathcal{S}_{\widetilde{\Psi}}, \mathbb{R}^M)$.

The operators Π_λ may be taken as the solution of the corresponding Riccati equation (15).

The constant $\widehat{\Upsilon} = \widehat{\Upsilon}(T_*)$ will depend on (and increase with) the cost of null controllability of systems (10) and (12) in intervals $(jT_*, (j+1)T_*)$, $j \in \mathbb{N}$, which are used to construct recursively an open-loop stabilizing control.

Since a key tool for the procedure is the null controllability of the linearized system, we would like to refer to a short list of works related with null controllability, observability inequalities, and exact controllability to trajectories. Namely to [6, 26–28, 40, 51], see also references therein.

Though the details must be checked, it is plausible that the entire procedure can be followed (adapted) for systems of several coupled parabolic equations, provided we have the null controllability of the linearized systems [1, 2].

We are particularly interested in stabilization to time-dependent trajectories, which are important for applications where external forces depend on time, $f_0 = f_0(t)$. Notice that in such cases, the free-dynamics (uncontrolled) trajectories are necessarily time-dependent. That is, the uncontrolled system has no equilibria (steady states). Of course, when time-independent solutions do exist, then it makes sense to consider the problem of stabilization to an equilibrium, which has been studied for the last years by many authors for several systems and is by now quite well understood, we refer to [7–12, 23, 24, 34, 36, 37, 39, 48] and references therein. At this point we must say that the spectral approach used in the case of a targeted time-independent solution are not (or, seem not to be) appropriate to deal with the case of time-dependent targeted solutions, as the examples in [50] do suggest.

1.3 Contents and notation

The paper mainly focuses on the case of boundary actuators. Section 2 concerns the boundary stabilization to zero of the linearized system (11), provided the pair $(\hat{a}, \hat{b})$ satisfies (6). We prove that there exists a family of actuators $\{\Psi_i \mid i \in \{1, 2, \dots, M\}\}$ (satisfying the conditions in Main Theorem 1.2) and $\kappa \in L^2(\mathbb{R}_0, \mathbb{R}^M)$ such that the solution of the system (13) satisfies, in particular, $(v, \kappa) \in L^2(\mathbb{R}_0, (H^1_0(\Omega) \cap H^2(\Omega)) \times \mathbb{R}^M)$, if $(v(0), \kappa(0)) \in H^1_0(\Omega) \times \mathbb{R}^M$. In Section 3 we deal with the feedback boundary stabilization to zero of the nonlinear (full) system

$$\frac{\partial}{\partial t}\begin{bmatrix} v \\ \kappa \end{bmatrix} + \begin{bmatrix} -\nu\Delta + \widehat{L} & \widehat{L}_\varsigma B_{\widetilde{\Psi}} \\ 0 & \varsigma \end{bmatrix} \begin{bmatrix} v \\ \kappa \end{bmatrix} + \begin{bmatrix} -B_{\widetilde{\Psi}} \\ 1 \end{bmatrix} \begin{bmatrix} -B_{\widetilde{\Psi}} \\ 1 \end{bmatrix}^* \Pi_\lambda \begin{bmatrix} v \\ \kappa \end{bmatrix} = \begin{bmatrix} \widehat{\mathcal{N}}(v + B_{\widetilde{\Psi}}\kappa) \\ 0 \end{bmatrix}$$

provided the nonlinearity $\widehat{\mathcal{N}}$ satisfies (7). Recall that if (v, κ) satisfies the dynamics of the latter system, then (y, κ) with $y = v + \hat{y} + B_{\widetilde{\Psi}}\kappa$ satisfies the dynamics of the system in the Main Theorem 1.2. The case of internal controls is briefly revisited in Section 4. In Section 5 we show that our condition $(\hat{y}, f) \in \mathfrak{C}$, that is, (6) and (7), is satisfied for regular enough $\hat{y}$ and for some polynomial nonlinearities, which appear in several models of real world evolution processes. Finally, Section 6 contains the results of some numerical simulations, for both internal and boundary actuators, showing that the feedback control can stabilize systems whose free dynamics is unstable. The Appendix gathers the proofs of auxiliary results needed in the main text.

Notation. We write $\mathbb{R}$ and $\mathbb{N}$ for the sets of real and nonnegative integer numbers, respectively, and we define $\mathbb{R}_a := (a, +\infty)$ for all $a \in \mathbb{R}$, and $\mathbb{N}_0 := \mathbb{N} \setminus \{0\}$.

We denote by $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}_0$, a bounded C^∞-smooth domain with boundary $\Gamma = \partial\Omega$.

Given an open interval $I \subseteq \mathbb{R}$, and Banach spaces X and Y, we write $W(I, X, Y) := \{f \in L^2(I, X) \mid \frac{\partial}{\partial t} f \in L^2(I, Y)\}$, where the derivative $\frac{\partial}{\partial t} f$ is taken in the sense of distributions. This space is endowed with the natural norm $|f|_{W(I,X,Y)} := \left(|f|^2_{L^2(I,X)} + |\frac{\partial}{\partial t} f|^2_{L^2(I,Y)}\right)^{1/2}$. If the inclusion $X \subseteq Y$ is continuous, we write $X \hookrightarrow Y$; we write $X \overset{\mathrm{d}}{\hookrightarrow} Y$, respectively $X \overset{\mathrm{c}}{\hookrightarrow} Y$, if the inclusion is also dense, respectively compact. If $X \hookrightarrow \mathcal{H}$ and $Y \hookrightarrow \mathcal{H}$ for a Hausdorff topological space $\mathcal{H}$, then $X \cap Y$ is a Banach space, with $|\cdot|_{X \cap Y} := (|\cdot|^2_X + |\cdot|^2_Y)^{\frac{1}{2}}$.

Given a Hilbert space H, with scalar product $(\cdot, \cdot)_H$ and a subset $S \subseteq H$, the subspace orthogonal to S will be denoted $S^\perp := \{h \in H \mid (h, s)_H = 0 \text{ for all } s \in S\}$, as usual.

$\overline{C}_{[a_1,\dots,a_k]}$ denotes a function of nonnegative variables a_j that increases in each of its arguments, and C, C_i, $i = 1, 2, \dots$, stand for positive constants.

2 Boundary stabilization of the linearized system

We start by briefly recalling some classical results in Sections 2.1 and 2.2. Then in Sections 2.2, 2.3 and 2.4 we construct a stabilizing control provided a suitable stabilizability condition depending on $1 - P_M$ is satisfied (cf. Main Theorem 1.2 and Theorem 2.4). This stabilizability condition is one of the main results of the paper. In Section 2.5 we present a family of actuators satisfying the stabilizability condition. Finally in Section 2.6 we briefly revisit the procedure on the construction of a stabilizing feedback rule once the existence of a stabilizing control is known.

We consider a system in the form (4)-(5), without the nonlinearity $\widehat{\mathcal{N}}$. In order to study such system we start by denoting the Hilbert space $H \coloneqq L^2(\Omega, \mathbb{R})$ which we will consider as a pivot space, $H' = H$. We also denote $V \coloneqq H^1_0(\Omega, \mathbb{R})$ and $\mathrm{D}(\Delta) \coloneqq V \cap H^2(\Omega, \mathbb{R})$, which are supposed to be endowed with the scalar products

$$(v, w)_V \coloneqq (\nabla v, \nabla w)_{L^2(\Omega,\mathbb{R}^d)} \quad \text{and} \quad (v, w)_{\mathrm{D}(\Delta)} \coloneqq (\Delta v, \Delta w)_H,$$

and corresponding norms $|v|_V \coloneqq (v, v)_V^{\frac{1}{2}}$ and $|v|_{\mathrm{D}(\Delta)} \coloneqq (v, v)_{\mathrm{D}(\Delta)}^{\frac{1}{2}}$. We have

$$\mathrm{D}(\Delta) \xhookrightarrow{\mathrm{d,c}} V \xhookrightarrow{\mathrm{d,c}} H \xhookrightarrow{\mathrm{d,c}} V' \xhookrightarrow{\mathrm{d,c}} \mathrm{D}(\Delta)',$$

and the sequence of repeated eigenvalues α_i, $i = 1, 2, \dots$, of $-\Delta$ satisfies

$$0 < \alpha_1 \le \alpha_2 \le \alpha_3 \le \dots, \qquad \lim_{i\to+\infty} \alpha_i = +\infty.$$

Furthermore, $\langle v, w\rangle_{V',V} = (v, w)_H$, for all $(v, w) \in H \times V$.

Boundedness assumption. For $m \in \mathbb{N}_0$, for simplicity we denote

$$\mathcal{W} \coloneqq L^\infty(\mathbb{R}_0, L^d(\Omega, \mathbb{R})) \times L^\infty_w(\mathbb{R}_0, L^\infty(\Omega, \mathbb{R}^d)). \tag{19}$$

We fix $\hat{a}$ and $\hat{b}$, and a constant $C_{\mathcal{W}} \ge 0$, satisfying

$$\left|(\hat{a}, \hat{b})\right|^2_{\mathcal{W}} \coloneqq |\hat{a}|^2_{L^\infty(\mathbb{R}_0,L^d(\Omega,\mathbb{R}))} + \left|\hat{b}\right|^2_{L^\infty_w(\mathbb{R}_0,L^\infty(\Omega,\mathbb{R}^d))} \le C_{\mathcal{W}}. \tag{20}$$

Remark 2.1 Notice that (20) is weaker than (6). Condition (20) is sufficient for the existence and uniqueness of weak solutions (for the linearized system). We use (6) to derive the existence and uniqueness of strong solutions.

Throughout this paper I stands for the bounded time interval $I = (s_0, s_1)$, with $0 \le s_0 < s_1 < +\infty$, whose length is denoted by $|I| \coloneqq s_1 - s_0$.

We will look for weak solutions in $W(I, H^1(\Omega), V')$ and strong solutions in $W(I, H^2(\Omega), H)$. The corresponding traces on the boundary are denoted

$$G^1(I, \Gamma) \coloneqq W(I, H^1(\Omega), V')\big|_\Gamma \quad \text{and} \quad G^2(I, \Gamma) \coloneqq W(I, H^2(\Omega), H)\big|_\Gamma\,,$$

respectively. As usual, we endow the trace spaces with the norms

$$|\gamma|_{G^1(I,\Gamma)} \coloneqq \inf_{\gamma=v|_\Gamma} |v|_{W(I,H^1(\Omega),V')}\,, \qquad |\gamma|_{G^2(I,\Gamma)} \coloneqq \inf_{\gamma=v|_\Gamma} |v|_{W(I,H^2(\Omega),H)}\,.$$

2.1 Weak solutions

Here we recall some regularity results for the weak solutions for systems as (4). We start considering the more general system

$$\tfrac{\partial}{\partial t}z - \nu\Delta z + \hat{a}z + \nabla\cdot(\hat{b}z) + h = 0, \tag{21a}$$
$$z\big|_{\Gamma} = \gamma, \qquad z(s_0) = z_0. \tag{21b}$$

where the control is replaced by a general external force.

Existence and uniqueness of week solutions can be derived by standard arguments, by using the estimates in the following Lemma, whose proof is also standard and is omitted.

Lemma 2.1 *We have, for $z \in H^1(\Omega)$ and $y \in V$,*

$$\begin{aligned}
\langle \hat{a}z, y\rangle_{V',V} &\le C\,|\hat{a}|_{L^d}\,|z|_H^{\frac12}\,|z|_{H^1(\Omega)}^{\frac12}\,|y|_H^{\frac12}\,|y|_V^{\frac12}\,, &&\textit{for}\quad d\in\{1,2\}.\\
\langle \hat{a}z, y\rangle_{V',V} &\le C\,|\hat{a}|_{L^d}\,|z|_H\,|y|_V\,, &&\textit{for}\quad d\ge 3.\\
\langle \nabla\cdot(\hat{b}z), y\rangle_{V',V} &\le C|\hat{b}|_{L^\infty}\,|z|_H\,|y|_V\,, &&\textit{for}\quad d\ge 1.
\end{aligned}$$

for a suitable constant $C \ge 0$, depending only on (Ω, d).

Lemma 2.2 *Given $(\hat{a},\hat{b}) \in \mathcal{W}$ satisfying (20), $h \in L^2(I,V')$, $\gamma = 0$, and $z_0 \in H$, there is a weak solution $z \in W(I,V,V')$ for (21), which is unique and depends continuously on the data:*

$$|z|^2_{W(I,V,V')} \le \overline{C}_{\left[|I|,C_{\mathcal{W}},\frac{1}{\nu}\right]}\left(|z_0|_H^2 + |h|^2_{L^2(I,V')}\right).$$

Furthermore, $|z(s)|_H^2 \le \mathrm{e}^{\frac{C_1}{\nu}C_{\mathcal{W}}(s-s_0)}\left(|z_0|_H^2 + \frac{1}{\nu}\,|h|^2_{L^2((s_0,s),V')}\right)$, for $s \in I$.

Lemma 2.3 *With $(\hat{a},\hat{b})$, h, and z_0 as in Lemma 2.2, and $\gamma \in G^1(I,\Gamma)$, there is a weak solution $z \in W(I,H^1(\Omega,\mathbb{R}),V')$ for (21), which is unique and depends continuously on the data:*

$$|z|^2_{W(I,H^1(\Omega,\mathbb{R}),V')} \le \overline{C}_{\left[|I|,C_{\mathcal{W}},\frac{1}{\nu}\right]}\left(|z_0|_H^2 + |h|^2_{L^2(I,V')} + |\gamma|^2_{G^1(I,\Gamma)}\right).$$

In the Appendix, Section A.1 we present the proof of Lemma 2.2. For the nonhomogeneous boundary case $\gamma \ne 0$ we recall that we can define weak solutions by a standard lifting argument (e.g., see [40]).

2.2 Strong solutions

For strong solutions we need further regularity for $(\hat{a},\hat{b})$. Roughly, we will need further regularity for the reference trajectory $\hat{y}$ (cf. system (5)). We denote

$$\mathcal{W}_{\mathrm{st}} \coloneqq \{(a,b)\in\mathcal{W} \mid \nabla\cdot b \in L^\infty_w(\mathbb{R}_0, L^r(\Omega,\mathbb{R}))\}, \tag{22a}$$
$$\text{with}\quad r=2\quad\text{if}\quad d\in\{1,2,3\},\quad\text{and}\quad r=\infty\quad\text{if}\quad d\ge 4, \tag{22b}$$

(cf. (19) and (6)). We fix $\hat a$ and $\hat b$, and a constant $C_{\mathcal{W}_{\rm st}} \ge 0$, satisfying

$$|(\hat a, \hat b)|^2_{\mathcal{W}_{\rm st}} := \left((|(a,b)|^2_{\mathcal{W}} + |\nabla\cdot b|^2_{L^\infty_w(\mathbb{R}_0, L^r(\Omega,\mathbb{R}))} \right)^{\frac12} \le C_{\mathcal{W}_{\rm st}}. \tag{23}$$

Now, we have the following estimates for the convection term

$$|\nabla\cdot(\hat b z)|_{L^2} = |(\nabla\cdot\hat b)z + \hat b\cdot\nabla z|_{L^2} \le |(\nabla\cdot\hat b)|_{L^2}\, |z|_{L^\infty} + |\hat b|_{L^\infty}\, |z|_{H^1}$$

and, by using the Agmon inequalities,

$$|\nabla\cdot(\hat b z)|_{L^2} \le C|(\nabla\cdot\hat b)|_{L^2}\, |z|^{\frac12}_{L^2}\, |z|^{\frac12}_{H^1} + |\hat b|_{L^\infty}\, |z|_{H^1}, \quad \text{for} \quad d=1. \tag{24a}$$

$$|\nabla\cdot(\hat b z)|_{L^2} \le C|(\nabla\cdot\hat b)|_{L^2}\, |z|^{\frac12}_{L^2}\, |z|^{\frac12}_{H^2} + |\hat b|_{L^\infty}\, |z|_{H^1}, \quad \text{for} \quad d=2. \tag{24b}$$

$$|\nabla\cdot(\hat b z)|_{L^2} \le C|(\nabla\cdot\hat b)|_{L^2}\, |z|^{\frac12}_{H^1}\, |z|^{\frac12}_{H^2} + |\hat b|_{L^\infty}\, |z|_{H^1}, \quad \text{for} \quad d=3. \tag{24c}$$

The Agmon inequalities can be found in [45, chapter II, Section 1.4] for $d \ge 2$. For $d = 1$ and $\Omega = (l, r)$ with $l < r$, the inequality reads $|z|_{L^\infty} \le 2^{\frac12}\, |z|^{\frac12}_H\, |z|^{\frac12}_V$ and follows from the fact that, for all $x_1 \in \Omega$,

$$|z(x_1)|^2_{\mathbb{R}} = |z(x_1)|^2_{\mathbb{R}} - |z(l)|^2_{\mathbb{R}} = 2\int_l^{x_1} z(\rho)\tfrac{\mathrm d}{\mathrm dx} z(\rho)\,\mathrm d\rho \le 2\,|z|_{L^2(\Omega)}\,\left|\tfrac{\mathrm d}{\mathrm dx} z\right|_{L^2(\Omega)},$$

For $d \ge 4$, the Agmon inequality does not allow us to bound the L^∞-norm by the $\mathrm{D}(\Delta)$-norm. This is the reason we (need to) take different spaces in (22). Notice that

$$|\nabla\cdot(bz)|_{L^2} \le |\nabla\cdot b|_{L^\infty}\, |z|_{L^2} + |b|_{L^\infty}\, |z|_{H^1}, \quad \text{for all} \quad d \ge 1, \tag{24d}$$

We have the following results concerning the existence of strong solutions, whose proofs are standard.

Lemma 2.4 *Given $(\hat a, \hat b) \in \mathcal{W}_{\rm st}$ satisfying (23), $h \in L^2(I, H)$, $\gamma = 0$, and $z_0 \in V$, then there is a strong solution $z \in W(I, \mathrm{D}(\Delta), H)$ for (21), which is unique and depends continuously on the data*

$$|z|^2_{W(I,\mathrm{D}(\Delta),H)} \le \overline{C}_{\left[|I|, C_{\mathcal{W}_{\rm st}}, \frac1\nu\right]} \left(|z_0|^2_V + |h|^2_{L^2(I,H)} \right).$$

Lemma 2.5 *With $(\hat a, \hat b)$ and h as in Lemma 2.4, $\gamma \in G^2(I, \Gamma)$, and $z_0 \in H^1(\Omega, \mathbb{R})$, with $z_0|_\Gamma = \gamma(0)$, there is a strong solution $z \in W(I, H^2(\Omega,\mathbb{R}), H)$ for (21), which is is unique and depends continuously on the data*

$$|z|^2_{W(I,H^2(\Omega,\mathbb{R}),H)} \le \overline{C}_{\left[|I|, C_{\mathcal{W}_{\rm st}}, \frac1\nu\right]} \left(|z_0|^2_{H^1(\Omega)} + |h|^2_{L^2(I,H)} + |\gamma|^2_{G^2(I,\Gamma)} \right).$$

We also have a smoothing property as follows.

Lemma 2.6 *Let $(\hat a, \hat b)$, h, and γ be as in Lemma 2.5, and let $z_0 \in H$. Then the weak solution z of system (21) satisfies $(\cdot - s_0)z \in W(I, H^2(\Omega), H)$, and*

$$(|(\cdot - s_0)z|^2_{W(I,H^2(\Omega),H))} \le \overline{C}_{\left[|I|, C_{\mathcal{W}_{\rm st}}, \frac1\nu\right]} \left(|z_0|^2_H + |h|^2_{L^2(I,H)} + |\gamma|^2_{G^2(I,\Gamma)} \right).$$

Proof Since z solves (21), then also $w = (\cdot - s_0)z$ does, with different data:

$$\begin{aligned}&\tfrac{\partial}{\partial t}w - \nu\Delta w + \hat{a}w + \nabla\cdot(\hat{b}w) + (\cdot - s_0)h - z = 0,\\ &w|_{\Gamma} = (\cdot - s_0)\gamma, \qquad w(0) = 0.\end{aligned}$$

From Lemma 2.5, we can derive that, with $\breve{h} \coloneqq (\cdot - s_0)h$ and $\breve{\gamma} \coloneqq (\cdot - s_0)\gamma$,

$$|w|^2_{W(I,H^2(\Omega),H)} \le \overline{C}_{\left[|I|,C_{\mathcal{W}_{\rm st}},\frac{1}{\nu}\right]}\left(\left|\breve{h}\right|^2_{L^2(I,H)} + |z|^2_{L^2(I,H)} + |\breve{\gamma}|^2_{G^2(I,\Gamma)}\right).$$

The result follows from $|z|^2_{L^2(I,H)} \le |z|^2_{W(I,H^1(\Omega),H^{-1}(\Omega))}$ and Lemma 2.2. □

2.3 Controls supported in a subset

Consider, in the cylinder $I \times \Omega$, the controlled system (21)

$$\tfrac{\partial}{\partial t}z - \nu\Delta z + \hat{a}z + \nabla\cdot(\hat{b}z) = 0, \tag{25a}$$
$$z|_{\Gamma} = \mathcal{B}\zeta, \qquad z(s_0) = z_0, \tag{25b}$$

where $\mathcal{B} \in \mathcal{L}(\mathcal{Z}, G^1(I,\Gamma))$ is to be seen as a control operator, and $\mathcal{Z}$ a given Hilbert space. Given an open subset $\Gamma_{\rm c} \subseteq \Gamma$, we define the spaces

$$G^i_{\rm c}(I,\Gamma) \coloneqq \{\gamma \in G^i(I,\Gamma) \mid \gamma|_{\Gamma\setminus\overline{\Gamma_{\rm c}}} = 0\}, \quad \text{with} \quad i \in \{1,2\}.$$

Following a standard argument, see [40, Section 4] and also [23] and references therein, we can construct open subsets $\widetilde{\omega}$ with

$$\Omega \cap \widetilde{\omega} = \emptyset, \quad \Gamma \cap \partial\widetilde{\omega} \subseteq \overline{\Gamma_{\rm c}}, \quad \text{and} \quad \widetilde{\Omega} \coloneqq \Omega \cup \widetilde{\omega} \cup (\Gamma \cap \partial\widetilde{\omega}), \tag{26}$$

such that $\widetilde{\Omega}$ is still a smooth domain.

Let us be given $(\hat{a},\hat{b}) \in \mathcal{W}$, and let $\widetilde{a}$ and $\widetilde{b}$ be, respectively, the extensions of $\hat{a}$ and $\hat{b}$ by zero outside Ω. Notice that we still have $(\widetilde{a},\widetilde{b}) \in \widetilde{\mathcal{W}}$ (cf. (19), (20))

$$\widetilde{\mathcal{W}} \coloneqq L^\infty(\mathbb{R}_0, L^d(\widetilde{\Omega},\mathbb{R})) \times L^\infty_w(\mathbb{R}_0, L^\infty(\widetilde{\Omega},\mathbb{R}^d)), \quad \left|(\widetilde{a},\widetilde{b})\right|^2_{\widetilde{\mathcal{W}}} \le C_{\mathcal{W}}. \tag{27}$$

However, for given $(\hat{a},\hat{b}) \in \mathcal{W}_{\rm st}$, see (22), we cannot guarantee that we still have $(\tilde{a},\tilde{b}) \in \widetilde{\mathcal{W}_{\rm st}}$,

$$\widetilde{\mathcal{W}_{\rm st}} \coloneqq \left\{(a,b) \in \widetilde{\mathcal{W}} \mid \nabla\cdot b \in L^\infty_w(\mathbb{R}_0, L^r(\widetilde{\Omega},\mathbb{R}))\right\}, \tag{28}$$

with r as in (22). We need a smoother extension for the vector field b, given by the following proposition whose proof is given in the Appendix, Section A.2.

Proposition 2.1 *There exists $\widetilde{\omega}$ satisfying* (26) *and there exists an extension $\overline{b}$ of $\hat{b}$ such that the linear mapping $(\hat{a},\hat{b}) \mapsto (\tilde{a},\overline{b})$ is continuous.*

Remark 2.2 In the literature we may find some results allowing us to construct/extend vectors fields satisfying some divergence constraint. See the results in [3, 4, 14], for $d \in \{2, 3\}$. See also [46, chapter 1, Theorem 2.4]. We were not able to use those results to "construct" an extension $\mathcal{W}_{\rm st} \to \widetilde{\mathcal{W}}_{\rm st}$. Proposition 2.1, in some sense, generalizes some results in [14, Section 3] to higher dimensions and for essentially bounded vectors $\hat{b}$. Furthermore, the extension $\bar{b}$ constructed in the Appendix will be divergence free if so is $\hat{b}$, see (A.4) in the Appendix. Thus, Proposition 2.1 also generalizes to higher dimensions the result in Proposition 4.2 presented in [40], for $d = 3$.

It is known, see for example [21], that we can find a family of internal controls $\{\tilde{\eta}(w) \mid w \in L^2(\widetilde{\Omega})\}$, with $\tilde{\eta} \in \mathcal{L}(H, L^2(I, L^2(\widetilde{\Omega})))$, such that the solution z^e of the system

$$\tfrac{\partial}{\partial t} z^e - \nu \Delta z^e + \tilde{a} z^e + \nabla \cdot (\bar{b} z^e) = 1_{\widetilde{\omega}} \tilde{\eta}(z_0^e), \tag{29a}$$

$$z^e|_{\widetilde{\Gamma}} = 0, \qquad z^e(s_0) = z_0^e, \tag{29b}$$

where z_0^e is the extension of z_0 by zero outside Ω, satisfies $z^e(s_1) = 0$ and

$$|\tilde{\eta}(z_0^e)|^2_{L^2(I, L^2(\widetilde{\Omega}))} \le \mathrm{e}^{C_{\widetilde{\omega},\Omega} \Theta_\nu^I} |z_0^e|^2_{L^2(\widetilde{\Omega})} = \mathrm{e}^{C_{\widetilde{\omega},\Omega} \Theta_\nu^I} |z_0|^2_H, \tag{30}$$

with $\Theta_\nu^I := \Theta\left(\nu |I|, \left|\frac{\tilde{a}}{\nu}\right|_{L^\infty(I, L^d(\widetilde{\Omega}))}, \left|\frac{\bar{b}}{\nu}\right|_{L^\infty_w(I, L^\infty(\widetilde{\Omega}))}, d\right)$, and

$$\Theta(r, \theta_1, \theta_2, d) := 1 + \theta_1^2 + d\theta_2^2 + \frac{1}{r} + r\left(\theta_1 + d\theta_2^2\right), \tag{31}$$

and where $C_{\widetilde{\omega},\Omega}$ is a constant depending on $\widetilde{\omega}$ and Ω.

Moreover, if $z_0 \in V$ then $z_0^e \in H_0^1(\widetilde{\Omega})$, and we have that z^e is a strong solution, which implies that $z := z^e|_\Omega$ solves (25), with $\mathcal{Z} = G_{\rm c}^2(I, \Gamma)$, $\zeta = z^e|_\Gamma$, and the inclusion operator

$$\mathcal{B} = \iota^2_{\Gamma_{\rm c}} \in \mathcal{L}(G_{\rm c}^2(I, \Gamma), G^2(I, \Gamma)), \qquad \mathcal{B}\zeta = \zeta.$$

Furthermore, we find that

$$\begin{aligned} |\zeta|^2_{G_{\rm c}^2(I,\Gamma)} &\le |z^e|^2_{W(I, H^2(\widetilde{\Omega}), L^2(\widetilde{\Omega}))} \le \overline{C}_{\left[|I|, C_{\mathcal{W}_{\rm st}}, \frac{1}{\nu}\right]} \left(1 + \mathrm{e}^{C_{\widetilde{\omega},\Omega}\Theta_\nu^I}\right) |z^e(s_0)|^2_{H_0^1(\widetilde{\Omega})} \\ &\le 2\overline{C}_{\left[|I|, C_{\mathcal{W}_{\rm st}}, \frac{1}{\nu}\right]} \mathrm{e}^{C_{\widetilde{\omega},\Omega}\Theta_\nu^I} |z(s_0)|^2_V \end{aligned}$$

and, since the choice of such subset $\widetilde{\omega}$ is at our disposal, we can conclude that there exists a constant $C_{\Gamma_c,\Omega} > 0$ depending on Γ_c and Ω, such that

$$|\zeta|^2_{G_{\rm c}^2(I,\Gamma)} \le \overline{C}_{\left[|I|, C_{\mathcal{W}_{\rm st}}, \frac{1}{\nu}\right]} \mathrm{e}^{C_{\Gamma_{\rm c},\Omega}\Theta_\nu^I} |z(s_0)|^2_V.$$

Therefore we have the following.

Theorem 2.1 *Let $(\hat{a}, \hat{b}) \in \mathcal{W}_{\rm st}$ and $\mathcal{B} = \iota^2_{\Gamma_{\rm c}}$. Then, there is a family $\{\bar{\zeta}(z_0) \mid z_0 \in V\}$, with $\bar{\zeta} \in \mathcal{L}(H, G^2_{\rm c}(I, \Gamma))$, such that the solutions $z = z(z_0, \bar{\zeta}(z_0))$ to* (25) *satisfy $z(z_0, \bar{\zeta}(z_0))(s_1) = 0$ and, for a constant $\widehat{C}_0 = C(\Gamma_c, \Omega)$, we have*

$$\left|\bar{\zeta}(z_0)\right|^2_{G^2_{\rm c}(I,\Gamma)} \le \overline{C}_{\left[|I|, C_{\mathcal{W}_{\rm st}}, \frac{1}{\nu}\right]} \mathrm{e}^{\widehat{C}_0 \Theta^I_\nu} \left|z_0\right|^2_V .$$

Theorem 2.2 *Let $(\hat{a}, \hat{b}) \in \mathcal{W}_{\rm st}$, $\mathcal{B} = \iota^2_{\Gamma_{\rm c}}$, and $s_{1/2} \in I$. Then, there is a family $\{\bar{\zeta}_{\rm c}(z_0) \mid z_0 \in H\}$, with $\bar{\zeta}_{\rm c} \in \mathcal{L}(H, G^2_{\rm c}(I, \Gamma))$, such that the solutions $z = z(z_0, \bar{\zeta}_{\rm c}(z_0))$ to* (25) *satisfy $z(z_0, \bar{\zeta}_{\rm c}(z_0))(s_1) = 0$ and, for a constant $\widehat{C}_0 = C(\Gamma_c, \Omega)$, we have that*

$$\left|\bar{\zeta}_{\rm c}(z_0)\right|^2_{G^2_{\rm c}(I,\Gamma)} \le |I_1|^{-2} \overline{C}_{\left[|I|, C_{\mathcal{W}_{\rm st}}, \frac{1}{\nu}\right]} \mathrm{e}^{\widehat{C}_0 \Theta^{I_2}_\nu} \left|z_0\right|^2_H .$$

where $I_1 = (s_0, s_{1/2})$ and $I_2 = (s_{1/2}, s_1)$.

Proof Firstly we apply zero control for time $t \in I_1$ in this way, see Lemma 2.6, we arrive at a vector $z(s_{1/2}) = z(z_0, 0)(s_{1/2}) \in V$ and the mapping $z(s_0) = z_0 \mapsto z(z_0, 0)(s_{1/2})$ is linear and continuous:

$$\left|z(z_0, 0)(s_{1/2})\right|^2_V \le |I_1|^{-2} \overline{C}_{\left[|I_1|, C_{\mathcal{W}_{\rm st}}, \frac{1}{\nu}\right]} \left|z_0\right|^2_H .$$

Next we apply, in I_2, the control $\bar{\zeta}(z(s_{1/2}))$ given by Theorem 2.1. Thus

$$\begin{aligned} \left|\bar{\zeta}(z(s_{1/2}))\right|^2_{G^2_{\rm c}(I_2,\Gamma)} &\le \overline{C}_{\left[|I_2|, C_{\mathcal{W}_{\rm st}}, \frac{1}{\nu}\right]} \mathrm{e}^{\widehat{C}_0 \Theta^{I_2}_\nu} \left|z(s_{1/2})\right|^2_V . \\ &\le |I_1|^{-2} \overline{C}_{\left[|I_1|, C_{\mathcal{W}_{\rm st}}, \frac{1}{\nu}\right]} \overline{C}_{\left[|I_2|, C_{\mathcal{W}_{\rm st}}, \frac{1}{\nu}\right]} \mathrm{e}^{\widehat{C}_0 \Theta^{I_2}_\nu} \left|z_0\right|^2_H . \end{aligned}$$

It remain to check that the concatenated control

$$\bar{\zeta}_{\rm c}(z_0) := \begin{cases} 0, & \text{if } t \in I_1 \\ \zeta(z(z_0, 0)(s_{1/2})), & \text{if } t \in I_2 \end{cases}$$

is in $G^2_{\rm c}(I, \Gamma)$. It is clear that the control is supported in $\overline{\Gamma}_{\rm c}$. It is enough to check that the following weighted concatenation of the corresponding solutions

$$\breve{z}(t) := \begin{cases} \psi(t) z(z_0, 0)(t), & \text{if } t \in I_1 \\ z(z(s_{1/2}), \bar{\zeta}(z(s_{1/2})))(t), & \text{if } t \in I_2 \end{cases}$$

is in $W(I, H^2(\Omega), L^2(\Omega))$, for some smooth function ψ vanishing for $t \in [s_0, r_1]$ and taking the value 1 for $t \in [r_2, s_{1/2}]$, with $s_0 < r_1 < r_2 < s_{1/2}$. Notice that ψ does not change the trace on the boundary, $\breve{z}|_\Gamma = z|_\Gamma$. Since

$$\breve{z}|_{I_1} \in W(I_1, H^2(\Omega), L^2(\Omega)) \quad \text{and} \quad \breve{z}|_{I_2} W(I_1, H^2(\Omega), L^2(\Omega)),$$

and recalling that (cf. [32, chapter 1, sections 3.2 and 9.3])

$$\begin{aligned} &\{v(s_{1/2}) \mid v \in W(I_1, H^2(\Omega), L^2(\Omega))\} \\ &= H^1(\Omega) = \{v(s_{1/2}) \mid v \in W(I_2, H^2(\Omega), L^2(\Omega))\}, \end{aligned}$$

it follows that the concatenation is in $W(I, H^2(\Omega), L^2(\Omega))$, because by construction $(\breve{z}|_{I_1})(s_{1/2}) = (\breve{z}|_{I_2})(s_{1/2}) \in V \subset H^1(\Omega)$. □

2.4 Stabilization to zero by finite dimensional controls

Here $(\hat{a}, \hat{b}) \in \mathcal{W}_{\mathrm{st}}$. We look for stabilizing controls, of the form $B_\Psi u(t) = \sum_{i=1}^{M} u_i(t)\Psi_i(x)$, with $u \in H^1(\mathbb{R}_0, \mathbb{R}^M)$. Each actuator $\Psi_i \in H^{\frac{3}{2}}(\Gamma)$ is supposed to satisfy $1_{\Gamma_{\mathrm{c}}}\Psi_i(x) = \Psi_i(x)$, that is, $\operatorname{supp}\Psi_i(x) \subseteq \overline{\Gamma_{\mathrm{c}}}$.

We will construct the finite-dimensional stabilizing control from the control $\bar{\zeta}_{\mathrm{c}}(z_0) \in G^2_{\mathrm{c}}(I, \Gamma)$, given by Theorem 2.2. Notice that the range of $\bar{\zeta}_{\mathrm{c}}(z_0)$ is not necessarily finite-dimensional. Note that, if we take $P_M\bar{\zeta}_{\mathrm{c}}(z_0)$ instead, then such control takes values in the span $\mathcal{S}_\Psi = \operatorname{span}\{\Psi_i \mid i \in \{1, 2, \dots, M\}\}$ of the actuators. Recall that P_M is the orthogonal projection in $L^2(\Gamma, \mathbb{R})$ onto $\mathcal{S}_\Psi$. Moreover, writing $u = (B_\Psi)^{-1}P_M\bar{\zeta}_{\mathrm{c}}(z_0)$, that is, $P_M\bar{\zeta}_{\mathrm{c}}(z_0) = B_\Psi u$, we do not necessarily have that $u \in H^1(I, \mathbb{R}^M)$, as we show now by recalling the characterization of $G^2(J, \Gamma)$ in terms of (fractional) Sobolev-Bochner spaces.

We consider (cf. [25, section 2.1] and [38, section 2.2], see also [33, chapter 4, section 2]) the following subspace of $W(J, H^1(\Omega), H^{-1}(\Omega))$ defined by

$$\begin{aligned}\underline{W}(J, H^1(\Omega), H^{-1}(\Omega)) &\coloneqq W(\mathbb{R}, H^1(\mathbb{R}^d), H^{-1}(\mathbb{R}^d))|_{J\times\Omega}\\ &\hookrightarrow W(J, H^1(\Omega), H^{-1}(\Omega)),\end{aligned}$$

and the corresponding trace space

$$\underline{G}^1(J, \Gamma) \coloneqq \underline{W}(J, H^1(\Omega), H^{-1}(\Omega))|_\Gamma \hookrightarrow G^1(J, \Gamma).$$

Analogously, we consider the space

$$\begin{aligned}\underline{W}(J, H^2(\Omega), L^2(\Omega)) &\coloneqq W(\mathbb{R}, H^2(\mathbb{R}^d), L^2(\mathbb{R}^d))|_{J\times\Omega}\\ &\hookrightarrow W(J, H^2(\Omega), L^2(\Omega)),\end{aligned}$$

and the corresponding trace space

$$\underline{G}^2(J, \Gamma) \coloneqq \underline{W}(J, H^2(\Omega), L^2(\Omega))|_\Gamma \hookrightarrow G^2(J, \Gamma).$$

Notice that (cf. [38, section 2.2]) for a general domain $\Omega \subset \mathbb{R}^d$, we have

$$\begin{aligned}\underline{W}(J, H^2(\Omega), L^2(\Omega)) &= W(J, H^2(\Omega), L^2(\Omega)), & \underline{G}^2(J, \Gamma) &= G^2(J, \Gamma),\\ \underline{W}(J, H^1(\Omega), H^{-1}(\Omega)) &\neq W(J, H^1(\Omega), H^{-1}(\Omega)), & \underline{G}^1(J, \Gamma) &\neq G^1(J, \Gamma).\end{aligned}$$

We have the following characterizations in [25, Theorem 3.1],

$$\underline{G}^1(J, \Gamma) = \mathcal{G}^1(J, \Gamma) \coloneqq L^2(J, H^{\frac{1}{2}}(\Gamma)) \cap H^{\frac{1}{2}}(J, H^{-\frac{1}{2}}(\Gamma)), \tag{32a}$$

$$G^2(J, \Gamma) = \mathcal{G}^2(J, \Gamma) \coloneqq L^2(J, H^{\frac{3}{2}}(\Gamma)) \cap H^{\frac{3}{4}}(J, L^2(\Gamma)). \tag{32b}$$

That is, by taking $P_M\bar{\zeta}_{\mathrm{c}}(z_0)$ we will obtain a control u in $H^{\frac{3}{4}}(I, \mathbb{R}^M)$, but not necessarily in $H^1(I, \mathbb{R}^M)$. In order to obtain the desired regularity $H^1(I, \mathbb{R}^M)$ for the control, we will take controls of the form $Q_{\widetilde{M}}P_M\bar{\zeta}_{\mathrm{c}}(z_0)$ where $Q_{\widetilde{M}}$ is a suitable orthogonal projection in $L^2(I)$ with range contained in $H^1_0(I)$.

2.4.1 Further regularity in time variable for the control

The stabilizing control in $\mathbb{R}_{s_0} = (s_0, +\infty)$, $s_0 \geq 0$, will be constructed recursively in intervals of the same length, as $J^j \coloneqq (s_0 + jT_*, s_0 + (j+1)T_*)$, with $j \in \mathbb{N}$, where the length T_* will be fixed.

Let $J \coloneqq (0, T_*)$ and let $\xi_j \in H_0^1(J)$ be the orthonormalized eigenfunctions of the Dirichlet Laplacian $\Delta_J \coloneqq -\frac{\partial}{\partial t}\frac{\partial}{\partial t}$ in $L^2(J)$, that is,

$$\xi_j(t) \coloneqq (\tfrac{2}{T_*})^{\frac{1}{2}} \sin(\tfrac{j\pi t}{T_*}), \quad \Delta_J \xi_j = \beta_j \xi_j, \quad \text{with} \quad 0 < \beta_j \coloneqq (\tfrac{\pi}{T_*})^2 j^2 \to +\infty.$$

We define $Q_{\widetilde{M}}$ as the orthogonal projection in $L^2(J)$ onto the finite dimensional subspace $\operatorname{span}\{\xi_j \mid j \in \{1, 2, \dots, \widetilde{M}\}\}$.

Then for each interval J^j, with $j \in \mathbb{N}$, we define the orthogonal projection $Q^j_{\widetilde{M}}$ in $L^2(J^j)$, with range $Q^j_{\widetilde{M}}\big(L^2(J^j)\big) \subset H_0^1(J^j)$, by

$$Q^j_{\widetilde{M}} f \coloneqq \mathcal{T}_{s_0 + jT_*} Q_{\widetilde{M}} \mathcal{T}_{-s_0 - jT_*} f \tag{33}$$

where $\mathcal{T}_r$ is the translation operator $\mathcal{T}_r f = f(\cdot - r)$.

Let us now fix $\lambda \geq 0$ and $j \in \mathbb{N}$, and consider, in $J^j \times \Omega$, the system:

$$\tfrac{\partial}{\partial t} z - \nu \Delta z + (\hat{a} - \tfrac{\lambda}{2}) z + \nabla \cdot (\hat{b} z) = 0, \tag{34a}$$

$$z|_\Gamma = Q^j_{\widetilde{M}} P_M \bar{\zeta}_{\rm c}(z_0), \qquad z(s_0 + jT_*) = z_0^j, \tag{34b}$$

where $z_0^j \in H$ and $\bar{\zeta}_{\rm c}(z_0^j) \in G^2_{\rm c}(J^j, \Gamma)$ is given by Theorem 2.2, with $(\hat{a} - \frac{\lambda}{2})$ in the place of $\hat{a}$. To fix ideas we take the point $s_{1/2} = s_0 + (j + \frac{1}{2})T_*$ of J^j in Theorem 2.2, $I_1 = (s_0 + jT_*, s_0 + (j + \frac{1}{2})T_*)$ and $I_2 = (s_0 + (j + \frac{1}{2})T_*, s_0 + (j+1)T_*)$.

Proposition 2.2 *$P_M \in \mathcal{L}(G^2_{\rm c}(I, \Gamma))$ and $Q^j_{\widetilde{M}} P_M \in \mathcal{L}(G^2_{\rm c}(J^j, \Gamma))$.*

The proof of Proposition 2.2 is given in the Appendix, Section A.3.

Let z solve (34) with the identity in the place of $Q^j_{\widetilde{M}} P_M$, and let z_M solve (34). Then, $d = z - z_M$ solves

$$\begin{aligned} &\tfrac{\partial}{\partial t} d - \nu \Delta d + (\hat{a} - \tfrac{\lambda}{2}) d + \nabla \cdot (\hat{b} d) = 0, \\ &d|_\Gamma = (1 - Q^j_{\widetilde{M}} P_M) \bar{\zeta}_{\rm c}(z_0^j), \qquad d(s_0) = 0. \end{aligned}$$

Notice that $z_M(s_0 + (j+1)T_*) = -d(s_0 + (j+1)T_*)$. From Lemma 2.3 and Theorem 2.2, it follows, with $\mathcal{R} \coloneqq 1 - Q^j_{\widetilde{M}} P_M$,

$$\left| \bar{\zeta}_{\rm c}(z_0^j) \right|^2_{G^2_{\rm c}(I,\Gamma)} \leq (\tfrac{T_*}{2})^{-2} \overline{C}_{\left[T_*, C_{\mathcal{W}_{\rm st}}, \frac{1}{\nu}\right]} {\rm e}^{\widehat{C}_0 \Theta^{I_2}_\nu} \left| z_0^j \right|^2_H . \tag{35a}$$

$$|z_M(s_0 + (j+1)T_*)|^2_H \leq \Xi(T_*) \, |\mathcal{R}|^2_{\mathcal{L}(G^2_{\rm c}(J^j,\Gamma), G^1_{\rm c}(J^j,\Gamma))} \left| z_0^j \right|^2_H , \tag{35b}$$

$$|z_M|^2_{L^\infty(J^j,H)} \leq \Big(\overline{C} + \Xi(T_*) \, |\mathcal{R}|^2_{\mathcal{L}(G^2_{\rm c}(J^j,\Gamma), G^1_{\rm c}(J^j,\Gamma))} \Big) \left| z_0^j \right|^2_H , \tag{35c}$$

with $\overline{C} = \overline{C}_{\left[T_*, C_{\mathcal{W}_{\rm st}}, \lambda, \frac{1}{\nu}\right]}$ and, where for $\tau > 0$,

$$\begin{aligned}\varXi(\tau) &:= 4\tau^{-2}\overline{C}_{\left[\tau, C_{\mathcal{W}_{\rm st}}, \lambda, \frac{1}{\nu}\right]}\mathrm{e}^{\widehat{D}\Theta\left(\frac{\nu\tau}{2}, \left|\frac{\hat{a}}{\nu} - \frac{\lambda}{2\nu}\right|_{L^\infty(\mathbb{R}_0, L^d(\Omega))}, \left|\frac{\hat{b}}{\nu}\right|_{L^\infty_w(\mathbb{R}_0, L^\infty(\Omega))}, d\right)}\\ &\geq \left|\bar{\zeta}_{\rm c}\right|^2_{\mathcal{L}(H, G^2_{\rm c}(I,\Gamma))}, \quad |I| = \tau.\end{aligned}$$

We can see that when $(\hat{a} - \frac{\lambda}{2}, \hat{b}) = (0, 0)$ then the system

$$\tfrac{\partial}{\partial t} z - \nu\Delta z + (\hat{a} - \tfrac{\lambda}{2})z + \nabla\cdot(\hat{b}z) = 0, \qquad z|_\Gamma = 0,$$

(cf. system(34)) is exponentially stable. Therefore from now we consider the case $(\hat{a} - \frac{\lambda}{2}, \hat{b}) \neq (0, 0)$ where we can see that it holds

$$\lim_{\tau\to+\infty} \varXi(\tau) = +\infty \quad \text{and} \quad \lim_{\tau\to 0} \varXi(\tau) = +\infty.$$

Hence we can set $T_* > 0$ such that

$$\varXi(T_*) = \min_{\tau>0} \varXi(\tau) =: \varUpsilon. \tag{36}$$

2.4.2 The first stabilizability condition

We show that a stabilizing control can be constructed, under a boundedness condition on the operator $Q^j_{\widetilde{M}} P_M$. Let us consider, in $\mathbb{R}_{s_0} \times \Omega$, the system,

$$\tfrac{\partial}{\partial t} z - \nu\Delta z + (\hat{a} - \tfrac{\lambda}{2})z + \nabla\cdot(\hat{b}z) = 0, \tag{37a}$$

$$z|_\Gamma = \breve{\zeta}_{\rm c}(z_0), \qquad z(s_0) = z_0, \tag{37b}$$

where the control $\breve{\zeta}_{\rm c}(z_0)$ is defined recursively as follows.

1. In the interval $J^0 = (s_0, s_0 + T_*)$ we take the control as in system (34), $\breve{\zeta}_{\rm c}(z_0)|_{J^0} = Q^0_{\widetilde{M}} P_M \bar{\zeta}_{\rm c}(z_0)$.
2. Once the control has been constructed for time $t \in (s_0, s_0 + jT_*)$, $j \geq 1$, we solve the system and take the final state $z_j = z(s_0 + jT_*)$. Then we take again the control $\breve{\zeta}_{\rm c}(z_0)|_{J^j} = Q^j_{\widetilde{M}} P_M \bar{\zeta}_{\rm c}(z_j)$ as in system (34).

In Theorem 2.3 we will give a stabilizability condition in terms of the norm $\xi := \left|1 - Q_{\widetilde{M}} P_M\right|^2_{\mathcal{L}(G^2_{\rm c}(J,\Gamma), G^1_{\rm c}(J,\Gamma))}$. Denoting also, for each $j \in \mathbb{N}$, $\xi^j := \left|1 - Q^j_{\widetilde{M}} P_M\right|^2_{\mathcal{L}(G^2_{\rm c}(J^j,\Gamma), G^1_{\rm c}(J^j,\Gamma))}$, we observe that the dependence of ξ^j on j is only terms of the length $|J^j|$ of J^j. Since $|J^j| = |J| = T_*$, then $\xi^j = \xi$.

Theorem 2.3 *The system* (37) *is exponentially stable with rate* $\delta > 0$, *if*

$$\left|1 - Q_{\widetilde{M}} P_M\right|^2_{\mathcal{L}(G^2_{\rm c}(J,\Gamma), G^1_{\rm c}(J,\Gamma))} \leq \mathrm{e}^{-\delta T_*}\varUpsilon^{-1}. \tag{38}$$

Furthermore, $\left|\breve{\zeta}_{\rm c}(z_0)\right|^2_{G^2_{\rm c}(\mathbb{R}_{s_0},\Gamma)} \leq \frac{C}{1-\mathrm{e}^{-\delta T_*}} |z_0|^2_H$, *for a suitable constant* $C > 0$.

Proof From (35) and (38), we find that the solution of (37) satisfies

$$\begin{aligned}
|z(s_0+(j+1)T_*)|_H^2 &\le \mathrm{e}^{-\delta T_*}\,|z(s_0+jT_*)|_H^2 \le \mathrm{e}^{-(j+1)\delta T_*}\,|z_0|_H^2\,,\\
|z|^2_{L^\infty(J^j,H)} &\le \left(\overline{C}+\mathrm{e}^{-\delta T_*}\right)\mathrm{e}^{-j\delta T_*}\,|z_0|_H^2\,.
\end{aligned}$$

Since for $t\in J^j$ we have that $t=s_0+jT_*+rT_*$ with $r\in(0,1)$, it follows that $jT_*=t-s_0-rT_*$, and

$$|z(t)|_H^2 \le \left(\overline{C}+\mathrm{e}^{-\delta T_*}\right)\mathrm{e}^{r\delta T_*}\mathrm{e}^{-\delta(t-s_0)}\,|z(s_0)|_H^2\,,\quad t\ge s_0.$$

Finally we observe that $\left|\breve\zeta_{\mathrm{c}}(z_0)\right|^2_{G^2_{\mathrm{c}}(\mathbb{R}_{s_0},\Gamma)} \le \sum_{j\in\mathbb{N}}\left|\bar\zeta_{\mathrm{c}}(z(s_0+jT_*))\right|^2_{G^2_{\mathrm{c}}(J^j,\Gamma)}$, that is, $\left|\breve\zeta_{\mathrm{c}}(z_0)\right|^2_{G^2_{\mathrm{c}}(\mathbb{R}_{s_0},\Gamma)} \le C\sum_{j\in\mathbb{N}}\mathrm{e}^{-j\delta T_*}\,|z_0|_H^2 \le \frac{C}{1-\mathrm{e}^{-\delta T_*}}\,|z_0|_H^2$. □

2.4.3 The main stabilizability condition

Notice that (38) involves spaces of functions defined in the cylinder $J\times\Gamma$. Here we present the main stabilizability condition in terms of spaces of functions defined on Γ only.

The norms of the spaces $G^i(J,\Gamma)$ have been introduced as the trace norm in Section 2. Here, the (fractional) Sobolev-Bochner spaces $\mathcal{G}^1(J,\Gamma)$ and $\mathcal{G}^2(J,\Gamma)$, in (32) above, are supposed to be endowed with the usual norms (based on the Fourier Transform). See [41, Section A.2] and references therein. That is, the norms may not coincide, but they are equivalent

$$D_1^\flat\,|\cdot|_{\mathcal{G}^1(J,\Gamma)} \le |\cdot|^2_{\underline{G}^1(J,\Gamma)} \le D_1^\sharp\,|\cdot|_{\mathcal{G}^1(J,\Gamma)}\,, \tag{39a}$$

$$D_2^\flat\,|\cdot|_{\mathcal{G}^2(J,\Gamma)} \le |\cdot|^2_{G^2(J,\Gamma)} \le D_2^\sharp\,|\cdot|_{\mathcal{G}^2(J,\Gamma)}\,. \tag{39b}$$

The constants $D_i^\flat$ and $D_i^\sharp$, $i\in\{1,2\}$, depend on the length $|J|$ of the interval J.

Recall the space $\mathcal{H}(\Gamma_{\mathrm{c}})$ defined in (18). We have the following result.

Theorem 2.4 *The system* (37) *is stabilizable to zero with rate* $\delta>0$, *if*

$$|1-P_M|^2_{\mathcal{H}(\Gamma_{\mathrm{c}})} \le \frac{D_2^\flat}{4D_1^\sharp}\mathrm{e}^{-\delta T_*}\Upsilon^{-1}. \tag{40}$$

Furthermore, there exists $\widetilde{M}=\overline{C}_{[M]}$ *so that, with* $u^{\breve\zeta_{\mathrm{c}}(z_0)}=(B_\Psi)^{-1}\breve\zeta_{\mathrm{c}}(z_0)$, *where* $\breve\zeta_{\mathrm{c}}(z_0)$ *is as in Theorem 2.3, we have the estimate*

$$\left|z(z_0,B_\Psi u^{\breve\zeta_{\mathrm{c}}(z_0)})\right|^2_{L^2(\mathbb{R}_{s_0},H)}+\left|u^{\breve\zeta_{\mathrm{c}}(z_0)}\right|^2_{H^1(\mathbb{R}_{s_0},\mathbb{R}^M)} \le \overline{C}_{\left[C_{\mathcal{W}_{\mathrm{st}}},\lambda,\frac{1}{\nu},\frac{T_*}{\delta},\delta,\Upsilon\right]}\,|z_0|_H^2\,.$$

For the proof of Theorem 2.4 we will need the following auxiliary results.

Proposition 2.3 *Let us be given Banach spaces X_1, Y_1, X_2, Y_2 and Hausdorff topological spaces Z_1, Z_2 such that $X_i \hookrightarrow Z_i$ and $Y_i \hookrightarrow Z_i$, $i \in \{1, 2\}$. Then for any given $A \in \mathcal{L}(X_1, X_2) \cap \mathcal{L}(Y_1, Y_2)$, we have that $A \in \mathcal{L}(X_1 \cap Y_1, X_2 \cap Y_2)$ and $|A|_{\mathcal{L}(X_1\cap Y_1, X_2\cap Y_2)} \le |A|_{\mathcal{L}(X_1,X_2)} + |A|_{\mathcal{L}(Y_1,Y_2)}$.*

The proof is straightforward and is omitted.

Lemma 2.7 *We have the continuous inclusions*

$$G^2(J,\Gamma) \hookrightarrow H^{\frac{9}{16}}(J, H^{\frac{3}{8}}(\Gamma)) \bigcap H^{\frac{3}{8}}(J, H^{\frac{3}{4}}(\Gamma)) \hookrightarrow \underline{G}^1(J,\Gamma).$$

Proof The inclusion $H^{\frac{9}{16}}(J, H^{\frac{3}{8}}(\Gamma)) \bigcap H^{\frac{3}{8}}(J, H^{\frac{3}{4}}(\Gamma)) \hookrightarrow \underline{G}^1(J,\Gamma)$ follows easily from (32). From (32) and [41, Lemma A.12] we find that $G^2(J,\Gamma) \hookrightarrow H^{\frac{3}{8}}(J, H^{\frac{3}{4}}(\Gamma))$. Then, using again [41, Lemma A.12], we also obtain the inclusion $H^{\frac{3}{4}}(J, L^2(\Gamma) \bigcap H^{\frac{3}{8}}(J, H^{\frac{3}{4}}(\Gamma)) \hookrightarrow H^{\frac{9}{16}}(J, H^{\frac{3}{8}}(\Gamma))$. □

Lemma 2.8 *We have that for big enough $\widetilde{M} = \overline{C}_{[M]}$,*

$$\left|1 - Q_{\widetilde{M}} P_M\right|^2_{\mathcal{L}(G^2_{\rm c}(J,\Gamma), \underline{G}^1_{\rm c}(J,\Gamma))} \le 4 \tfrac{D_1^\sharp}{D_2^\flat} \left|1 - P_M\right|^2_{\mathcal{H}(\Gamma_{\rm c})},$$

where $D_1^\sharp$ and $D_2^\flat$ are as in (32).

Proof For simplicity we denote

$$\begin{aligned} &\underline{G}^1_{\rm c} := \underline{G}^1_{\rm c}(J,\Gamma) := \underline{G}^1(J,\Gamma) \textstyle\bigcap G^1_{\rm c}(J,\Gamma), && G^2_{\rm c} := G^2_{\rm c}(J,\Gamma),\\ &\mathcal{G}^1_{\rm c} := \{\gamma \in \mathcal{G}^1(J,\Gamma) \mid \gamma|_{\Gamma\setminus\overline{\Gamma_{\rm c}}} = 0\}, && \mathcal{G}^2_{\rm c} := \{\gamma \in \mathcal{G}^2(J,\Gamma) \mid \gamma|_{\Gamma\setminus\overline{\Gamma_{\rm c}}} = 0\}, \end{aligned}$$

and

$$\mathcal{I}(\Gamma_{\rm c}) := \left\{\gamma \in H^{\frac{9}{16}}(J, H^{\frac{3}{8}}(\Gamma)) \bigcap H^{\frac{3}{8}}(J, H^{\frac{3}{4}}(\Gamma)) \mid \operatorname{supp}\gamma \subseteq \overline{\Gamma_{\rm c}}\right\}.$$

From $\mathcal{R} := 1 - Q_{\widetilde{M}} P_M = 1 - P_M + (1 - Q_{\widetilde{M}}) P_M$, we have

$$|\mathcal{R}|^2_{\mathcal{L}(G^2_{\rm c}, \underline{G}^1_{\rm c})} \le 2\left|1 - P_M\right|^2_{\mathcal{L}(G^2_{\rm c}, \underline{G}^1_{\rm c})} + 2\left|(1 - Q_{\widetilde{M}}) P_M\right|^2_{\mathcal{L}(G^2_{\rm c}, \underline{G}^1_{\rm c})}. \tag{41}$$

Now recalling (32) and (18), we find

$$\begin{aligned} \left|1 - P_M\right|^2_{\mathcal{L}(G^2_{\rm c}, \underline{G}^1_{\rm c})} &\le |\iota|^2_{\mathcal{L}(\mathcal{G}^1_{\rm c}, \underline{G}^1_{\rm c})} \left|1 - P_M\right|^2_{\mathcal{L}(\mathcal{G}^2_{\rm c}, \mathcal{G}^1_{\rm c})} |\iota|^2_{\mathcal{L}(G^2_{\rm c}, \mathcal{G}^2_{\rm c})}\\ &\le \tfrac{D_1^\sharp}{D_2^\flat} \left|1 - P_M\right|^2_{\mathcal{H}(\Gamma_{\rm c})}, \end{aligned} \tag{42}$$

because, for any $w \in \mathcal{G}^2_{\rm c}$, and denoting for simplicity $P_M^\perp := 1 - P_M$,

$$\begin{aligned} &|(1 - P_M)w|^2_{\mathcal{G}^1_{\rm c}} = \left|P_M^\perp w\right|^2_{L^2(J, H^{\frac{1}{2}}(\Gamma))} + \left|P_M^\perp w\right|^2_{H^{\frac{1}{2}}(J, H^{-\frac{1}{2}}(\Gamma))}\\ &\le \left|P_M^\perp\right|^2_{\mathcal{L}(H^{\frac{3}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma))} |w|^2_{L^2(J, H^{\frac{3}{2}}(\Gamma))} + \left|P_M^\perp\right|^2_{\mathcal{L}(L^2(\Gamma), H^{-\frac{1}{2}}(\Gamma))} |w|^2_{H^{\frac{1}{2}}(J, L^2(\Gamma))}\\ &\le \left|P_M^\perp\right|^2_{\mathcal{L}(H^{\frac{3}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma))} |w|^2_{L^2(J, H^{\frac{3}{2}}(\Gamma))} + \left|P_M^\perp\right|^2_{\mathcal{L}(L^2(\Gamma), H^{-\frac{1}{2}}(\Gamma))} |w|^2_{H^{\frac{3}{4}}(J, L^2(\Gamma))}\\ &\le \left|P_M^\perp\right|^2_{\mathcal{H}(\Gamma_{\rm c})} |w|^2_{\mathcal{G}^2_{\rm c}}. \end{aligned}$$

Recalling Lemma 2.7 and Proposition 2.3, we find

$$\begin{aligned}
&\left|(1-Q_{\widetilde{M}})P_M\right|^2_{\mathcal{L}(G^2_{\rm c},\underline{G}^1_{\rm c})} \le \left|1-Q_{\widetilde{M}}\right|^2_{\mathcal{L}(P_M\mathcal{I}(\Gamma_{\rm c}),\underline{G}^1_{\rm c})} \left|P_M\right|^2_{\mathcal{L}(G^2_{\rm c},\mathcal{I}(\Gamma_{\rm c}))}\\
&\le \left|1-Q_{\widetilde{M}}\right|^2_{\mathcal{L}\left(H^{\frac{3}{8}}(J,P_M H^{\frac{3}{4}}(\Gamma)),L^2(J,P_M H^{\frac{1}{2}}(\Gamma))\right)} \left|P_M\right|^2_{\mathcal{L}(G^2_{\rm c},\mathcal{I}(\Gamma_{\rm c}))}\\
&\quad + \left|1-Q_{\widetilde{M}}\right|^2_{\mathcal{L}(H^{\frac{9}{16}}(J,P_M H^{\frac{3}{8}}(\Gamma)),H^{\frac{1}{2}}(J,P_M H^{-\frac{1}{2}}(\Gamma)))} \left|P_M\right|^2_{\mathcal{L}(G^2_{\rm c},\mathcal{I}(\Gamma_{\rm c}))}\\
&\le \overline{C}_{[M]}\left(\left|1-Q_{\widetilde{M}}\right|^2_{\mathcal{L}(H^{\frac{3}{8}}(J),L^2(J))} + \left|1-Q_{\widetilde{M}}\right|^2_{\mathcal{L}(H^{\frac{9}{16}}(J),H^{\frac{1}{2}}(J))}\right).
\end{aligned}$$

Above we considered the fractional Sobolev spaces $H^s(J)$, $s\in[0,1]$, being endowed with the norm defined through the Fourier Transform. Recall that we can also see $H^s(J)$ as being the domain $H^s_{\rm f}(J)$ of the fractional operator $(-\Delta_J+1)^{\frac{s}{2}}$, where we recall $-\Delta_J+1\colon H^2(J)\cap H^1_0(J)\to L^2(J)$, with the equivalent norm

$$|g|^2_{H^s_{\rm f}(J)} = \sum_{i=1}^{+\infty}(1+\beta_i)^s g_i^2, \quad \text{with} \quad w =: \sum_{i=1}^{+\infty} g_i\xi_i \in H^s(J) = H^s_{\rm f}(J).$$

In particular, observe that for $1\ge r\ge s\ge 0$, and

$$|g|^2_{H^s_{\rm f}(J)} = (1+\beta_{\widetilde{M}})^{s-r}\sum_{i=1}^{+\infty}(1+\beta_{\widetilde{M}})^{r-s}(1+\beta_i)^s g_i^2 \le (1+\beta_{\widetilde{M}})^{s-r}\,|g|^2_{H^r_{\rm f}(J)}.$$

This allow us to conclude that

$$\begin{aligned}
&\left|(1-Q_{\widetilde{M}})P_M\right|^2_{\mathcal{L}(G^2_{\rm c},G^1_{\rm c})}\\
&\le \overline{C}_{[M]}\left(C_1(1+\beta_{\widetilde{M}})^{-\frac{3}{8}} + C_2(1+\beta_{\widetilde{M}})^{-\frac{1}{16}}\right) \le 2C_3\overline{C}_{[M]}(1+\beta_{\widetilde{M}})^{-\frac{1}{16}}.
\end{aligned}$$

with $C_3=\max\{C_1,C_2\}$. Therefore, by setting $\widetilde{M}\in\mathbb{N}$ so that

$$1+\beta_{\widetilde{M}} \ge (2C_3\overline{C}_{[M]})^{16}(\tfrac{D_1^\sharp}{D_2^\flat})^{-16}\,|1-P_M|^{-32}_{\mathcal{H}(\Gamma_{\rm c})}, \tag{43}$$

we arrive at $\left|(1-Q_{\widetilde{M}})P_M\right|^2_{\mathcal{L}(G^2_{\rm c},G^1_{\rm c})} \le \frac{D_1^\sharp}{D_2^\flat}\,|1-P_M|^2_{\mathcal{H}(\Gamma_{\rm c})}$ and, by using (41) and (42), we obtain $\left|1-Q_{\widetilde{M}}P_M\right|^2_{\mathcal{L}(G^2_{\rm c},G^1_{\rm c})} \le 4\frac{D_1^\sharp}{D_2^\flat}\,|1-P_M|^2_{\mathcal{H}(\Gamma_{\rm c})}$. □

Now, Theorem 2.4 follows as a corollary of Theorem 2.3 and Lemma 2.8.

Proof of Theorem 2.4 Let us set $\widetilde{M}$ as in Lemma 2.8. From (40), we find that the stability condition (38) is satisfied. Then by Theorem 2.3 we have that system (37) is exponentially stable with rate $\frac{\delta}{2}$, and the control $u^{\check{\xi}_{\rm c}(z_0)} :=$

$(B_\Psi)^{-1}\breve\zeta_{\rm c}(z_0)$ (i.e., $z|_\Gamma=\breve\zeta_{\rm c}(z_0)=B_\Psi u^{\breve\zeta_{\rm c}(z_0)}$) satisfies

$$\begin{aligned}&\left|(B_\Psi)^{-1}\breve\zeta_{\rm c}(z_0)\right|^2_{H^1(\mathbb{R}_{s_0},\mathbb{R}^M)}=\sum_{j\in\mathbb{N}}\left|(B_\Psi)^{-1}\breve\zeta_{\rm c}(z_0)|_{J^j}\right|^2_{H^1(J^j,\mathbb{R}^M)}\\ &\le\sup_{i\in\mathbb{N}}\left|(B_\Psi)^{-1}\right|^2_{\mathcal{L}(Q^i_{\widetilde M}P_MG^2_{\rm c}(J^i,\Gamma),H^1(J^i,\mathbb{R}^M))}\sum_{j\in\mathbb{N}}\left|\breve\zeta_{\rm c}(z_0)|_{J^j}\right|^2_{G^2_{\rm c}(J^j,\Gamma)}\\ &=C_1\left|\breve\zeta_{\rm c}(z_0)\right|^2_{G^2_{\rm c}(\mathbb{R}_{s_0},\Gamma)}\le C_1\frac{C}{1-{\rm e}^{-\delta T_*}}\left|z_0\right|^2_H.\end{aligned}$$

Note that $\left|(B_\Psi)^{-1}\right|^2_{\mathcal{L}(Q^i_{\widetilde M}P_MG^2_{\rm c}(J^i,\Gamma),H^1(J^i,\mathbb{R}^M))}$ is independent of i, that is, the supremum coincides with $C_1:=\left|(B_\Psi)^{-1}\right|^2_{\mathcal{L}(Q_{\widetilde M}P_MG^2_{\rm c}(J,\Gamma),H^1(J,\mathbb{R}^M))}$. □

2.5 An example of suitable actuators. Estimate on the number of actuators.

Here we present a set of actuators which allow us to stabilize the system, that is, such that the stabilizability condition (40) is satisfied.

Let us take as actuators the functions (cf. [41,42]),

$$\operatorname{span}\{\Psi_i=\phi\mathbb{E}^0_{\Gamma_{\rm c}}\psi_i(\bar x)\mid 1\le i\le M\}\tag{44}$$

where the ψ_is are the first eigenfunctions of the Laplace (Laplace–de Rham) operator $\Delta_{\Gamma_{\rm c}}$ under homogeneous Dirichlet boundary conditions on $\Gamma_{\rm c}$

$$\Delta_{\Gamma_{\rm c}}\psi_i=\sigma_i\psi_i,\quad\text{with}\quad 0\le\sigma_i\le\sigma_{i+1}\to+\infty,$$

and where $\mathbb{E}^0_{\Gamma_{\rm c}}$ stands for the extension, to Γ, by zero outside $\Gamma_{\rm c}$, and $\phi\in C^2(\Gamma)$ is function with $\operatorname{supp}\phi=\overline{\Gamma_{\rm c}}$. For simplicity we suppose that the boundary $\partial\Gamma_{\rm c}$ of $\Gamma_{\rm c}$ in Γ is either empty or C^∞-smooth.

We see that

$$\begin{aligned}|1-P_M|^2_{\mathcal{H}(\Gamma_{\rm c})}&=|1-P_M|^2_{\mathcal{L}(H^{\frac32}(\Gamma),H^{\frac12}(\Gamma))}+|1-P_M|^2_{\mathcal{L}(L^2(\Gamma),H^{-\frac12}(\Gamma))}\\ &\le C_\phi((1+\sigma_M)^{-1}+(1+\sigma_M)^{-\frac12})\le 2C_\phi(1+\sigma_M)^{-\frac12},\end{aligned}$$

for a suitable constant depending on $|\phi|_{C^2(\Gamma)}$. Therefore, the stabilizability condition (40) is satisfied if

$$1+\sigma_M>\left(\tfrac{8C_\phi D_1^\sharp\Upsilon{\rm e}^{\delta T_*}}{D_2^\flat}\right)^2.\tag{45}$$

We may expect (as it is the case when $\Gamma_{\rm c}$ is flat, see [31]) that, $\sigma_M\ge C_dM^{\frac{2}{d-1}}$, for a suitable constant $C_d>0$. Then, we would obtain a sufficient condition for stabilizability in terms of the number of actuators as $1+C_dM^{\frac{2}{d-1}}>\left(\frac{8C_\phi D_1^\sharp\Upsilon{\rm e}^{\delta T_*}}{D_2^\flat}\right)^2$, which follows from $M\ge\left(\frac{8C_\phi D_1^\sharp\Upsilon{\rm e}^{\delta T_*}}{C_d^{\frac12}D_2^\flat}\right)^{d-1}$.

Remark 2.3 Note that the mapping $\phi\cdot\colon f \mapsto \phi f$ is in $\mathcal{L}(H^2(\Gamma))\bigcap\mathcal{L}(L^2(\Gamma))$ with $|\phi\cdot|_{\mathcal{L}(H^s(\Gamma))} \le C\,|\phi|_{C^2(\Gamma)}$, for all $s\in\{0,2\}$. Then by an interpolation argument, we can also show that the inequality holds for all $s\in[0,2]$, and from $\langle \phi f, g\rangle_{H^{-s}(\Gamma),H^s(\Gamma)} \coloneqq \langle f, \phi g\rangle_{H^{-s}(\Gamma),H^s(\Gamma)}$, we can conclude that $|\phi\cdot|_{\mathcal{L}(H^s(\Gamma))} \le C\,|\phi|_{C^2(\Gamma)}$ for all $s\in[-2,2]$.

Remark 2.4 In the beginning of Section 2.4, we ask the regularity $H^{\frac{3}{2}}(\Gamma)$ for the actuators, in order to have strong solutions. We take the function ϕ above, because the functions $\operatorname{span}\{\mathbb{E}^0_{\Gamma_c}\psi_i(\bar{x}) \mid 1\le i\le M\}$ as above are not in $H^{\frac{3}{2}}(\Gamma)$ in general, though we have $\psi_i\in H^2(\Gamma_c)\bigcap H^1_0(\Gamma_c)$.

2.6 Feedback stabilizing rule

We assume, in this section, that the actuators $\{\Psi_i \mid i\in\{1,2,\dots,M\}\}$ allow us to stabilize the system. That is we assume that there exists a family of controls $u=u(z_0)\in H^1(\mathbb{R}_{s_0},\mathbb{R}^M)$, so that the system

$$\tfrac{\partial}{\partial t}z - \nu\Delta z - \tfrac{\lambda}{2}z + \widehat{L}z = 0,\quad z|_\Gamma = B_\Psi u(z_0),\quad z(s_0)=z_0 \tag{46}$$

is exponentially stable, with $|z|^2_{L^2(\mathbb{R}_{s_0},H^1(\Omega))} + |u|^2_{H^1(\mathbb{R}_{s_0},\mathbb{R}_M)} \le C_2\,|z_0|^2_H$. We can rewrite (46) in the variables $(v,\kappa)=(z-B_{\widetilde{\Psi}}u,u)$, in the form of the extended system (13). Then as explained in the Introduction we can follow a standard procedure to find a stabilizing feedback control operator $\mathcal{F}_0=\mathcal{F}_0(t)$, in the form $\mathcal{F}_0(t)\begin{bmatrix} v\\ \kappa\end{bmatrix} = -\begin{bmatrix}-B_{\widetilde{\Psi}}\\ 1\end{bmatrix}\varkappa = \begin{bmatrix}-B_{\widetilde{\Psi}}\\ 1\end{bmatrix}\begin{bmatrix}-B_{\widetilde{\Psi}}{}^* & 1\end{bmatrix}\Pi\begin{bmatrix} v\\ \kappa\end{bmatrix}$, where $\Pi=\Pi(s)$ can be taken as the solution of a differential Riccati equation as (15). That is, the solution of

$$\tfrac{\partial}{\partial t}\begin{bmatrix} v\\ \kappa\end{bmatrix} + \begin{bmatrix}-\nu\Delta+\widehat{L} & \widehat{L}_\varsigma B_{\widetilde{\Psi}}\\ 0 & \varsigma\end{bmatrix}\begin{bmatrix} v\\ \kappa\end{bmatrix} + \mathcal{F}_0\begin{bmatrix} v\\ \kappa\end{bmatrix} = 0,\qquad \begin{bmatrix} v(s_0)\\ \kappa(s_0)\end{bmatrix} = \begin{bmatrix} v_0\\ \kappa_0\end{bmatrix} \tag{47}$$

satisfies, with C independent of (v_0,κ_0), the estimate

$$|(v,\kappa)(t)|^2_{\mathbf{H}} \le C\mathrm{e}^{-\lambda t}|(v_0,\kappa_0)|^2_{\mathbf{H}},\quad \text{for all}\quad t\ge 0. \tag{48}$$

Recall the notations $\mathbf{H}=H\times\mathbb{R}^M,\quad \mathbf{V}=V\times\mathbb{R}^M,\quad \mathbf{D}(\Delta)=\mathrm{D}(\Delta)\times\mathbb{R}^M$.

Theorem 2.5 *If $(v_0,\kappa_0)\in\mathbf{V}$, then the solution of (47) satisfies*

$$\left|\mathrm{e}^{\frac{\lambda}{2}t}(v,\kappa)(t)\right|^2_{\mathbf{V}} \le C|(v_0,\kappa_0)|^2_{\mathbf{V}},\quad \textit{for all } t\ge 0,$$

$$\sup_{t\ge 0}\left|\mathrm{e}^{\frac{\lambda}{2}\bullet}(v,\kappa)\right|^2_{L^2((t,t+1),\mathbf{D}(\Delta))} \le C|(v_0,\kappa_0)|^2_{\mathbf{V}},$$

with C independent of (v_0,κ_0). The solution (v,κ) is, and is unique, in the space $L^2_{\mathrm{loc}}(\mathbb{R}_0,\mathbf{D}(\Delta))\cap C([0,+\infty),\mathbf{V})$.

The proof is omitted. It follows from (48) and from the smoothing property for parabolic equations (cf. (2.6)). For further details we refer to [42]. Here we just note that we can rewrite the system (47) with a general external forcing $\mathbf{h}$ in the place of the controller as

$$\tfrac{\partial}{\partial t}\mathbf{z} + \mathbf{A}\mathbf{z} + \mathbf{R}\mathbf{z} + \mathbf{C}\mathbf{z} + \mathbf{h} = 0, \quad \mathbf{z}(s_0) = \mathbf{z}_0. \tag{49}$$

with $\mathbf{z} = \begin{bmatrix} v \\ \kappa \end{bmatrix}$, $\mathbf{z}_0 = \begin{bmatrix} v_0 \\ \kappa_0 \end{bmatrix}$, the diffusion term $\mathbf{A} = \begin{bmatrix} -\nu\Delta v & 0 \\ 0 & \varsigma\kappa \end{bmatrix}$, the reaction term $\mathbf{R}\mathbf{z} \coloneqq \begin{bmatrix} \hat{a}(v + B_{\widetilde{\Psi}}\kappa) - 2\varsigma B_{\widetilde{\Psi}}\kappa \\ 0 \end{bmatrix}$, and the convection term $\mathbf{C}\mathbf{z} \coloneqq \begin{bmatrix} \nabla\cdot(\hat{b}(v + B_{\widetilde{\Psi}}\kappa)) \\ 0 \end{bmatrix}$. In this case, from (24) and Lemma 2.1, we can obtain the analogous estimates

$$\begin{aligned}
&\langle \mathbf{R}\mathbf{z}, \tilde{\mathbf{z}}\rangle_{\mathbf{V}',\mathbf{V}} \le C\,|\hat{a}|_{L^d}\,|\mathbf{z}|_{\mathbf{H}}^{\frac12}\,|\mathbf{z}|_{\mathbf{V}}^{\frac12}\,|\tilde{\mathbf{z}}|_{\mathbf{H}}^{\frac12}\,|\tilde{\mathbf{z}}|_{\mathbf{V}}^{\frac12}, && \text{for} \quad d \in \{1,2\}.\\
&\langle \mathbf{R}\mathbf{z}, \tilde{\mathbf{z}}\rangle_{\mathbf{V}',\mathbf{V}} \le C\,|\hat{a}|_{L^d}\,|\mathbf{z}|_{\mathbf{H}}\,|\tilde{\mathbf{z}}|_{\mathbf{V}}, && \text{for} \quad d \ge 3.\\
&\langle \mathbf{C}\mathbf{z}, \tilde{\mathbf{z}}\rangle_{\mathbf{V}',\mathbf{V}} \le C|\hat{b}|_{L^\infty}\,|\mathbf{z}|_{\mathbf{H}}\,|\tilde{\mathbf{z}}|_{\mathbf{V}}, && \text{for} \quad d \ge 1.
\end{aligned}$$

for all $(\mathbf{z}, \tilde{\mathbf{z}}) \in \mathbf{V}\times\mathbf{V}$, and

$$\begin{aligned}
&|\mathbf{C}\mathbf{z}|_{\mathbf{H}} \le C|(\nabla\cdot\hat{b})|_{L^2}\,|\mathbf{z}|_{\mathbf{H}}^{\frac12}\,|\mathbf{z}|_{\mathbf{V}}^{\frac12} + |\hat{b}|_{L^\infty}\,|\mathbf{z}|_{\mathbf{V}}, && \text{for} \quad d = 1.\\
&|\mathbf{C}\mathbf{z}|_{\mathbf{H}} \le C|(\nabla\cdot\hat{b})|_{L^2}\,|z|_{\mathbf{H}}^{\frac12}\,|\mathbf{z}|_{\mathbf{D}(\Delta)}^{\frac12} + |\hat{b}|_{L^\infty}\,|\mathbf{z}|_{\mathbf{V}}, && \text{for} \quad d = 2.\\
&|\mathbf{C}\mathbf{z}|_{\mathbf{H}} \le C|(\nabla\cdot\hat{b})|_{L^2}\,|\mathbf{z}|_{\mathbf{V}}^{\frac12}\,|\mathbf{z}|_{\mathbf{D}(\Delta)}^{\frac12} + |\hat{b}|_{L^\infty}\,|\mathbf{z}|_{\mathbf{V}}, && \text{for} \quad d = 3.\\
&|\mathbf{C}\mathbf{z}|_{\mathbf{H}} \le C\,|\nabla\cdot b|_{L^\infty}\,|\mathbf{z}|_{\mathbf{H}} + |b|_{L^\infty}\,|\mathbf{z}|_{\mathbf{V}}, && \text{for} \quad d \ge 1,
\end{aligned}$$

for all $(\mathbf{z}, \tilde{\mathbf{z}}) \in \mathbf{D}(\Delta)\times\mathbf{D}(\Delta)$. Where $C > 0$ is a positive constant. In particular, notice that $\mathbf{D}(\Delta) \xhookrightarrow{\text{d,c}} \mathbf{V} \xhookrightarrow{\text{d,c}} \mathbf{H}$. The estimates above allow us to derive the analogous regularity properties for system (49) as in Lemmas 2.2 and 2.4.

3 The nonlinear systems

To derive the local stabilization result for the nonlinear system, we consider (47) with $\mathcal{N}$ as a perturbation:

$$\tfrac{\partial}{\partial t}\begin{bmatrix} v \\ \kappa \end{bmatrix} + \mathbb{A}_0 \begin{bmatrix} v \\ \kappa \end{bmatrix} + \mathcal{F}_0 \begin{bmatrix} v \\ \kappa \end{bmatrix} = \mathcal{N}\left(\begin{bmatrix} v \\ \kappa \end{bmatrix}\right), \quad \begin{bmatrix} v(0) \\ \kappa(0) \end{bmatrix} = \begin{bmatrix} v_0 \\ \kappa_0 \end{bmatrix}, \tag{50}$$

with

$$\mathbb{A}_0 \coloneqq \begin{bmatrix} -\nu\Delta + \widehat{L} & \widehat{L}_\varsigma B_{\widetilde{\Psi}} \\ 0 & \varsigma \end{bmatrix} \quad \text{and} \quad \mathcal{N}\left(\begin{bmatrix} v \\ \kappa \end{bmatrix}\right) \coloneqq \begin{bmatrix} \widehat{\mathcal{N}}(v + B_{\widetilde{\Psi}}\kappa) \\ 0 \end{bmatrix}.$$

The procedure is analogous to the one in [13, 30], however, since we are considering a general class of nonlinearities, we will recall the main steps.

Let us define $\mathbf{z} := \begin{bmatrix} v \\ \kappa \end{bmatrix}$. Note that we we can identify $v + B_{\widetilde{\Psi}}\kappa \in V \oplus B_{\widetilde{\Psi}}\mathbb{R}^M$, with its components (v, κ).

It follows that $\mathcal{N}$ satisfy, for a suitable $\widehat{C}_1 > 0$, estimates (7) in the form

$$\begin{aligned} |\mathcal{N}(\mathbf{z}) - \mathcal{N}(\tilde{\mathbf{z}})|_{\mathbf{H}}^2 &\le \widehat{C}_1 |\mathbf{z} - \tilde{\mathbf{z}}|_{\mathbf{V}}^2 \left(1 + |\mathbf{z}|_{\mathbf{V}}^{\varepsilon_1} + |\tilde{\mathbf{z}}|_{\mathbf{V}}^{\varepsilon_2}\right) \left(|\mathbf{z}|_{\mathbf{D}(\Delta)}^2 + |\tilde{\mathbf{z}}|_{\mathbf{D}(\Delta)}^2\right) \\ &\quad + \widehat{C}_1 |\mathbf{z} - \tilde{\mathbf{z}}|_{\mathbf{D}(\Delta)}^2 \left(|\mathbf{z}|_{\mathbf{V}}^{\varepsilon_3} + |\tilde{\mathbf{z}}|_{\mathbf{V}}^{\varepsilon_4}\right), \end{aligned} \tag{51a}$$

and

$$\begin{aligned} &(\mathcal{N}(\mathbf{z}) - \mathcal{N}(\tilde{\mathbf{z}}), \mathbf{z} - \tilde{\mathbf{z}})_{\mathbf{H}} \\ &\le \widehat{C}_1 (1 + |\mathbf{z}|_{\mathbf{V}}^{\varepsilon_5} + |\tilde{\mathbf{z}}|_{\mathbf{V}}^{\varepsilon_6})^{\frac{1}{2}} (1 + |\mathbf{z}|_{\mathbf{D}(\Delta)}^2 + |\tilde{\mathbf{z}}|_{\mathbf{D}(\Delta)}^2)^{\frac{1}{2}} \, |\mathbf{z} - \tilde{\mathbf{z}}|_{\mathbf{V}} \, |\mathbf{z} - \tilde{\mathbf{z}}|_{\mathbf{H}} \\ &\quad + \widehat{C}_1 (1 + |\mathbf{z}|_{\mathbf{V}}^{\varepsilon_5} + |\tilde{\mathbf{z}}|_{\mathbf{V}}^{\varepsilon_6})(1 + |\mathbf{z}|_{\mathbf{D}(\Delta)}^2 + |\tilde{\mathbf{z}}|_{\mathbf{D}(\Delta)}^2) \, |\mathbf{z} - \tilde{\mathbf{z}}|_{\mathbf{H}}^2 \end{aligned} \tag{51b}$$

with $\{\varepsilon_1, \varepsilon_2\} \in [0, +\infty)$ and $\{\varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6\} \in [2, +\infty)$.

With (51) we will be able to follow the argument used in the internal case in [13, 30] to prove the following result, saying that the feedback control stabilizes the nonlinear system to zero *locally*. Notice that Theorem 2.5 says that the feedback controller stabilizes the linearized system to zero *globally*.

Theorem 3.1 *We assume that* (47) *is stable, with* $\mathcal{F}_0 \in \mathcal{L}(\mathbf{H})$, *and that the estimates in Theorem 2.5 hold true. Then there exists* $\epsilon > 0$ *with the following property: if we have that* $|(v_0, \kappa_0)|_{\mathbf{V}} \le \epsilon$, *then there is a solution for system* (50), *in* $\mathbb{R}_0 \times \Omega$, *which is in* $L^2_{\text{loc}}(\mathbb{R}_0, \mathbf{D}(\Delta)) \cap C([0, +\infty), \mathbf{V})$, *is unique, and satisfies*

$$|(v, \kappa)(t)|_{\mathbf{V}} \le C \mathrm{e}^{-\lambda t} |(v_0, \kappa_0)|_{\mathbf{V}}, \quad \textit{for all } t \ge 0, \tag{52}$$

for a suitable constant C *independent of* $(\epsilon, (v_0, \kappa_0))$.

Proof We sketch/recall the main steps. We define the Banach space

$$\mathcal{Z}^\lambda := \left\{ \mathbf{z} \in L^2_{\text{loc}}(\mathbb{R}_0, \mathbf{H}) \;\middle|\; |\mathbf{z}|_{\mathcal{Z}^\lambda} < \infty \right\}$$

endowed with the norm $|\mathbf{z}|_{\mathcal{Z}^\lambda} := \sup_{r \ge 0} \left| \mathrm{e}^{\frac{\lambda}{2} \cdot} \mathbf{z} \right|_{W((r, r+1), \mathbf{D}(\Delta), \mathbf{H})}$. We also set

$$\mathcal{Z}^\lambda_{\text{loc}} := \left\{ \mathbf{z} \in L^2_{\text{loc}}(\mathbb{R}_0, \mathbf{H})) \;\middle|\; \left| \mathrm{e}^{\frac{\lambda}{2} \cdot} \mathbf{z} \right|_{W((r, r+1), \mathbf{D}(\Delta), \mathbf{H})} < \infty, \text{ for all } r \ge 0 \right\}.$$

For a given constant $\varrho > 0$ we define the subset

$$\mathcal{Z}^\lambda_\varrho := \left\{ \mathbf{z} \in \mathcal{Z}^\lambda \mid |\mathbf{z}|^2_{\mathcal{Z}^\lambda} \le \varrho |\mathbf{z}_0|^2_{\mathbf{V}} \right\},$$

with $\mathbf{z}_0 = \begin{bmatrix} v_0 \\ \kappa_0 \end{bmatrix} \in \mathbf{V}$, and the mapping $\Psi \colon \mathcal{Z}^\lambda_\varrho \to \mathcal{Z}^\lambda_{\text{loc}}$, $\bar{\mathbf{z}} \mapsto \mathbf{z}$, taking a given vector $\bar{\mathbf{z}}$ to the solution $\mathbf{z} = \begin{bmatrix} v \\ \kappa \end{bmatrix}$ of

$$\tfrac{\partial}{\partial t} \mathbf{z} + \mathbb{A}_0 \mathbf{z} + \mathcal{F}_0 \mathbf{z} = \mathcal{N}(\bar{\mathbf{z}}), \quad \mathbf{z}(0) = \mathbf{z}_0. \tag{53}$$

ⓢ Step 1: *a preliminary estimate.* Proceeding as in [13] we can conclude that the solution of the system (53) with a general $g \in L^2_{\mathrm{loc}}(\mathbb{R}_0, \mathbf{H})$ in the place of $\mathcal{N}(\bar{\mathbf{z}})$ satisfies

$$\sup_{r\geq 0} |\mathrm{e}^{\frac{\lambda}{2}\cdot}\mathbf{z}(\cdot)|^2_{W((r,r+1),\mathbf{D}(\Delta),\mathbf{H})} \leq \overline{C}\left(|\mathbf{z}_0|^2_{\mathbf{V}} + \sup_{k\in\mathbb{N}}\int_k^{k+1} \mathrm{e}^{2\lambda s}|g(s)|^2_{\mathbf{H}}\,\mathrm{d}s\right). \quad (54)$$

with $\overline{C} = \overline{C}_{\left[C_{\mathcal{W}_{\mathrm{st}}},\lambda,\frac{1}{\lambda},\frac{T_*}{\delta},\delta,\Upsilon\right]}$.

ⓢ Step 2: *Ψ maps $\mathcal{Z}^\lambda_\varrho$ into itself, if $|\mathbf{z}_0|_{\mathbf{V}}$ is small.* Now we will replace g by $\mathcal{N}(\bar{\mathbf{z}})$ in (54). From (51a), with $(\mathbf{z}, \tilde{\mathbf{z}}) = (\bar{\mathbf{z}}, 0)$, and from $\mathrm{e}^{\frac{\lambda}{2}s\varepsilon} \geq 1$ since $\frac{\lambda}{2}s\varepsilon \geq 0$, we find that

$$\sup_{k\in\mathbb{N}}\int_k^{k+1} \mathrm{e}^{2\lambda s}|\mathcal{N}(\bar{\mathbf{z}})(s)|^2_{\mathbf{H}}\,\mathrm{d}s \leq \sup_{k\in\mathbb{N}}\widetilde{C}_k\int_k^{k+1}|\mathrm{e}^{\frac{\lambda}{2}r}\bar{\mathbf{z}}(r)|^2_{\mathbf{D}(\Delta)}\,\mathrm{d}r$$

with $\widetilde{C}_k := \sup_{s\in[k,k+1]}\widehat{C}_1\left(|\mathrm{e}^{\frac{\lambda}{2}s}\bar{\mathbf{z}}(s)|^2_{\mathbf{V}} + |\mathrm{e}^{\frac{\lambda}{2}s}\bar{\mathbf{z}}(s)|^{2+\varepsilon_1}_{\mathbf{V}} + |\mathrm{e}^{\frac{\lambda}{2}s}\bar{\mathbf{z}}(s)|^{\varepsilon_3}_{\mathbf{V}}\right)$. Therefore

$$\sup_{k\in\mathbb{N}}\int_k^{k+1} \mathrm{e}^{2\lambda s}|\mathcal{N}(\bar{\mathbf{z}})(s)|^2_{\mathbf{H}}\,\mathrm{d}s \leq \widehat{C}_1\left(|\bar{\mathbf{z}}|^4_{\mathcal{Z}^\lambda} + |\bar{\mathbf{z}}|^{4+\varepsilon_1}_{\mathcal{Z}^\lambda} + |\bar{\mathbf{z}}|^{\varepsilon_3+2}_{\mathcal{Z}^\lambda}\right),$$

because $W((k,k+1),\mathbf{D}(\Delta),\mathbf{H}) \hookrightarrow C([k,k+1],\mathbf{V})$ uniformly with respect to $k \in \mathbb{N}$. Thus, inequality (54) with $g = \mathcal{N}(\bar{\mathbf{z}})$ and $\bar{\mathbf{z}} \in \mathcal{Z}^\lambda_\varrho$ gives us

$$\begin{aligned}|\Psi(\bar{\mathbf{z}})|^2_{\mathcal{Z}^\lambda} &\leq \overline{C}\left(|\mathbf{z}_0|^2_V + \widehat{C}_1\left(|\bar{\mathbf{z}}|^4_{\mathcal{Z}^\lambda} + |\bar{\mathbf{z}}|^{4+\varepsilon_1}_{\mathcal{Z}^\lambda} + |\bar{\mathbf{z}}|^{\varepsilon_3+2}_{\mathcal{Z}^\lambda}\right)\right)\\ &\leq C_2\left(1 + \varrho^2|\mathbf{z}_0|^2_{\mathbf{V}} + \varrho^{\frac{4+\varepsilon_1}{2}}|\mathbf{z}_0|^{2+\varepsilon_1}_{\mathbf{V}} + \varrho^{\frac{\varepsilon_3+2}{2}}|\mathbf{z}_0|^{\varepsilon_3}_{\mathbf{V}}\right)|\mathbf{z}_0|^2_{\mathbf{V}}\end{aligned}$$

and if we set $\varrho = 4C_2$ and $\epsilon < \min\left\{\varrho^{-1}, \varrho^{-\frac{4+\varepsilon_1}{2(2+\varepsilon_1)}}, \varrho^{-\frac{\varepsilon_3+2}{2\varepsilon_3}}\right\}$, then we obtain

$$|\Psi(\bar{\mathbf{z}})|^2_{\mathcal{Z}^\lambda} \leq C_2\left(1 + \varrho^2\epsilon^2 + \varrho^{\frac{4+\varepsilon_1}{2}}\epsilon^{2+\varepsilon_1} + \varrho^{\frac{\varepsilon_3+2}{2}}\epsilon^{\varepsilon_3}\right) \leq 4C_2|\mathbf{z}_0|^2_{\mathbf{V}} = \varrho|\mathbf{z}_0|^2_{\mathbf{V}}, \quad (55)$$

if $|\mathbf{z}_0|_{\mathbf{V}} \leq \epsilon$, which means that $\Psi(\bar{\mathbf{z}}) \in \mathcal{Z}^\lambda_\varrho$.

ⓢ Step 3: *Ψ is a contraction, if $|\mathbf{z}_0|_{\mathbf{V}}$ is smaller.* Let us take two functions $\bar{\mathbf{z}}_1, \bar{\mathbf{z}}_2 \in \mathcal{Z}^\lambda_\varrho$ and let $\Psi(\bar{\mathbf{z}}_1)$ and $\Psi(\bar{\mathbf{z}}_2)$ be the corresponding solutions for (53). Set $e = \bar{\mathbf{z}}_1 - \bar{\mathbf{z}}_2$ and $d^\Psi = \Psi(\bar{\mathbf{z}}_1) - \Psi(\bar{\mathbf{z}}_2)$. Then d^Ψ solves (53) with $d^\Psi(0) = 0$ and $g = \mathcal{N}(\bar{\mathbf{z}}_1) - \mathcal{N}(\bar{\mathbf{z}}_2)$ in the place of $\mathcal{N}(\bar{\mathbf{z}})$. Therefore, by (54), we have

$$|d^\Psi|^2_{\mathcal{Z}^\lambda} \leq \overline{C}\sup_{t\geq 0}\int_t^{t+1} \mathrm{e}^{2\lambda s}|\mathcal{N}(\bar{\mathbf{z}}_1) - \mathcal{N}(\bar{\mathbf{z}}_2)|^2_{\mathbf{H}}\,\mathrm{d}s,$$

and from $\mathrm{e}^{2\lambda s}|\mathcal{N}(\bar{\mathbf{z}}_1)-\mathcal{N}(\bar{\mathbf{z}}_2)|_{\mathbf{H}}^2 \le |\mathrm{e}^{\frac{\lambda}{2}s}e(s)|_{\mathbf{V}}^2\Xi_1 + |\mathrm{e}^{\frac{\lambda}{2}s}e(s)|_{\mathbf{D}(\Delta)}^2\Xi_2$, with

$$\begin{aligned}\Xi_1 &= \left(1+\left|\mathrm{e}^{\frac{\lambda}{2}s}\bar{\mathbf{z}}_1(s)\right|_{\mathbf{V}}^{\varepsilon_1}+\left|\mathrm{e}^{\frac{\lambda}{2}s}\bar{\mathbf{z}}_2(s)\right|_{\mathbf{V}}^{\varepsilon_2}\right)\left(\left|\mathrm{e}^{\frac{\lambda}{2}s}\bar{\mathbf{z}}_1(s)\right|_{\mathbf{D}(\Delta)}^2+\left|\mathrm{e}^{\frac{\lambda}{2}s}\bar{\mathbf{z}}_2(s)\right|_{\mathbf{D}(\Delta)}^2\right),\\ \Xi_2 &= \left(|\mathrm{e}^{\frac{\lambda}{2}s}\bar{\mathbf{z}}_1(s)|_{\mathbf{V}}^{\varepsilon_3}+|\mathrm{e}^{\frac{\lambda}{2}s}\bar{\mathbf{z}}_2(s)|_{\mathbf{V}}^{\varepsilon_4}\right),\end{aligned}$$

it follows that

$$|d^\Psi|_{\mathcal{Z}^\lambda}^2 \le C_3|e|_{\mathcal{Z}^\lambda}^2\left(1+|\bar{\mathbf{z}}_1|_{\mathcal{Z}^\lambda}^{\varepsilon_1}+|\bar{\mathbf{z}}_2|_{\mathcal{Z}^\lambda}^{\varepsilon_2}\right)\left(|\bar{\mathbf{z}}_1|_{\mathcal{Z}^\lambda}^2+|\bar{\mathbf{z}}_2|_{\mathcal{Z}^\lambda}^2+|\bar{\mathbf{z}}_1|_{\mathcal{Z}^\lambda}^{\varepsilon_3}+|\bar{\mathbf{z}}_2|_{\mathcal{Z}^\lambda}^{\varepsilon_4}\right),$$

and since $\bar{\mathbf{z}}_1$ and $\bar{\mathbf{z}}_2$ are both in $\mathcal{Z}_\varrho^\lambda$, we arrive to

$$|d^\Psi|_{\mathcal{Z}^\lambda}^2 \le C_3|e|_{\mathcal{Z}^\lambda}^2\Big(1+\varrho^{\frac{\varepsilon_1}{2}}|\mathbf{z}_0|_{\mathbf{V}}^{\varepsilon_1}+\varrho^{\frac{\varepsilon_2}{2}}|\mathbf{z}_0|_{\mathbf{V}}^{\varepsilon_2}\Big)\Big(2\varrho|\mathbf{z}_0|_{\mathbf{V}}^2+\varrho^{\frac{\varepsilon_3}{2}}|\mathbf{z}_0|_{\mathbf{V}}^{\varepsilon_3}+\varrho^{\frac{\varepsilon_4}{2}}|\mathbf{z}_0|_{\mathbf{V}}^{\varepsilon_4}\Big).$$

Choosing $\epsilon>0$, smaller than the one in Step 2, such that

$$\epsilon<\min\Big\{\varrho^{-1},\varrho^{-\frac{4+\varepsilon_1}{2(2+\varepsilon_1)}},\varrho^{-\frac{\varepsilon_3+2}{2\varepsilon_3}},\varrho^{-\frac{1}{2}},(\tfrac{\gamma^2}{18C_3})^{\frac{1}{2}}\varrho^{-\frac{1}{2}},(\tfrac{\gamma^2}{9C_3})^{\frac{1}{\varepsilon_3}}\varrho^{-\frac{1}{2}},(\tfrac{\gamma^2}{9C_3})^{\frac{1}{\varepsilon_4}}\varrho^{-\frac{1}{2}}\Big\},$$

then we have that Ψ maps $\mathcal{Z}_\varrho^\lambda$ into itself and

$$\begin{aligned}|d^\Psi|_{\mathcal{Z}^\lambda}^2 &\le C_3|e|_{\mathcal{Z}^\lambda}^2\left(1+\varrho^{\frac{\varepsilon_1}{2}}\epsilon^{\varepsilon_1}+\varrho^{\frac{\varepsilon_2}{2}}\epsilon^{\varepsilon_2}\right)\left(2\varrho\epsilon^2+\varrho^{\frac{\varepsilon_3}{2}}\epsilon^{\varepsilon_3}+\varrho^{\frac{\varepsilon_4}{2}}\epsilon^{\varepsilon_4}\right)\\ &< C_3|e|_{\mathcal{Z}^\lambda}^2 3\tfrac{3\gamma^2}{9C_3},\end{aligned}$$

provided $|\mathbf{z}_0|_{\mathbf{V}}^2\le\epsilon$. That is $|\Psi(\bar{\mathbf{z}}_1)-\Psi(\bar{\mathbf{z}}_2)|_{\mathcal{Z}^\lambda}^2<\gamma^2|\bar{\mathbf{z}}_1-\bar{\mathbf{z}}_2|_{\mathcal{Z}^\lambda}^2$. Furthermore, we can see that ϵ can be taken independent of $(\varepsilon_1,\varepsilon_2,\dots,\varepsilon_6)\in\mathbb{R}_0^2\times\mathbb{R}_2^2$, because the function $t\mapsto\frac{c_1t+c_2}{c_3t+c_4}$, $t>0$ and $(c_1,c_2,c_3,c_4)\in\mathbb{R}^4$, is monotone if $c_3t+c_4\ne 0$ for all $t>0$. Indeed, we can take $\epsilon=\min\Big\{\varrho^{-1},\varrho^{-\frac{1}{2}},(\frac{\gamma^2}{18C_3})^{\frac{1}{2}}\varrho^{-\frac{1}{2}}\Big\}$. That is, ϵ depends essentially on the constant $\overline{C}$ in (54) and on the constant $\hat{C}_1$ in (51). From [13, Section 4.2] we can see that the constant $\overline{C}$ in (54) will essentially depend on the constant C in Theorem 2.5.

Ⓢ Step 4: Fixed point argument. We can conclude that if $\mathbf{z}_0\in\mathbf{V}$ is sufficiently small, $|\mathbf{z}_0|_V^2<\epsilon$, then there exists a unique fixed point $\mathbf{z}=\Psi(\bar{\mathbf{z}})=\bar{\mathbf{z}}\in\mathcal{Z}_\varrho^\epsilon$ for Ψ. It follows from the definitions of Ψ and Z_ϱ^ϵ that $\mathbf{z}$ solves the system (53), with $\bar{\mathbf{z}}=\mathbf{z}$. We can conclude that $\mathbf{z}$ solves (50). Further, inequality (52) can be concluded from (55).

Ⓢ Step 5: Uniqueness. Finally we show the uniqueness of the solution for (50) in the space $Z:=L^2_{\mathrm{loc}}(\mathbb{R}_0,\mathbf{D}(\Delta))\cap C([0,+\infty),\mathbf{V})\supset\mathcal{Z}_\varrho^\epsilon$. Let $\mathbf{z}_1$ and $\mathbf{z}_2$ be two solutions, in Z, for (50). It turns out that $e=\mathbf{z}_1-\mathbf{z}_2$ solves (53) with $g=\mathcal{N}(\mathbf{z}_1)-\mathcal{N}(\mathbf{z}_2)$ in the place of $\mathcal{N}(\bar{\mathbf{z}})$. Using (51b), and following standard arguments, we can obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}|e|_{\mathbf{H}}^2\le C_4(1+|\mathbf{z}_1|_{\mathbf{V}}^{\varepsilon_5}+|\mathbf{z}_2|_{\mathbf{V}}^{\varepsilon_6})(1+|\mathbf{z}_1|_{\mathbf{D}(\Delta)}^2+|\mathbf{z}_2|_{\mathbf{D}(\Delta)}^2)\,|e|_{\mathbf{H}}^2\,.$$

Notice that the function

$$s \mapsto \mathcal{G}(s) := C_4 \left(1 + |\mathbf{z}_1(s)|_{\mathbf{V}}^{\varepsilon_5} + |\mathbf{z}_2(s)|_{\mathbf{V}}^{\varepsilon_6}\right) \left(1 + |\mathbf{z}_1(s)|_{\mathbf{D}(\Delta)}^2 + |\mathbf{z}_2(s)|_{\mathbf{D}(\Delta)}^2\right)$$

is locally integrable, which allow us to write

$$|e(t)|_{\mathbf{H}}^2 \le \mathrm{e}^{\int_0^t \mathcal{G}(s)\,\mathrm{d}s} |e(0)|_{\mathbf{H}}^2 = 0, \quad \text{for all} \quad t \ge 0.$$

That is, the uniqueness holds true: $\mathbf{z}_1 - \mathbf{z}_2 = e = 0$. □

4 Proofs of Main Theorems 1.1 and 1.2.

Let us be given a solution $\hat{y}$ for the uncontrolled system (2) (with $u = 0$) with $s_0 = 0$ and $\hat{y}_0 := \hat{y}(0) \in H$. We suppose that $(\hat{y}, f) \in \mathfrak{C}$, with $\mathfrak{C}$ defined as in (8). That is, we suppose that $\hat{\mathcal{N}}$ and $(\hat{a}, \hat{b})$ defined in (5) satisfy (7) and (23), respectively, for suitable nonnegative constants $\widehat{C}$ and $C_{\mathcal{W}_{\mathrm{st}}}$.

By taking the Riccati feedback $\mathcal{F}_0(t) \begin{bmatrix} v \\ \kappa \end{bmatrix} = \begin{bmatrix} -B_{\widetilde{\Psi}} \\ 1 \end{bmatrix} \begin{bmatrix} -B_{\widetilde{\Psi}}^* & 1 \end{bmatrix} \Pi \begin{bmatrix} v \\ \kappa \end{bmatrix}$, as in Section 2.6, we see that the Main Theorem 1.2, in the Introduction, is a corollary of Theorems 3.1, 2.5 and 2.4.

Notice that the condition $(v_0, \kappa_0) \in V \times \mathbb{R}^M$ in Theorem 3.1 leads to the the compatibility condition $z(0)|_\Gamma = B_\Psi \kappa(0) \in \mathcal{S}_\Psi$, that is, $(y(0) - \hat{y}(0))|_\Gamma \in \mathcal{S}_\Psi$.

The case of internal controls. The analogous of Theorems 2.4 and 2.5, for the case of internal controls can be found in [17, Theorem 2.10 and Corollary 2.15], where the stabilizability condition reads (see also [29]),

$$|1_\omega(1 - P_M)1_\omega|_{\mathcal{L}(H,V')}^2 < \Upsilon^{-1}. \tag{56}$$

Observe that the computations in Section 3 can be followed by just using (7) instead of (51) and replacing the triple $(\mathbf{H}, \mathbf{V}, \mathbf{D}(\Delta))$ by $(H, V, \mathrm{D}(\Delta))$. Therefore, we arrive at the analogous of Theorem 2.4 for the internal case. As a corollary we obtain the Main Theorem 1.1 as in the Introduction.

5 Examples of covered nonlinearities

Many systems modeling real evolutions involve polynomial nonlinearities, for example Fisher-like equations [22,49] modeling population dynamics, Burgers-like equations [18,30] modeling fluid (e.g., traffic) flow, and the Schlögl equations [43] modeling certain chemical reactions. Here, we check the property

$$(\hat{y}, f) \in \mathfrak{C}$$

we ask/assume for the pair $(\hat{y}, f)$. See (8). That is, we investigate whether both (6) and (7) hold true, in case the function $f(y, \nabla y)$ takes (or can be written in) the form

$$f(y, \nabla y) = f_{\mathrm{r}}(y) + f_{\mathrm{c}}(y) \cdot \nabla y,$$

where $f_{\rm r}$ and $f_{\rm c} = [f_{{\rm c}1} \;\; f_{{\rm c}2} \;\; \dots \;\; f_{{\rm c}d}]^\top$, are polynomials:

$$f_{\rm r}(y) = \sum_{j=0}^{\bar p} \bar r_j y^j \quad \text{and} \quad f_{{\rm c}k}(y) = \sum_{j=0}^{p_k} r_{k,j} y^j,$$

with $\bar r_j$ and $r_{k,j}$ real numbers, and $p_k \in \mathbb{N}$ for $k \in \{1, \dots, d\}$.

It is enough to analyze the case of monomials, with degree greater than or equal to 2:

$$f(y) = y^{\bar n} \text{ with } \bar n \geq 2$$

and

$$f(y, \nabla y) = y^n \partial_{x_{\bar k}} y = \tfrac{1}{n+1} \partial_{x_{\bar k}} y^{n+1}, \quad \text{with } n \geq 1 \text{ for some } \bar k \in \{1, \dots, d\}.$$

In this case, recalling the notation in Section 1.1, for a given trajectory $\hat y$, we obtain respectively

$$\hat a = \bar n \hat y^{\bar n - 1}, \quad \hat b = 0,$$

and

$$\hat a = n \hat y^{n-1} \partial_{x_{\bar k}} \hat y - \nabla \cdot \hat b = 0, \quad \hat b = [\hat b_1 \;\; \hat b_2 \;\; \dots \;\; \hat b_d]^\top, \quad \text{with } \hat b_k = \begin{cases} 0 & \text{if } k \neq \bar k, \\ \hat y^n & \text{if } k = \bar k. \end{cases}$$

For illustration, we consider here the case $d = 3$. The following estimates will be also valid for $d \in \{1, 2\}$, though in those cases better estimates may hold true. On the other hand, some of the following arguments will not work in dimension $d \geq 4$, in that case some changes are needed.

5.1 Checking the conditions on the pair $(\hat a, \hat b)$. Case $d = 3$

Observe that in the case of a reaction nonlinearity $f(y) = y^{\bar n}$, we find that condition (6) is satisfied provided $\hat y \in L^\infty(\mathbb{R}_0, L^{3(\bar n - 1)})$. In the case of a convection nonlinearity $f(y, \nabla y) = y^n \partial_{x_{\bar k}} y$, we find that conditions (6) is satisfied provided $\hat y \in L^\infty_w(\mathbb{R}_0, L^\infty)$ and $\partial_{x_{\bar k}} \hat y \in L^\infty(\mathbb{R}_0, L^2)$.

5.2 Checking the conditions on the nonlinearity $\mathcal{N}$. Case $d = 3$

For simplicity, we restrict ourselves to the case $\hat y \in L^\infty_w(\mathbb{R}_0, L^\infty)$.

Example 1 In case $\mathcal{N}(z) = \hat y^m z^2$, $m \in \mathbb{N}$, conditions (7) hold true. We may write

$$|\mathcal{N}(z) - \mathcal{N}(\tilde z)|_H^2 = |\hat y^m (z - \tilde z)(z + \tilde z)|_H^2 \leq |z - \tilde z|_H^2 \, |\hat y^m|_{L^\infty}^2 \, |z + \tilde z|_{L^\infty}^2 ,$$

and

$$|\mathcal{N}(z) - \mathcal{N}(\tilde z)|_H^2 \leq C \, |\hat y^m|_{L^\infty}^2 \, |z - \tilde z|_V^2 \left(|z|_{{\rm D}(\Delta)}^2 + |\tilde z|_{{\rm D}(\Delta)}^2 \right),$$

which shows that (7a) holds true. Furthermore, we find

$$\begin{aligned}
&(\mathcal{N}(z)-\mathcal{N}(\tilde{z}), z-\tilde{z})_H \leq |\hat{y}^m (z-\tilde{z})^2 (z+\tilde{z})|_{L^1} \leq |\hat{y}^m|_{L^\infty} |z+\tilde{z}|_{L^\infty} |z-\tilde{z}|^2_{L^2}\\
&\leq C_1(|z|_{L^\infty}+|\tilde{z}|_{L^\infty})|z-\tilde{z}|^2_{L^2} \leq 2^{\frac{1}{2}} C_1(|z|^2_{L^\infty}+|\tilde{z}|^2_{L^\infty})^{\frac{1}{2}}|z-\tilde{z}|^2_{L^2}\\
&\leq C_2(|z|_V\,|z|_{\mathrm{D}(\varDelta)}+|\tilde{z}|_V\,|\tilde{z}|_{\mathrm{D}(\varDelta)})^{\frac{1}{2}}|z-\tilde{z}|^2_{L^2}\\
&\leq C_3(|z|^2_V+|\tilde{z}|^2_V+|z|^2_{\mathrm{D}(\varDelta)}+|\tilde{z}|^2_{\mathrm{D}(\varDelta)})^{\frac{1}{2}}|z-\tilde{z}|^2_{L^2}\\
&\leq C_4(1+|z|^2_V+|\tilde{z}|^2_V)^{\frac{1}{2}}(1+|z|^2_{\mathrm{D}(\varDelta)}+|\tilde{z}|^2_{\mathrm{D}(\varDelta)})^{\frac{1}{2}}|z-\tilde{z}|_H|z-\tilde{z}|_V,
\end{aligned}$$

which shows that (7b) holds true.

Example 2 In case $\mathcal{N}(z)=\hat{y}^m z^n$, $m\in\mathbb{N}$ and $n=\{3,4,5\}$, (7) holds true. We may write, for suitable nonzero constants r_j,

$$\mathcal{N}(z)-\mathcal{N}(\tilde{z})=\hat{y}^m(z-\tilde{z})\sum_{j=0}^{n-1} r_j z^j \tilde{z}^{n-1-j},$$

where in the sum we have monomials of degree $n-1$. For example for $z^1\tilde{z}^{n-2}$, by standard (yet appropriate) Young, Hölder, Sobolev, and Agmon inequalities, we may write

$$\begin{aligned}
&\left|(z-\tilde{z})z^1\tilde{z}^{n-2}\right|^2_H=\left|(z-\tilde{z})^2 z^2\tilde{z}^{2n-4}\right|_{L^1}\leq|z-\tilde{z}|^2_{L^\infty}\,|z|_{L^\infty}\,|\tilde{z}|_{L^\infty}\,\left|z\tilde{z}^{2n-5}\right|_{L^1}\\
&\leq C_1\,|z-\tilde{z}|_V\,|z-\tilde{z}|_{\mathrm{D}(\varDelta)}\,|z|_V^{\frac{1}{2}}\,|z|^{\frac{1}{2}}_{\mathrm{D}(\varDelta)}\,|\tilde{z}|_V^{\frac{1}{2}}\,|\tilde{z}|^{\frac{1}{2}}_{\mathrm{D}(\varDelta)}\,|z|_{L^6}\,|\tilde{z}|^{2n-5}_{L^{\frac{6(2n-5)}{5}}}\,,
\end{aligned}$$

and, since $H^1(\varOmega)\hookrightarrow L^6(\varOmega)\hookrightarrow L^{\frac{6(2n-5)}{5}}(\varOmega)$,

$$\begin{aligned}
&\left|(z-\tilde{z})z^1\tilde{z}^{n-2}\right|^2_H\leq\tfrac{C_1}{2}\,|z-\tilde{z}|^2_V\,|z|_{\mathrm{D}(\varDelta)}|\tilde{z}|_{\mathrm{D}(\varDelta)}+C_2\,|z-\tilde{z}|^2_{\mathrm{D}(\varDelta)}|z|^3_V\,|\tilde{z}|^{1+2(2n-5)}_V\\
&\leq\tfrac{C_1}{4}\,|z-\tilde{z}|^2_V\left(|z|^2_{\mathrm{D}(\varDelta)}+|\tilde{z}|^2_{\mathrm{D}(\varDelta)}\right)+C_3\,|z-\tilde{z}|^2_{\mathrm{D}(\varDelta)}\left(|z|^{4n-6}_V+|\tilde{z}|^{4n-6}_V\right).
\end{aligned}$$

Furthermore, we find

$$\begin{aligned}
&((z-\tilde{z})z^1\tilde{z}^{n-2}, z-\tilde{z})_H\leq|z^1\tilde{z}^{n-2}|_{L^\infty}|z-\tilde{z}|^2_{L^2}\leq C_4(|z|^{n-1}_{L^\infty}+|\tilde{z}|^{n-1}_{L^\infty})|z-\tilde{z}|^2_{L^2}\\
&\leq C_5(|z|_V^{\frac{n-1}{2}}\,|z|^{\frac{n-1}{2}}_{\mathrm{D}(\varDelta)}+|\tilde{z}|_V^{\frac{n-1}{2}}\,|\tilde{z}|^{\frac{n-1}{2}}_{\mathrm{D}(\varDelta)})|z-\tilde{z}|^2_{L^2}
\end{aligned}$$

which implies the inequality

$$\begin{gathered}
((z-\tilde{z})z^1\tilde{z}^{n-2}, z-\tilde{z})_H\leq C_7(|z|^{\delta_n}_V+|\tilde{z}|^{\delta_n}_V)(|z|^2_{\mathrm{D}(\varDelta)}+|\tilde{z}|^2_{\mathrm{D}(\varDelta)})|z-\tilde{z}|^2_{L^2},\\
\text{with}\quad \delta_n=2,\quad \text{for } n=5,\quad \text{and}\quad \delta_n=\tfrac{2(n-1)}{5-n}\quad \text{for } n\in\{3,4\}.
\end{gathered}$$

where we have used, for $n\in\{3,4\}$, the Young inequality

$$|z|_V^{\frac{n-1}{2}}\,|z|^{\frac{n-1}{2}}_{\mathrm{D}(\varDelta)}\leq C_6(|z|_V^{\frac{n-1}{2}\frac{4}{4-n+1}}+|z|^{\frac{n-1}{2}\frac{4}{n-1}}_{\mathrm{D}(\varDelta)}).$$

For the other monomials we can obtain analogous estimates, which give us

$$\begin{aligned}|\mathcal{N}(z)-\mathcal{N}(\tilde z)|_H^2 &\le C_8\,|\hat y^m|_{L^\infty}^2\,|z-\tilde z|_V^2\left(|z|_{\mathrm{D}(\varDelta)}^2+|\tilde z|_{\mathrm{D}(\varDelta)}^2\right)\\ &\quad + C_8\,|\hat y^m|_{L^\infty}^2\,|z-\tilde z|_{\mathrm{D}(\varDelta)}^2\left(|z|_V^{4n-6}+|\tilde z|_V^{4n-6}\right),\\ (\mathcal{N}(z)-\mathcal{N}(\tilde z),z-\tilde z)_H &\le C_9(1+|z|_V^6+|\tilde z|_V^6)(|z|_{\mathrm{D}(\varDelta)}^2+|\tilde z|_{\mathrm{D}(\varDelta)}^2)|z-\tilde z|_{L^2}^2,\end{aligned}$$

which show that (7) hold true.

Example 3 In the case $\mathcal{N}(z)=\hat y^m z^6$, we were not able to derive (7a). Proceeding as above, for suitable nonzero constants r_j,

$$\mathcal{N}(z)-\mathcal{N}(\tilde z)=\hat y^m(z-\tilde z)\sum_{j=0}^{4} r_j z^j\tilde z^{5-j},$$

where in the sum we have now monomials of degree 5. If for example for $z^1\tilde z^4$, we proceed as above and write

$$\begin{aligned}\left|(z-\tilde z)z^1\tilde z^4\right|_H^2 &= \left|(z-\tilde z)^2z^2\tilde z^8\right|_{L^1}\le |z-\tilde z|_{L^\infty}^2\,|z|_{L^\infty}\,|\tilde z|_{L^\infty}\left|z\tilde z^7\right|_{L^1}\\ &\le C_1\,|z-\tilde z|_V\,|z-\tilde z|_{\mathrm{D}(\varDelta)}\,|z|_V^{\frac12}\,|z|_{\mathrm{D}(\varDelta)}^{\frac12}\,|\tilde z|_V^{\frac12}\,|\tilde z|_{\mathrm{D}(\varDelta)}^{\frac12}\left|z\tilde z^7\right|_{L^1},\end{aligned}$$

we cannot bound the term $\left|z\tilde z^7\right|_{L^1}$ by the V-norms of z and $\tilde z$ (for $d=3$). Trying to use again the $\mathrm{D}(\varDelta)$-norms, we were not able to arrive to (7a) (the $\mathrm{D}(\varDelta)$-norms will appear with a power strictly greater than 2).

Example 4 In the case $\mathcal{N}(z)=\nabla\cdot(g(\hat y)z^n)$, where $n\in\{2,3\}$ and $g\colon\mathbb{R}\to\mathbb{R}^3$ is a smooth function, estimates (7) hold true provided $g(\hat y)\in\mathcal{W}_{\mathrm{st}}$. We will consider the cases $n=2$ and $n=3$ separately.

The case $n=3$. We write, for suitable nonzero constants r_j,

$$\mathcal{N}(z)-\mathcal{N}(\tilde z)=\nabla\cdot\left(g(\hat y)(z-\tilde z)\sum_{j=0}^{2} r_j z^j\tilde z^{2-j}\right)$$

where in the sum we have monomials of degree 2. For example for $z\tilde z$ we find

$$\begin{aligned}&|\nabla\cdot(g(\hat y)(z-\tilde z)z\tilde z)|_H^2\\ &\le \left|(\nabla\cdot g(\hat y))^2(z-\tilde z)^2z^2\tilde z^2\right|_{L^1}+|g(\hat y)|_{L^\infty}^2\left|(\nabla((z-\tilde z)z\tilde z))^2\right|_{L^1}\\ &\le |(\nabla\cdot g(\hat y))|_{L^3}^2\,|z-\tilde z|_{L^6}^2\left|z^2\tilde z^2\right|_{L^\infty}\\ &\quad + |g(\hat y)|_{L^\infty}^2\left(\left|(\nabla(z-\tilde z))^2\right|_{L^1}\left|z^2\tilde z^2\right|_{L^\infty}+|z-\tilde z|_{L^\infty}^2\left|(\nabla(z\tilde z))^2\right|_{L^1}\right)\\ &\le C\,|z-\tilde z|_V^2\,|z|_{L^\infty}^2\,|\tilde z|_{L^\infty}^2+C\,|z-\tilde z|_V\,|z-\tilde z|_{\mathrm{D}(\varDelta)}\left(|z|_V^2\,|\tilde z|_{L^\infty}^2+|z|_{L^\infty}^2\,|\tilde z|_V^2\right)\\ &\le C_1\,|z-\tilde z|_V^2\left(|z|_V^2+|\tilde z|_V^2\right)\left(|z|_{\mathrm{D}(\varDelta)}^2+|\tilde z|_{\mathrm{D}(\varDelta)}^2\right)\\ &\quad + C_1\,|z-\tilde z|_V\,|z-\tilde z|_{\mathrm{D}(\varDelta)}\left(|z|_V^2\,|\tilde z|_V\,|\tilde z|_{\mathrm{D}(\varDelta)}+|z|_V\,|z|_{\mathrm{D}(\varDelta)}\,|\tilde z|_V^2\right)\\ &\le C_2\,|z-\tilde z|_V^2\left(|z|_V^2+|\tilde z|_V^2+1\right)\left(|z|_{\mathrm{D}(\varDelta)}^2+|\tilde z|_{\mathrm{D}(\varDelta)}^2\right)\\ &\quad + C_2\,|z-\tilde z|_{\mathrm{D}(\varDelta)}^2\left(|z|_V^6+|\tilde z|_V^6\right).\end{aligned}$$

We can obtain analogous estimates for the other monomials, and obtain

$$\begin{aligned}|\mathcal{N}(z)-\mathcal{N}(\tilde z)|_H^2 \le C_3\,|z-\tilde z|_V^2 \left(|z|_V^2+|z|_V^2+1\right)\left(|z|_{\mathrm{D}(\Delta)}^2+|\tilde z|_{\mathrm{D}(\Delta)}^2\right)\\ + C_3\,|z-\tilde z|_{\mathrm{D}(\Delta)}^2 \left(|z|_V^6+|\tilde z|_V^6\right).\end{aligned}$$

which shows that (7a) holds true. Furthermore, we also obtain

$$\begin{aligned}&(\nabla\cdot(g(\hat y)(z-\tilde z)z\tilde z)\,,z-\tilde z)_H=(g(\hat y)z\tilde z(z-\tilde z),\nabla(z-\tilde z))_{L^2(\Omega,\mathbb{R}^d)}\\ &\le |g(\hat y)|_{L^\infty(\Omega,\mathbb{R}^d)}|z\tilde z|_{L^\infty(\Omega,\mathbb{R})}|z-\tilde z|_H|z-\tilde z|_V\\ &\le C_4(|z|_V|z|_{\mathrm{D}(\Delta)}|\tilde z|_V|\tilde z|_{\mathrm{D}(\Delta)})^{\frac12}|z-\tilde z|_H|z-\tilde z|_V\\ &\le C_5(|z|_V^2+|\tilde z|_V^2)^{\frac12}(|z|_{\mathrm{D}(\Delta)}^2+|\tilde z|_{\mathrm{D}(\Delta)}^2)^{\frac12}|z-\tilde z|_H|z-\tilde z|_V\end{aligned}$$

and

$$(\mathcal{N}(z)-\mathcal{N}(\tilde z),z-\tilde z)_H\le C_6(|z|_V^2+|\tilde z|_V^2)^{\frac12}(|z|_{\mathrm{D}(\Delta)}^2+|\tilde z|_{\mathrm{D}(\Delta)}^2)^{\frac12}|z-\tilde z|_H|z-\tilde z|_V$$

which shows that (7b) holds true.

The case $n=2$. We write, for suitable nonzero constants r_j,

$$\mathcal{N}(z)-\mathcal{N}(\tilde z)=\nabla\cdot\left(g(\hat y)(z-\tilde z)(r_0\tilde z^1+r_1z^1)\right)$$

Again, we consider the monomials in the last sum separately, and find

$$\begin{aligned}&|\nabla\cdot(g(\hat y)(z-\tilde z)z)|_H^2\\ &\le C\,|z-\tilde z|_V^2\,|z|_{L^\infty}^2+C\,|z-\tilde z|_V\,|z-\tilde z|_{\mathrm{D}(\Delta)}\,|z|_V^2\\ &\le C_1\,|z-\tilde z|_V^2\,|z|_{\mathrm{D}(\Delta)}^2+\tfrac{C}{2}(|z-\tilde z|_V^2+|z-\tilde z|_{\mathrm{D}(\Delta)}^2)\,|z|_V^2\\ &\le C_2\,|z-\tilde z|_V^2\,|z|_{\mathrm{D}(\Delta)}^2+\tfrac{C}{2}\,|z-\tilde z|_{\mathrm{D}(\Delta)}^2\,|z|_V^2\,.\end{aligned}$$

We can obtain analogous estimates for the other monomial, and conclude that

$$|\mathcal{N}(z)-\mathcal{N}(\tilde z)|_H^2\le C_3\,|z-\tilde z|_V^2\left(|z|_{\mathrm{D}(\Delta)}^2+|\tilde z|_{\mathrm{D}(\Delta)}^2\right)+C_3\,|z-\tilde z|_{\mathrm{D}(\Delta)}^2\left(|z|_V^2+|\tilde z|_V^2\right).$$

which shows that (7a) holds true. Furthermore, we also obtain

$$\begin{aligned}&(\nabla\cdot(g(\hat y)(z-\tilde z)z)\,,z-\tilde z)_H=(g(\hat y)z(z-\tilde z),\nabla(z-\tilde z))_{L^2(\Omega,\mathbb{R}^d)}\\ &\le C_4|z|_{L^\infty}|z-\tilde z|_H|z-\tilde z|_V\le C_5|z|_{\mathrm{D}(\Delta)}|z-\tilde z|_H|z-\tilde z|_V\end{aligned}$$

and

$$(\mathcal{N}(z)-\mathcal{N}(\tilde z),z-\tilde z)_H\le C_6(|z|_{\mathrm{D}(\Delta)}^2+|\tilde z|_{\mathrm{D}(\Delta)}^2)^{\frac12}|z-\tilde z|_H|z-\tilde z|_V$$

which shows that (7b) holds true.

6 A numerical example

The simulations below have been done by considering a finite element approximation for the space variable based on the classical piecewise linear *hat* functions. For the time variable we have used the Crank-Nicolson scheme. Since the manuscript is already long we skip the details on the discretization.

We consider the following nonlinear parabolic equation, for time $t \in [0, T]$, with $T = 8$, in the unit ball $\Omega = \mathbb{D} = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1\}$.

$$\begin{aligned}\tfrac{\partial}{\partial t} y - \nu \Delta y + c_3 y^3 + c_2 y^2 + c_1 y + \frac{1}{2}\nabla \cdot \left(y^2, y^2\right) + f_0 = 0, \qquad y|_\Gamma = g,\\ y(0) = y_0,\end{aligned} \tag{57}$$

where c_1, c_2, and c_3 are constants in $\mathbb{R}$, and f_0 is a fixed appropriate function.

Let us fix a smooth function $\hat{y}$ which we will take as our reference trajectory. Then, as external forces, we (must) take the functions

$$\begin{aligned}f_0 = f_0(\hat{y}) &= -\left(\tfrac{\partial}{\partial t}\hat{y} - \nu \Delta \hat{y} + c_3 \hat{y}^3 + c_2 \hat{y}^2 + c_1 \hat{y} + \frac{1}{2}\nabla \cdot \left(\hat{y}^2, \hat{y}^2\right)\right),\\ g = g(\hat{y}) &= \hat{y}|_\Gamma\,,\end{aligned} \tag{58}$$

We will also set the parameters

$$\nu = 0.2, \quad (c_1,\, c_2, c_3) = (-2, -1, -3), \quad \lambda = 1.$$

In the simulations below we will take reference trajectory

$$\hat{y}(t) = (2x_1^3 + x_2^2)\sin t,$$

and the external forces as in (58).

Remark 6.1 To make a smooth function $\hat{y}(t, x)$ a solution of (57), we have just to set the appropriate external forces f_0 and g as in (58).

Our internal actuators will be defined as follows. We define the rectangle

$$\omega \coloneqq \left(0, \tfrac{1}{2}\right) \times \left(0, \tfrac{1}{3}\right). \tag{59}$$

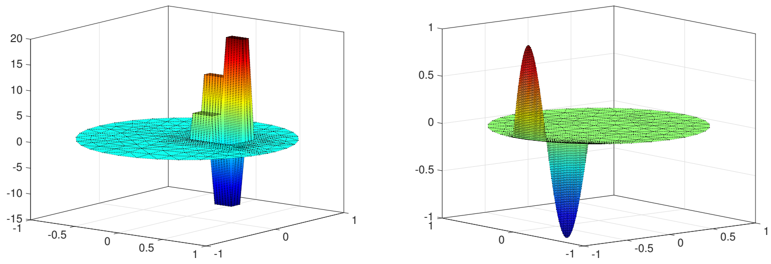
Then, we take a regular partition of ω into $M = mn$ subrectangles

$$\omega_{l_1,l_2} \coloneqq \left(\tfrac{l_1-1}{2m}, \tfrac{l_1}{2m}\right) \times \left(\tfrac{l_2-1}{3n}, \tfrac{l_2}{3n}\right), \qquad (l_1, l_2) \in \{1, 2, \dots, m\} \times \{1, 2, \dots, n\}.$$

We take the M actuators $1_{\omega_{l_1,l_2}}$. Thus, at any given time instant, in each subrectangle ω_{l_1,l_2} the control is constant. As an illustration, we plot a linear combination of 4 piecewise-constant actuators in Figure 1(a), corresponding to the arrangement $(m, n) = (2, 2)$.

For the boundary control case, our boundary, once parametrized by arc length, is $\Gamma = [0, 2\pi)$. We use M boundary actuators whose form is

$$\Psi_i(\theta) = 1_{(\theta_0,\theta_1)} \sin\left(\frac{i(\theta - \theta_0)}{\theta_1 - \theta_0}\right), \quad i = 1, 2, \dots, M, \tag{60}$$



(a) Piecewise-constant actuators.

(b) The boundary actuator Ψ_2.

Fig. 1 Internal and boundary actuators.

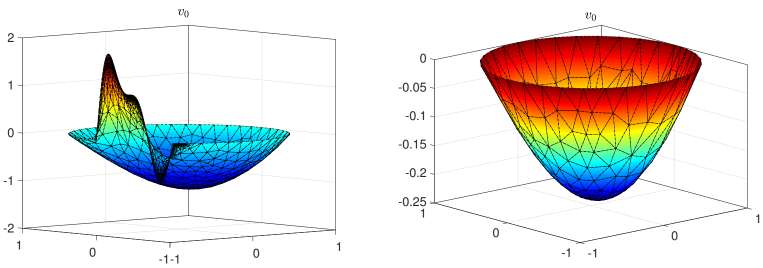
with $\theta_0 = \pi$ and $\theta_1 = \frac{5\pi}{4}$. As an illustration, the boundary actuator Ψ_2 is plotted in Figure 1(b).

The feedback operators have been computed by solving the Riccati equation 15, with $\hat{a}$ and $\hat{b}$ as in (5b), with $M = 6$, and with $\lambda = 1$. In the boundary case we have taken $\varsigma = 8$. In the internal case the actuators are those corresponding to the arrangement $(m, n) = (3, 2)$.

The initial condition has been taken in the form $y_0 = \hat{y}(0) + \epsilon v_0$, where v_0 is the (numerical) solution of the elliptic system

$$-0.5\Delta v_0 + 0.1 v_0 = \cos(3x_2)^2 + \sin(x_1) + 2; \quad v_0|_{\Gamma} = \sum_{i=1}^{M} \varrho_i \Psi_i,$$

and $\varrho = (1, 1, 0, 0.5, 0, 0)$ in the boundary case and $\varrho = (0, 0, 0, 0, 0, 0)$ in the internal case. The corresponding functions are shown in Figure 2



(a) For boundary case.

(b) For internal case

Fig. 2 Perturbations of initial condition.

6.1 The case of boundary actuators

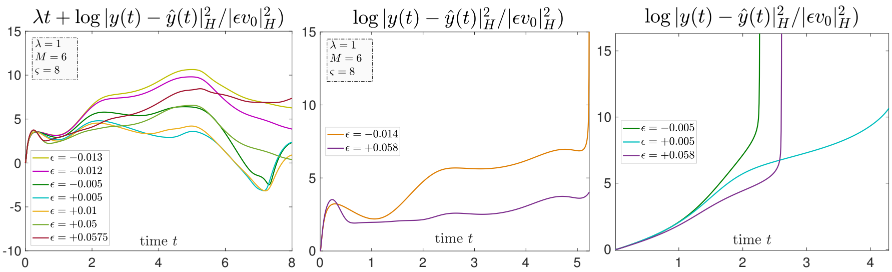
We will confirm that the feedback control is able to stabilize locally system (57) to the targeted trajectory $\hat{y}$ with exponential rate $\frac{\lambda}{2}$, see Main Theorem 1.2. That is, the solutions of the system

$$\tfrac{\partial}{\partial t} y - \nu \Delta y + f(y, \nabla y) + f_0 = 0, \quad y|_\Gamma = g + B_\Psi \kappa, \quad y(0) = y_0, \tag{61a}$$

$$\tfrac{\partial}{\partial t} \kappa + \varsigma\kappa = \begin{bmatrix} B_{\widetilde{\Psi}}^* & -1 \end{bmatrix} \Pi_\lambda \begin{bmatrix} y - \hat{y} - B_{\widetilde{\Psi}}\kappa \\ \kappa \end{bmatrix}, \quad \kappa(0) = \epsilon\varrho \tag{61b}$$

with $y_0 = \hat{y}(0) + \epsilon v_0$, go exponentially to $\hat{y}$, with rate $\frac{\lambda}{2}$, provided $\epsilon\, |v_0|_{H^1(\Omega)} = |y_0 - \hat{y}(0)|_{H^1(\Omega)}$ is small enough, that is, provided ϵ is small enough.

In Figure 3(a) we observe that under the boundary feedback control, the system is stable and the solution y goes exponentially to $\hat{y}$ with rate $\frac{\lambda}{2}$, for small $|\epsilon|_\mathbb{R}$. The feedback fails to stabilize the system to $\hat{y}$ for bigger magnitudes of ϵ, as we see in Figure 3(b). In Figure 3(c) we see that the uncontrolled system is not stable, and the solution may explode even for small ϵ.



(a) With feedback. Small ϵ. (b) With feedback. Larger ϵ. (c) Uncontrolled solution.

Fig. 3 Boundary actuators. Controlled versus uncontrolled solutions.

Remark 6.2 In the theoretical results we have asked the boundary actuators to be in $H^{\frac{3}{2}}(\Gamma)$. The actuators in (60) are in $H^s(\Gamma)$ for all $s < \frac{3}{2}$, but not necessarily in $H^{\frac{3}{2}}(\Gamma)$ (cf [32, Chapter 1, Section 11.3, Theorem 11.4]). This lack of regularity was neglected for the simulations.

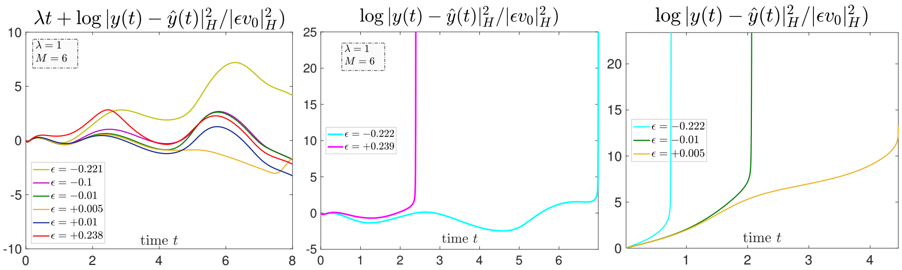
6.2 The case of internal actuators

We will confirm that, with $B_M = 1_\omega P_M 1_\omega = P_M$, the solutions of the system

$$\begin{aligned} \tfrac{\partial}{\partial t} y - \nu \Delta y + f(y, \nabla y) + f_0 = -B_M B_M^* \Pi_\lambda (y - \hat{y}), \qquad y|_\Gamma = g, \\ y(0) = \hat{y}(0) + \epsilon v_0, \end{aligned} \tag{62}$$

go exponential to the targeted trajectory $\hat{y}$, with rate $\frac{\lambda}{2}$, provided $\epsilon\, |v_0|_{H^1(\Omega)} = |y_0 - \hat{y}(0)|_{H^1(\Omega)}$ is small enough.

In Figure 4(a), we observe that for small $|\epsilon|_{\mathbb{R}}$ the feedback control is able to stabilize the system to $\hat{y}$ with exponential rate $\frac{\lambda}{2}$. Figure 4(a) shows that for bigger magnitudes of ϵ the feedback controller is not able to stabilize the system to $\hat{y}$. In Figure 4(c), the uncontrolled system is unstable and exploding, even for small ϵ.



(a) With feedback. Small ϵ. (b) With feedback. Larger ϵ. (c) Uncontrolled solution.

Fig. 4 Internal actuators. Controlled versus uncontrolled solutions.

6.3 On the computation of the solution of the differential Riccati equations

We cannot solve numerically the differential Riccati equations backwards in the time interval $[0, +\infty)$. So we solve them in a finite interval $[0, T]$, and we refer to [30, sections 5.3.2 and 5.3.3] for a procedure to find an appropriate final condition $\Pi(T)$. Here we have followed essentially the same procedure where in particular $\Pi(T)$ will be the solution of a suitable algebraic Riccati equation. To solve such equations we use the software in [15] (see also [16]).

Though, the numerical issues are not the subject of this work we must, however, mention that it is well known that as our discretization is refined solving the Riccati equation becomes a very hard and challenging numerical problem.

Another remark is that the function f_0 in the discretized systems has been taken as the function which makes $\hat{y}$ a solution of the discrete system. This was done to somehow avoid the effects of the numerical error which is propagated over time. See the discussion on [30, Section 7.2].

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Appendix

A.1 Proof of Lemma 2.2

Observe that given z solving (21), with $\gamma = 0$, by rescaling time $t = \frac{\tau}{\nu}$ and defining $\breve{z}(\tau) = z(\frac{\tau}{\nu})$, $\breve{a}(\tau) = \hat{a}(\frac{\tau}{\nu})$, $\breve{b}(\tau) = \hat{b}(\frac{\tau}{\nu})$, and $\breve{h}(\tau) = h(\frac{\tau}{\nu})$, we find

$$\partial_\tau \breve{z} - \Delta \breve{z} + \tfrac{\breve{a}}{\nu}\breve{z} + \nabla \cdot (\tfrac{\breve{b}}{\nu}\breve{z}) + \tfrac{\breve{h}}{\nu} = 0, \qquad \breve{z}|_\Gamma = 0, \qquad \breve{z}(\nu s_0) = z_0.$$

Then, by standard arguments, we can find

$$\begin{aligned}
|\breve{z}(\nu s)|_H^2 &\le \mathrm{e}^{\frac{C^2}{\nu^2} C_{\mathcal{W}}(\nu s - \nu s_0)} \left(|\breve{z}(\nu s_0)|_H^2 + \tfrac{1}{\nu^2} \left|\breve{h}\right|^2_{L^2((\nu s_0, \nu s), V')} \right),\\
2\nu\, |\breve{z}|^2_{L^2(\nu I, V)} &= |\breve{z}(\nu s_0)|_H^2 - |\breve{z}(\nu s_1)|_H^2 + \int_{\nu s_0}^{\nu s_1} \langle \tfrac{\breve{a}}{\nu}\breve{z} + \nabla \cdot (\tfrac{\breve{b}}{\nu}\breve{z}) + \tfrac{\breve{h}}{\nu}, \breve{z} \rangle_{V',V} \,\mathrm{d}\tau\\
&\le C_1 \,|\breve{z}|^2_{L^\infty(\nu I, H)} + 3 \left| \tfrac{\breve{h}}{\nu} \right|^2_{L^2(\nu I, V')} + |\breve{z}|^2_{L^2(\nu I, V)}.
\end{aligned}$$

with $C_1 \coloneqq \left(1 + 3C^2 \left|\frac{\breve{a}}{\nu}\right|^2_{L^\infty(\nu I, L^d)} + 3C^2 \left|\frac{\breve{b}}{\nu}\right|^2_{L^\infty_w(\nu I, L^\infty)}\right)$. Therefore,

$$\begin{aligned}
|z(s)|_H^2 &\le \mathrm{e}^{\frac{C^2}{\nu} C_{\mathcal{W}}(s - s_0)} \left(|z(s_0)|_H^2 + \tfrac{1}{\nu} |h|^2_{L^2((s_0,s),V')} \right),\\
|z|^2_{L^2(I,V)} &\le \left(\tfrac{1}{2\nu} + \tfrac{3C^2}{2\nu^3} |(a,b)|^2_{\mathcal{W}} \right) |z|^2_{L^\infty(I,H)} + \tfrac{3}{2\nu^2} |h|^2_{L^2(I,V')},\\
\left| \tfrac{\partial}{\partial t} z \right|_{L^2(I,V')} &\le \left(\nu + C\,|a|_{L^\infty(I,L^d)} + C\,|b|_{L^\infty_w(I,L^\infty)} \right) |z|_{L^2(I,V)} + |h|_{L^2(I,V')},
\end{aligned}$$

which imply the statement of Lemma 2.2. □

A.2 Proof of Proposition 2.1

We construct an extension for $\hat{b} = \hat{b}(t,x)$ independent of t. We consider a different local system of space coordinates w, in order to flatten the boundary. We use some basic concepts from Riemannian manifolds, see [44, chapter 1, Section 13 and chapter 2, Section 2].

Change of coordinates. Up to a translation and rotation, we may suppose that locally the boundary $\Gamma = \partial\Omega$ is the graph of a smooth function Λ, with $\overline{w} \coloneqq (w_1, w_2, \dots, w_{d-1})$,

$$\begin{aligned}
\overline{w} &\mapsto G_\Lambda(\overline{w}) \coloneqq (\overline{w}, \Lambda(\overline{w})) \in \Gamma,\\
\overline{w} &\in \mathbb{D}_r^{d-1} \coloneqq \{\overline{w} \in \mathbb{R}^{d-1} \mid w_1^2 + w_2^2 + \dots + w_{d-1}^2 \le r\},
\end{aligned}$$

with (small) $r > 0$. Locally, a tubular neighborhood is given by

$$\mathcal{T} = \mathcal{T}_{r,l} \coloneqq \left\{ x \in \mathbb{R}^d \mid x = G_\Lambda(\overline{w}) + w_d \mathbf{n}_{G_\Lambda(\overline{w})}, \quad w \coloneqq (\overline{w}, w_d) \in \mathbb{D}_r^{d-1} \times (-l, l) \right\}, \quad \text{(A.1)}$$

with (small) $l > 0$. Where $\mathbf{n}_{G_\Lambda(\overline{w})}$ stands for the unit outward normal vector at $G_\Lambda(\overline{w}) \in \Gamma$.

Let us denote $\mathcal{O}^- = \mathbb{D}_r^{d-1} \times (-l, 0)$ and $\mathcal{O}^+ = \mathbb{D}_r^{d-1} \times (0, l)$. Notice that the points outside Ω correspond to those in $\mathcal{O}^+$. We may suppose that $x_0 = G_\Lambda(0) \in \Gamma_\mathrm{c} \subseteq \Gamma \cap \mathcal{T}$, that is, in Proposition 2.1 we may take $\widetilde{\omega}$ corresponding to a subset of $\mathcal{O}^+$.

The new coordinates $(w_1, w_2, \dots, w_d)$ induce the vector fields

$$\frac{\partial}{\partial_{w_i}} = \sum_{i=1}^{d} \frac{\partial x_j}{\partial_{w_i}} \frac{\partial}{\partial_{x_j}}, \quad i = 1, 2, \dots, d, \qquad \text{(A.2)}$$

defined in $\mathcal{O} := \mathbb{D}_r^{d-1} \times (-l, l)$. We can see the neighborhood $\mathcal{T}$ (endowed with the usual Euclidean scalar product) as the Riemannian manifold $(\mathcal{O}, g)$ by taking the metric tensor

$$g = \sum_{i=1}^{d} \sum_{j=1}^{d} g_{ij} \mathrm{d}w_j \otimes \mathrm{d}w_j, \qquad \text{with} \quad g_{ij} = \left(\frac{\partial}{\partial_{w_i}}, \frac{\partial}{\partial_{w_j}} \right)_{\mathbb{R}^d}$$

where $(\cdot, \cdot)_{\mathbb{R}^d}$ stands for the usual Euclidean scalar product in $\mathbb{R}^d$.

The divergence of a vector field $V = \sum_{i=1}^{d} V_i \frac{\partial}{\partial_{w_i}}$ reads, in the new coordinates,

$$\nabla_w \cdot V = (-1)^d \frac{1}{\sqrt{\bar{g}}} \sum_{j=1}^{d} \frac{\partial (V_j \sqrt{\bar{g}})}{\partial_{w_j}},$$

where $\bar{g} := \det[g_{ij}]$ stands for the determinant of the matrix whose entries are the coefficients of the metric tensor. Recall that in our setting $\sqrt{\bar{g}}$ coincides with the Jacobian $\left| \det \left[\frac{\partial x}{\partial w} \right] \right|$ of the the smooth diffeomorphism $w \mapsto x$, because

$$[g_{ij}] = \left[\sum_{k=1}^{d} \frac{\partial x_k}{\partial w_i} \frac{\partial x_k}{\partial w_j} \right] = \left[\frac{\partial x}{\partial w} \right] \left[\frac{\partial x}{\partial w} \right]^\top,$$

where $\left[\frac{\partial x}{\partial w} \right]^\top$ is the transpose matrix of $\left[\frac{\partial x}{\partial w} \right]$.

The extension. For a given vector field $V^- = \sum_{i=1}^{d} V_i^- \frac{\partial}{\partial_{w_i}}$, defined in $\mathcal{O}^-$, we consider the following vector field defined in $\mathcal{O}^+$:

$$V^+ := \sum_{i=1}^{d} V_i^+ \frac{\partial}{\partial_{w_i}}, \quad \text{with} \quad \begin{cases} V_i^+(\overline{w}, s) = -\mathcal{Q}(w) V_i^-(\overline{w}, -s), & \text{if } i \neq d, \\ V_d^+(\overline{w}, s) = \mathcal{Q}(w) V_d^-(\overline{w}, -s), \end{cases} \tag{A.3}$$

where $s \in (0, l)$ and

$$\mathcal{Q}(w) := \frac{\sqrt{\bar{g}}|_{(\overline{w}, -s)}}{\sqrt{\bar{g}}|_{(\overline{w}, s)}}.$$

Then, denoting the mapping $\sigma \colon \mathcal{O}^+ \to \mathcal{O}^-$, $(\overline{w}, s) \mapsto (\overline{w}, -s)$, we find

$$\begin{aligned} \sqrt{\bar{g}}|_{(\overline{w}, s)} (-1)^d (\nabla_w \cdot V^+)|_{(\overline{w}, s)} &= \sum_{j=1}^{d} \left. \frac{\partial (V_j^+ \sqrt{\bar{g}})}{\partial_{w_j}} \right|_{(\overline{w}, s)} \\ = \left. \frac{\partial ((V_d^- \sqrt{\bar{g}}) \circ \sigma)}{\partial_{w_d}} \right|_{(\overline{w}, s)} &- \sum_{j=1}^{d-1} \left. \frac{\partial ((V_d^- \sqrt{\bar{g}}) \circ \sigma)}{\partial_{w_j}} \right|_{(\overline{w}, s)} = - \sum_{j=1}^{d} \left. \frac{\partial (V_d^- \sqrt{\bar{g}})}{\partial_{w_j}} \right|_{(\overline{w}, -s)}, \end{aligned}$$

which gives us

$$(\nabla_w \cdot V^+)|_{(\overline{w}, s)} = -\mathcal{Q}(w) (\nabla_w \cdot V^-)|_{(\overline{w}, -s)}. \tag{A.4}$$

For $S \subseteq \mathcal{O}$, let us denote $\mathcal{W}_2(S) := \{ v \in L^\infty(S, \mathbb{R}^d) \mid (\nabla_w \cdot v) \in L_g^r(S, \mathbb{R}) \}$, with $r \in \{2, \infty\}$ (cf. (28)). In particular, since $\mathcal{Q}(w)$ is a smooth function, we observe that $V^+ \in \mathcal{W}_2(\mathcal{O}^+)$ if $V^- \in \mathcal{W}_2(\mathcal{O}^-)$. Here, $(f, h)_{L_g^2(S, \mathbb{R})} := \int_S f h \, \mathrm{d}\mathcal{O}$, $\mathrm{d}\mathcal{O} := \sqrt{\bar{g}} \mathrm{d}w_1 \wedge \mathrm{d}w_2 \wedge \ldots \wedge \mathrm{d}w_d$. It is also clear that the linear mapping $V^- \mapsto V^+$ is continuous. Finally, we prove that the function defined by

$$V^- \mapsto \overline{V}, \quad \text{with} \quad \overline{V}(w) := \begin{cases} V^-(w), & \text{if } w \in \mathcal{O}^-, \\ V^+(w), & \text{if } w \in \mathcal{O}^+, \end{cases}$$

maps $\mathcal{W}_2(\mathcal{O}^-)$ into $\mathcal{W}_2(\mathcal{O})$. We need to prove that $\nabla_w \cdot \overline{V} \in L^2_g(\mathcal{O},\mathbb{R})$. For a smooth function $\phi \in \mathcal{D}(\mathcal{O}) \coloneqq C^\infty_c(\mathcal{O},\mathbb{R})$ with support contained in $\mathcal{O}$ we find, in the distribution sense, with $\Gamma_0 \coloneqq \{w \in \mathcal{O} \mid w_d = 0\}$,

$$\langle \nabla_w \cdot \overline{V}, \phi\rangle_{\mathcal{D}(\mathcal{O})',\mathcal{D}(\mathcal{O})} \coloneqq -\langle \overline{V}, \nabla_w \phi\rangle_{\mathcal{D}(\mathcal{O})',\mathcal{D}(\mathcal{O})} = -(\overline{V}, \nabla_w \phi)_{L^2_g(\mathcal{O},\mathbb{R})}$$
$$= (\nabla_w \cdot \overline{V}, \phi)_{L^2_g(\mathcal{O}^-,\mathbb{R})} + (\nabla_w \cdot \overline{V}, \phi)_{L^2_g(\mathcal{O}^+,\mathbb{R})} - \int_{\Gamma_0} \phi g\left(V^-, \tfrac{\partial}{\partial_{w_d}}\right) - \phi g\left(V^+, \tfrac{\partial}{\partial_{w_d}}\right) \mathrm{d}_g \Gamma_0.$$

Notice that $\frac{\partial}{\partial_{w_d}}$ is the unit outward normal at $\Gamma_0 \subset \partial\mathcal{O}^-$.

Now, since $g\left(V^-, \frac{\partial}{\partial_{w_d}}\right) - g\left(V^+, \frac{\partial}{\partial_{w_d}}\right) = V^-_d - V^+_d$, from $V^+_d(\overline{w}, s) = \mathcal{Q}(w) V^-_d(\overline{w}, -s)$ for all $s > 0$ and $\mathcal{Q}(\overline{w}, 0) = 1$, we can conclude that $V^-_d - V^+_d$ necessarily vanishes at Γ_0. Hence, the boundary term vanishes, and we can conclude that $\nabla_w \cdot V \in L^2_g(\mathcal{O},\mathbb{R})$. We may write

$$\langle \nabla_w \cdot V, \phi\rangle_{\mathcal{D}(\mathcal{O})',\mathcal{D}(\mathcal{O})} = (\nabla_w \cdot V, \phi)_{L^2_g(\mathcal{O}^-,\mathbb{R})} + (\nabla_w \cdot V, \phi)_{L^2_g(\mathcal{O}^+,\mathbb{R})} = (\nabla_w \cdot V, \phi)_{L^2_g(\mathcal{O},\mathbb{R})}.$$

Therefore, if in addition we have $\nabla_w \cdot V^- \in L^\infty_g(\mathcal{O}^-,\mathbb{R})$ then from (A.4) it follows that $\nabla_w \cdot \overline{V} \in L^\infty_g(\mathcal{O},\mathbb{R})$.

It is also clear that the mapping $V^- \mapsto \overline{V}$ maps $\mathcal{W}_2(\mathcal{O}^-)$ into $\mathcal{W}_2(\mathcal{O})$ continuously.

In the original coordinates the extension above reads: given $\hat{b} = \hat{b}_i \frac{\partial}{\partial_{x_i}}$, we firstly rewrite $\hat{b}$ in the new coordinates $\hat{b} = \hat{b}^w_i \frac{\partial}{\partial_{w_i}} =: V^-$, next we extend V^- to $\overline{V} = \overline{V}_i \frac{\partial}{\partial_{w_i}}$ (through V^+ as above), finally we rewrite $\overline{V}$ in the original coordinates: $\overline{V} = \overline{V}^o_i \frac{\partial}{\partial_{x_i}} =: \bar{b}$.

The continuity of $(\tilde{a}, \tilde{b}) \mapsto (\tilde{a}, \bar{b})$ from $\mathcal{W}_{\mathrm{st}}$ into $\widetilde{\mathcal{W}}_{\mathrm{st}}$ follows straightforwardly. □

Remark A.3 In [40, Proposition 4.2] we find, for $d = 3$, the result we present here in Proposition 2.1. Our proof borrows the idea from [40, Appendix]. We still present the proof in here because in [40], when computing the vector fields $\frac{\partial}{\partial_{w_i}}$ for $i = 1, 2, \ldots, d-1$, as in (A.2) above, the terms $w_d \frac{\partial(\mathbf{n}_{G_{\Lambda(\overline{w})}})_j}{\partial_{w_i}} \frac{\partial}{\partial_{x_j}}$ have been missed, see [40, Eq. (A.2)].

A.3 Proof of Proposition 2.2

From [33, chapter 4, Section 2.5, Theorem 2.3] we know that $u \mapsto (u_0, u|_\Gamma)$, maps the space $W(\mathbb{R}_0, H^2(\Omega,\mathbb{R}), L^2(\Omega,\mathbb{R}))$ continuously onto the product space

$$\left\{(z,g) \in H^1(\Omega,\mathbb{R}) \times \left(L^2(\mathbb{R}_0, H^{\frac{3}{2}}(\Gamma,\mathbb{R})) \bigcap H^{\frac{3}{4}}(\mathbb{R}_0, L^2(\Gamma,\mathbb{R}))\right) \,\middle|\, g(0) = z|_\gamma\right\}.$$

In particular, this implies that $G^2(I,\Gamma)) = L^2(I, H^{\frac{3}{2}}(\Gamma,\mathbb{R})) \bigcap H^{\frac{3}{4}}(I, L^2(\Gamma,\mathbb{R}))$.

Now, it is not difficult to check that

$$P_M \in \mathcal{L}(L^2(I, H^{\frac{3}{2}}(\Gamma,\mathbb{R}))) \bigcap \mathcal{L}(H^1(I, L^2(\Gamma,\mathbb{R}))).$$

Indeed, we may suppose, without loss of generality, that the family of actuators is orthonormed in $L^2(\Gamma,\mathbb{R})$, and in that case we obtain, for a Hilbert space $X \hookrightarrow L^2(\Gamma,\mathbb{R})$,

$$|P_M\zeta|^2_{L^2(I,X)} = \int_I \left|\sum_{i=1}^{M} (\zeta(s), \Psi_i)_{L^2(\Gamma,\mathbb{R})} \Psi_i\right|^2_X \mathrm{d}s \le M \sum_{i=1}^{M} \int_I |\zeta(s)|^2_{L^2(\Gamma,\mathbb{R})} |\Psi_i|^2_X \,\mathrm{d}s$$
$$\le M \max_{1\le i\le M} |\Psi_i|^2_X \,|\zeta|^2_{L^2(I,L^2(\Gamma,\mathbb{R}))} \le C\,|\zeta|^2_{L^2(I,X)},$$
$$\left|\tfrac{\partial}{\partial t} P_M\zeta\right|^2_{L^2(I,X)} = \int_I \left|\tfrac{\partial}{\partial s} \sum_{i=1}^{M} (\zeta(s), \Psi_i)_{L^2(\Gamma,\mathbb{R})} \Psi_i\right|^2_X \mathrm{d}s = \left|P_M \tfrac{\partial}{\partial t}\zeta\right|^2_{L^2(I,X)} \le C\left|\tfrac{\partial}{\partial t}\eta\right|^2_{L^2(I,X)}.$$

Now, from $P_M \in \mathcal{L}(L^2(I, L^2(\Gamma,\mathbb{R}))) \bigcap \mathcal{L}(H^1(I, L^2(\Gamma,\mathbb{R})))$, by an interpolation argument, it follows that $P_M \in \mathcal{L}(H^{\frac{3}{4}}(I, L^2(\Gamma,\mathbb{R})))$. See [32, chapter 1, Section 5.1] and [33, chapter 4, Section 2.1]. Finally, $P_M \in \mathcal{L}(L^2(I, H^{\frac{3}{2}}(\Gamma,\mathbb{R}))) \bigcap \mathcal{L}(H^{\frac{3}{4}}(I, L^2(\Gamma,\mathbb{R})))$ implies, by Proposition 2.3, that $P_M \in \mathcal{L}(G^2(I,\Gamma))$.

Finally, we prove that $Q^j_{\widetilde{M}} P_M \in \mathcal{L}(G^2_{\rm c}(J^j,\Gamma))$, with $J^j = (s_0 + jT_*, s_0 + (j+1)T_*)$. Since, by setting $I = J^j$, we have $P_M \in \mathcal{L}(G^2(J^i,\Gamma))$, it is enough to prove that $Q^j_{\widetilde{M}} \in \mathcal{L}(P_M G^2_{\rm c}(J^j,\Gamma), G^2_{\rm c}(J^j,\Gamma))$. Notice that

$$P_M G^2_{\rm c}(J^j,\Gamma) = L^2(J^j, P_M H^{\frac{3}{2}}(\Gamma)) \cap H^{\frac{3}{4}}(J^j, P_M L^2(\Gamma)),$$

and $P_M H^{\frac{3}{2}}(\Gamma) = \mathcal{S}_\Psi = P_M L^2(\Gamma)$. Since the space $\mathcal{S}_\Psi = \operatorname{span}\{\Psi_i \mid i \in \{1,2,\dots,M\}\}$ is finite-dimensional, it remains to observe that $Q^j_{\widetilde{M}} \in \mathcal{L}\left(L^2(J^j,\mathbb{R}^M) \cap H^{\frac{3}{4}}(J^j,\mathbb{R}^M)\right)$, which follows from $Q^j_{\widetilde{M}} \in \mathcal{L}\big(L^2(J^j,\mathbb{R})\big) \cap \mathcal{L}\big(H^{\frac{3}{4}}(J^j,\mathbb{R})\big)$. Finally, observe that looking at $H^{\frac{3}{4}}(J^j,\mathbb{R}) =: H^{\frac{3}{4}}(J^j,\mathbb{R})$ as the domain of $(-\Delta_{J^j}+1)^{\frac{3}{8}}$ (cf. proof of Lemma 2.8) we can conclude that $\left|Q^j_{\widetilde{M}}\right|^2_{\mathcal{L}\left(L^2(J^j,\mathbb{R})\right)} = 1 = \left|Q^j_{\widetilde{M}}\right|^2_{\mathcal{L}\left(H^{\frac{3}{4}}_{\rm f}(J^j,\mathbb{R})\right)}$. □

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