POLYNOMIAL STABILITY OF A COUPLED WAVE-HEAT NETWORK

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ABSTRACT. We study the long-time asymptotic behaviour of a topologically non-trivial network of wave and heat equations. By analysing the simpler wave and the heat networks separately, and then applying recent results for abstract coupled systems, we establish energy decay at the rate t^{-4} as $t \to \infty$ for all classical solutions.

1. INTRODUCTION

In this paper we study the long-time asymptotic behaviour of the system of coupled one-dimensional wave and heat equations shown in Figure 1.

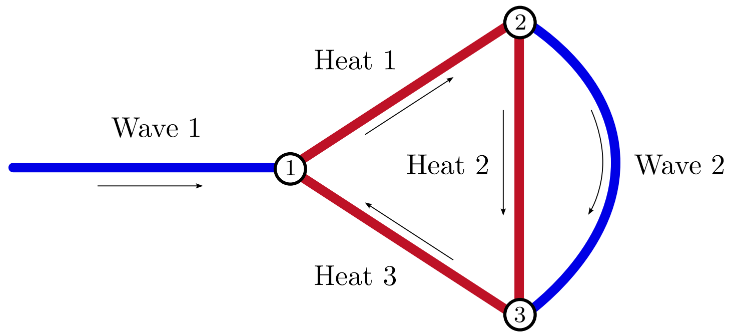

FIGURE 1. A wave-heat network with heat parts (red) and wave parts (blue). The arrows indicate the directions in which the spatial coordinates increase.

Along each of the blue edges we solve the wave equation $y_{tt}^k(x,t) = y_{xx}^k(x,t)$, for $x \in (0,1)$, $t > 0$ and $k = 1,2$, and along each of the red edges we solve the heat equation $w_t^k(x,t) = \beta_k w_{xx}^k(x,t)$, for $x \in (0,1)$, $t > 0$ and $k = 1,2,3$, where $\beta_1, \beta_2, \beta_3 > 0$ are constants. We impose the boundary condition $y_t^1(0,t) = 0$, together with the Kirchhoff-type coupling conditions

$$\begin{cases} y_t^1(1,t) = w^1(0,t) = w^3(1,t) \\ y_x^1(1,t) - \beta_1 w_x^1(0,t) + \beta_3 w_x^3(1,t) = 0 \end{cases}$$

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at vertex 1,

$$\begin{cases} w^1(1,t) = w^2(0,t) = y_t^2(0,t), \\ \beta_1 w_x^1(1,t) - \beta_2 w_x^2(0,t) - y_x^2(0,t) = 0 \end{cases}$$

at vertex 2, and

$$\begin{cases} w^2(1,t) = w^3(0,t) = y_t^2(1,t), \\ \beta_2 w_x^2(1,t) - \beta_3 w_x^3(0,t) + y_x^2(1,t) = 0 \end{cases}$$

at vertex 3, in all cases for $t > 0$. We study the total energy of the wave-heat system, defined as

$$E(t) = \frac{1}{2}\sum_{k=1}^{2}\int_0^1 |y_x^k(x,t)|^2 + |y_t^k(x,t)|^2 \,\mathrm{d}x + \frac{1}{2}\sum_{k=1}^{3}\int_0^1 |w^k(x,t)|^2 \,\mathrm{d}x$$

for $t \geq 0$, with the aim of establishing decay of this energy as $t \to \infty$ for initial data leading to either a mild or a classical solution of the corresponding abstract Cauchy problem. The following is our main result.

Theorem 1.1. *The energy of every solution of the wave-heat system satisfies $E(t) \to 0$ as $t \to \infty$, and for classical solutions we have $E(t) = o(t^{-4})$ as $t \to \infty$.*

Networks of PDEs have been studied extensively in the literature. Existence of solutions has been established for instance in [16, 18, 19, 22], while polynomial stability of networks of wave and heat equations has been studied in [20, 23] for certain particular network configurations and in [15] for general star-shaped networks. Stability of various other types of PDEs on networks has been considered in [2, 3, 4, 10, 13, 22, 26], to give just a small selection. This work also relates to the study of fluid-structure interactions and wave-heat systems on spatial domains of one- and higher-dimensional spatial domains; see for instance [5, 6, 7, 11, 12, 27, 28].

In the present paper, we establish well-posedness and stability of our wave-heat system by following the approach taken in [24] for composite systems subject to abstract boundary couplings. The corresponding results were already used in [24, Sec. 4] to investigate stability of certain types of networks of wave and heat equations. In this paper we investigate the structure depicted in Figure 1, which is more complicated than the PDE networks considered in [24, Sec. 4] on account of the fact that there are now multiple vertices at which wave equations are coupled with heat equations. We demonstrate how the abstract results in [24] can be applied, in a slightly more intricate way than was necessary in [24, Sec. 4], to analyse this more complicated model.

The method presented in [24] is based on a decomposition of the wave-heat system shown in Figure 1 into two smaller networks; in our situation, one "wave network" and one "heat network". The decomposition of the full network model and the interaction between the two subnetworks is illustrated in Figure 2. We analyse these two simpler networks separately in Section 3 in order to prove Theorem 1.1 in Section 4. First, in Section 2, we reproduce the theoretical results from [24] which will form the basis for our analysis.

We use standard notation and terminology throughout. All normed spaces will be considered over the complex field. Given normed spaces X and Y, we

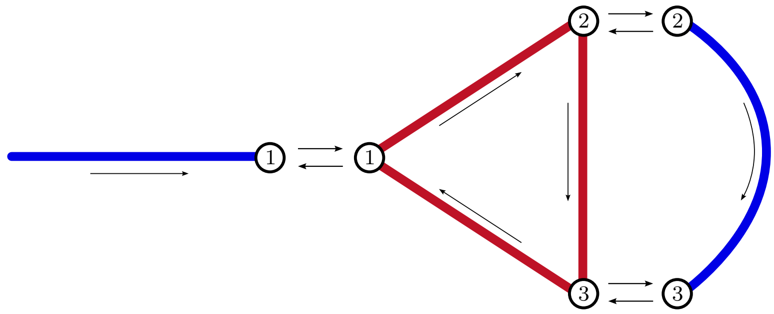

FIGURE 2. Decomposition of the network into a wave network (the two blue edges) and a heat network (the three red edges) which interact via coupling at the three vertices.

write $\mathcal{B}(X,Y)$ for the space of bounded linear operators from X to Y, and we write $\mathcal{B}(X)$ for $\mathcal{B}(X,X)$. We denote the domain of a linear operator L by $D(L)$, and we endow this space with the graph norm. We denote the kernel of a linear operator L by $\operatorname{Ker} L$, and we write $\rho(L)$ for the resolvent set of L. Furthermore, given a bounded linear operator $T \in \mathcal{B}(X)$ on a Hilbert space X, we denote by $\operatorname{Re} T$ the self-adjoint operator $\frac{1}{2}(T+T^*)$, and given two self-adjoint bounded linear operators $S, T \in \mathcal{B}(X)$ on a Hilbert space X we write $T \geq S$ to mean that $\langle Tx - Sx, x\rangle \geq 0$ for all $x \in X$. If p and q are two real-valued quantities we write $p \lesssim q$ to express that $p \leq Cq$ for some constant $C > 0$ which is independent of all parameters that are free to vary in a given situation. We shall also make use of standard 'big-O' and 'little-o' notation.

2. Boundary nodes and an abstract stability result

In this section we present the abstract setting we shall use to analyse our wave-heat network, and we recall the main result of [24]. Let X and U be Hilbert spaces, and let $L\colon D(L) \subseteq X \to X$, $G, K\colon D(L) \subseteq X \to U$ be linear operators. The triple (G, L, K) is said to be a *boundary node* on (U, X, U) if

(i) $G, K \in \mathcal{B}(D(L), U)$;
(ii) the restriction A of L to $\operatorname{Ker} G$ generates a C_0-semigroup on X;
(iii) the operator G has a bounded right-inverse, i.e. there exists $G^r \in \mathcal{B}(U, D(L))$ such that $GG^r = I$.

The boundary node (G, L, K) is said to be *impedance passive* if

$$\operatorname{Re}\langle Lx, x\rangle_X \leq \operatorname{Re}\langle Gx, Kx\rangle_U, \qquad x \in D(L),$$

and in this case the C_0-semigroup generated by A is necessarily contractive, by the Lumer–Phillips theorem [14, Thm. II.3.15].

If (G, L, K) is a boundary node on (U, X, U) and A denotes the restriction of L to $\operatorname{Ker} G$, we define the *transfer function* $P\colon \rho(A) \to \mathcal{B}(U)$ by $P(\lambda)u = Kx$ for all $\lambda \in \rho(A)$ and $u \in U$, where $x \in D(L)$ is the unique solution of the abstract elliptic problem

$$(\lambda - L)x = 0, \qquad Gx = u.$$

Existence of P follows from [25, Rem. 10.1.5 & Prop. 10.1.2]. Recall also from [24, Prop. 2.4(b)] that if the boundary node (G, L, K) is impedance passive then $\operatorname{Re} P(\lambda) \geq 0$ for all $\lambda \in \rho(A) \cap \overline{\mathbb{C}_+}$. As in [24], we do not distinguish notationally between P and its analytic extensions to larger domains. For more information on boundary nodes, see [17, Ch. 11], [21].

Let (G_1, L_1, K_1) and (G_2, L_2, K_2) be two impedance passive boundary nodes on (U, X_1, U) and (U, X_2, U), respectively, where X_1, X_2 and U are Hilbert spaces, and consider the coupled system

$$\begin{cases} \dot{z}_1(t) = L_1 z_1(t), & t \geq 0, \\ \dot{z}_2(t) = L_2 z_2(t), & t \geq 0, \\ G_1 z_1(t) = K_2 z_2(t), & t \geq 0, \\ G_2 z_2(t) = -K_1 z_1(t), & t \geq 0, \\ z_1(0) \in X_1, \ z_2(0) \in X_2. \end{cases}$$

We may reformulate this coupled system as an abstract Cauchy problem

$$\begin{cases} \dot{z}(t) = A z(t), & t \geq 0, \\ z(0) = z_0 \end{cases}$$

for $z(\cdot) = (z_1(\cdot), z_2(\cdot))^\top$ on $X = X_1 \times X_2$, where $z_0 = (z_1(0), z_2(0))^\top \in X$ and $A = \operatorname{diag}(L_1, L_2) \colon D(A) \subseteq X \to X$ with

$$D(A) = \big\{ (x_1, x_2)^\top \in D(L_1) \times D(L_2) : G_1 x_1 = K_2 x_2, \ G_2 x_2 = -K_1 x_1 \big\}.$$

For $j = 1, 2$ we denote by A_j the restriction of L_j to $\operatorname{Ker} G_j$ and we denote the transfer function of the boundary node (G_j, L_j, K_j) by P_j.

The following is a special case of [24, Thm. 3.1]. The result makes it possible to deduce a growth bound for the resolvent of A from information about the boundary nodes $(G_1 + K_1, L_1, K_1)$ and (G_2, L_2, K_2). Here we may think of $(G_1 + K_1, L_1, K_1)$ as a boundary-damped version of the boundary node (G_1, L_1, K_1). For instance, in our wave-heat network $(G_1 + K_1, L_1, K_1)$ describes a wave network with dissipative boundary conditions. Note that in the coupled system this boundary-damping is replaced by the interconnection with the boundary node (G_2, L_2, K_2).

Theorem 2.1. *Let (G_1, L_1, K_1) and (G_2, L_2, K_2) be two impedance passive boundary nodes on (U, X_1, U) and (U, X_2, U), respectively, and let A_0 denote the restriction of L_1 to $\operatorname{Ker}(G_1 + K_1)$. Suppose there exists a non-empty set $E \subseteq \{s \in \mathbb{R} : is \in \rho(A_0) \cap \rho(A_2)\}$ and that*

$$\sup_{s \in E} \|(is - A_j)^{-1}\| < \infty \tag{2.1}$$

for $j = 0, 2$. Suppose furthermore that there exists a function $\eta \colon E \to (0, \infty)$ such that $\operatorname{Re} P_2(is) \geq \eta(s) I$ for all $s \in E$. Then A generates a contraction semigroup on X, $iE \subseteq \rho(A)$ and

$$\|(is - A)^{-1}\| \lesssim \frac{\mu(s)}{\eta(s)}, \qquad s \in E,$$

where $\mu(s) = 1 + \|P_2(1 + is)\|^2$ for $s \in \mathbb{R}$.

Remark 2.2. Note that [24, Thm. 3.1] is in fact more general than this. In particular, in [24, Thm. 3.1] the space U is allowed to be different for each of the two boundary nodes, and in addition we may replace (2.1) by the weaker condition that the norms of the two resolvents can be estimated from above along the imaginary axis by functions which need not be bounded.

3. Analysis of the wave-heat network

We begin by expressing the dynamics of the wave-heat network as an abstract Cauchy problem on a Hilbert space $X = L^2(0,1)^7$. To this end, we define

$$z(t) = (y_x^1(\cdot,t), y_t^1(\cdot,t), y_x^2(\cdot,t), y_t^2(\cdot,t), w^1(\cdot,t), w^2(\cdot,t), w^3(\cdot,t))^\top$$

for $t \geq 0$. We note that the total energy of the solutions of the wave-heat network satisfies

$$E(t) = \frac{1}{2}\|z(t)\|^2, \qquad t \geq 0.$$

The evolution of the state $z(\cdot)$ is described by the abstract Cauchy problem

$$\begin{cases} \dot{z}(t) = Az(t), & t \geq 0, \\ z(0) = z_0 \in X, \end{cases} \tag{3.1}$$

where the generator $A\colon D(A) \subseteq X \to X$ is defined by

$$Ax = (v_1', u_1', v_2', u_2', \beta_1 w_1'', \beta_2 w_2'', \beta_3 w_3'')^\top$$

for $x = (u_1, v_1, u_2, v_2, w_1, w_2, w_3)^\top$ in the domain

$$\begin{aligned} D(A) = \big\{ (u_1, v_1, u_2, v_2, w_1, w_2, w_3)^\top \in H^1(0,1)^4 \times H^2(0,1)^3 : v_1(0) = 0,& \\ v_1(1) = w_1(0) = w_3(1), \quad u_1(1) - \beta_1 w_1'(0) + \beta_3 w_3'(1) = 0,& \\ w_1(1) = w_2(0) = v_2(0), \quad \beta_1 w_1'(1) - \beta_2 w_2'(0) - u_2(0) = 0,& \\ w_2(1) = w_3(0) = v_2(1), \quad \beta_2 w_2'(1) - \beta_3 w_3'(0) + u_2(1) = 0 \ \big\}.& \end{aligned}$$

As will become clear in Section 4 the operator A generates a C_0-semigroup $(T(t))_{t\geq 0}$ of contractions on X. We recall that the *mild solution* of the abstract Cauchy problem (3.1) is given by $z(t) = T(t)z_0$ for $t \geq 0$, and that $z(\cdot)$ is a *classical solution* if and only if $z_0 \in D(A)$; see [14, Sec. II.6].

In order to apply the results in Section 2, we express A and $D(A)$ in terms of two boundary nodes (G_1, L_1, K_1) and (G_2, L_2, K_2) on $(\mathbb{C}^3, L^2(0,1)^4, \mathbb{C}^3)$ and $(\mathbb{C}^3, L^2(0,1)^3, \mathbb{C}^3)$, respectively. We achieve this by decomposing the state $z(\cdot)$ into two parts, one related to the two wave equations, the second related to the three heat equations. More specifically, for $t \geq 0$ we write

$$z(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix}, \qquad z_1(t) = \begin{pmatrix} u_1(t) \\ v_1(t) \\ u_2(t) \\ v_2(t) \end{pmatrix}, \qquad z_2(t) = \begin{pmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \end{pmatrix}.$$

With respect to this decomposition the operator A can be expressed in the form $A = \operatorname{diag}(L_1, L_2)$, where

$$L_1 \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1' \\ u_1' \\ v_2' \\ u_2' \end{pmatrix}$$

with domain $D(L_1) = \{(u_1, v_1, u_2, v_2)^\top \in H^1(0,1)^4 : v_1(0) = 0\}$ and

$$L_2 \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} \beta_1 w_1'' \\ \beta_2 w_2'' \\ \beta_3 w_3'' \end{pmatrix}$$

with domain $D(L_2) = \{(w_1, w_2, w_3)^\top \in H^2(0,1)^3 : w_3(1) = w_1(0),\ w_1(1) = w_2(0), w_2(1) = w_3(0)\}$. In addition, we define $G_1, K_1 \colon D(L_1) \to \mathbb{C}^3$ by

$$G_1 \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} -u_1(1) \\ u_2(0) \\ -u_2(1) \end{pmatrix}, \qquad K_1 \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} -v_1(1) \\ -v_2(0) \\ -v_2(1) \end{pmatrix}$$

and $G_2, K_2 \colon D(L_2) \to \mathbb{C}^3$ by

$$G_2 \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} w_1(0) \\ w_2(0) \\ w_3(0) \end{pmatrix}, \qquad K_2 \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} \beta_3 w_3'(1) - \beta_1 w_1'(0) \\ \beta_1 w_1'(1) - \beta_2 w_2'(0) \\ \beta_2 w_2'(1) - \beta_3 w_3'(0) \end{pmatrix},$$

noting that these definitions ensure that $D(A) = \{(x_1, x_2)^\top \in D(L_1) \times D(L_2) : G_1 x_1 = K_2 x_2,\ G_2 x_2 = -K_1 x_1\}$, as required in Section 2.

We now show that (G_1, L_1, K_1) and (G_2, L_2, K_2) are impedance passive boundary nodes, and that the restrictions A_0 and A_2 of L_1 to $\operatorname{Ker}(G_1 + K_1)$ and of L_2 to $\operatorname{Ker} G_2$, respectively, have uniformly bounded resolvents along the imaginary axis. Furthermore, we derive bounds for the transfer function P_2 of the second node. Once these objectives have been met, Theorem 1.1 will be a straightforward consequence of Theorem 2.1 and the stability theory of strongly continuous semigroups.

3.1. **Analysis of the wave network.** In this section we analyse the wave network and show that (G_1, L_1, K_1) defines an impedance passive boundary node on $(\mathbb{C}^3, X_1, \mathbb{C}^3)$, where $X_1 = L^2(0,1)^4$. From the above definitions it is clear that $G_1, K_1 \in \mathcal{B}(D(L_1), \mathbb{C}^3)$, and since G_1 is surjective and $\mathbb{C}^3$ is finite-dimensional we know that G_1 has a bounded right-inverse. We note that the restriction A_1 of L_1 to $\operatorname{Ker} G_1$ has the structure $A_1 = \operatorname{diag}(A_{1,1}, A_{1,2})$, where $A_{1,1}, A_{1,2} \colon (u,v)^\top \mapsto (v', u')^\top$ with respective domains $D(A_{1,1}) = \{(u,v)^\top \in H^1(0,1)^2 : u(1) = 0, v(0) = 0\}$ and $D(A_{1,2}) = H_0^1(0,1) \times H^1(0,1)$. Since both $A_{1,1}$ and $A_{1,2}$ generate strongly continuous contraction semigroups on $L^2(0,1)^2$, A_1 generates a contraction semigroup on X_1. Thus (G_1, L_1, K_1) is a boundary node. A straightforward computation using integration by parts shows that $\operatorname{Re}\langle L_1 x, x\rangle = \operatorname{Re}\langle G_1 x, K_1 x\rangle$ for all $x \in D(L_1)$, so the boundary node (G_1, L_1, K_1) is impedance passive. Arguing as for instance in [17, Ex. 9.2.1] for the wave equations along each of the blue edges in Figure 2 it is possible to show that the semigroup generated by the restriction

A_0 of L_1 to $\operatorname{Ker}(G_1 + K_1)$ is uniformly exponentially stable. In fact, by adapting the argument given in [9, Sec. 1] we see that this semigroup is even nilpotent in our case. Either way, it follows that $i\mathbb{R} \subseteq \rho(A_0)$ and $\sup_{s\in\mathbb{R}} \|(is - A_0)^{-1}\| < \infty$.

3.2. **Analysis of the heat network.** It remains to analyse the heat network. In this section we show that (G_2, L_2, K_2) defines an impedance passive boundary node on $(\mathbb{C}^3, X_2, \mathbb{C}^3)$, where $X_2 = L^2(0,1)^3$, and we analyse the transfer function P_2 of this boundary node. It would certainly be possible to establish the boundary node property and stability of (G_2, L_2, K_2) directly from the definitions of L_2, G_2, and K_2. Instead, however, we demonstrate how the analysis can be simplified by viewing the boundary node as an *interconnection* of three individual heat equations. This approach allows us to use the interconnection theory developed in [1] to show the boundary node property and impedance passivity of (G_2, L_2, K_2), and also provides an efficient way of deriving a formula for the transfer function P_2. We begin, therefore, by analysing a single heat equation with two inputs and two outputs. In what follows we use the principal branch of the logarithm to define the complex square root.

Lemma 3.1. *Let $X = L^2(0,1)$ and $\beta > 0$, and define $L\colon D(L) \subseteq X \to X$ and $G, K\colon D(L) \to \mathbb{C}^2$ by $D(L) = H^2(0,1)$ and*

$$Lw = \beta w'', \qquad Gw = \begin{pmatrix} w(0) \\ w(1) \end{pmatrix}, \qquad Kw = \begin{pmatrix} -\beta w'(0) \\ \beta w'(1) \end{pmatrix}$$

for $w \in D(L)$. Then (G, L, K) is an impedance passive boundary node on $(\mathbb{C}^2, X, \mathbb{C}^2)$, the semigroup generated by the restriction A of L to $\operatorname{Ker} G$ is uniformly exponentially stable, and the map $w \mapsto (Gw, Kw)^\top$ is a surjection from $D(L)$ onto $\mathbb{C}^4$. Furthermore, the transfer function P of the boundary node (G, L, K) satisfies

$$P(\lambda) = \begin{pmatrix} p_1(\lambda) & p_2(\lambda) \\ p_2(\lambda) & p_1(\lambda) \end{pmatrix}, \qquad \lambda \in \overline{\mathbb{C}_+},$$

where

$$p_1(\lambda) = \frac{\sqrt{\lambda\beta}}{\tanh\sqrt{\lambda/\beta}}, \qquad p_2(\lambda) = -\frac{\sqrt{\lambda\beta}}{\sinh\sqrt{\lambda/\beta}}$$

for $\lambda \in \overline{\mathbb{C}_+} \setminus \{0\}$ and $p_1(0) = -p_2(0) = \beta$.

Proof. It is straightforward to establish that (G, L, K) is an impedance passive boundary node on $(\mathbb{C}^2, X, \mathbb{C}^2)$, that the semigroup generated by A is uniformly exponentially stable, and that the map $w \mapsto (Gw, Kw)^\top$ is a surjection from $D(L)$ onto $\mathbb{C}^4$. In order to determine the transfer function P, let $\lambda \in \overline{\mathbb{C}_+}$ and $u = (u_1, u_2)^\top \in \mathbb{C}^2$. Then $P(\lambda)u = (-\beta w'(0), \beta w'(1))^\top$, where $w \in H^2(0,1)$ is the unique solution of the problem $\lambda w - \beta w'' = 0$ subject to the boundary conditions $w(0) = u_1$ and $w(1) = u_2$. Solving this boundary-value problem yields $w(x) = u_1(1-x) + u_2 x$ for $x \in (0,1)$ if $\lambda = 0$, and

$$w(x) = u_1 \cosh(\nu x) + \left(\frac{u_2}{\sinh\nu} - \frac{u_1}{\tanh\nu}\right)\sinh(\nu x), \qquad x \in (0,1),$$

if $\lambda \in \overline{\mathbb{C}_+} \setminus \{0\}$, where $\nu = \sqrt{\lambda/\beta}$. The formula for $P(\lambda)$ follows at once. $\square$

The following is the key result in analysing the heat part of our system.

Proposition 3.2. *The triple (G_2, L_2, K_2) is an impedance passive boundary node on $(\mathbb{C}^3, X_2, \mathbb{C}^3)$, and the semigroup generated by the restriction A_2 of L_2 to $\operatorname{Ker} G_2$ is uniformly exponentially stable. Furthermore, the transfer function P_2 satisfies*

$$\|P_2(1+is)\| \lesssim 1 + |s|^{1/2}, \qquad s \in \mathbb{R},$$

and, given any $\varepsilon > 0$, there exists a constant $c_\varepsilon > 0$ such that

$$\operatorname{Re} P_2(is) \geq c_\varepsilon (1 + |s|^{1/2}) I, \qquad s \in \mathbb{R} \setminus (-\varepsilon, \varepsilon). \tag{3.2}$$

Proof. We begin by considering a boundary node (G, L, K) on $(\mathbb{C}^6, X_2, \mathbb{C}^6)$ defined by three separate heat equations, each of the form in Lemma 3.1. More precisely, let $L\colon D(L) \subseteq X_2 \to X_2$ and $G, K\colon D(L) \to \mathbb{C}^6$ be defined by

$$L\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} \beta_1 w_1'' \\ \beta_2 w_2'' \\ \beta_3 w_3'' \end{pmatrix}, \quad G\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} w_1(0) \\ w_1(1) \\ w_2(0) \\ w_2(1) \\ w_3(0) \\ w_3(1) \end{pmatrix}, \quad K\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} -\beta_1 w_1'(0) \\ \beta_1 w_1'(1) \\ -\beta_2 w_2'(0) \\ \beta_2 w_2'(1) \\ -\beta_3 w_3'(0) \\ \beta_3 w_3'(1) \end{pmatrix},$$

respectively, for $(w_1, w_2, w_3)^\top \in D(L) = H^2(0,1)^3$. We note that we may recover our original heat network from the decoupled heat equations by choosing three inputs u_1, u_2, u_3 and three outputs y_1, y_2, y_3 in a suitable way, namely $w_1(0) = w_3(1) = u_1$ and $y_1 = \beta_3 w_3'(1) - \beta_1 w_1'(0)$, $w_1(1) = w_2(0) = u_2$ and $y_2 = \beta_1 w_1'(1) - \beta_2 w_2'(0)$, and finally $w_2(1) = w_3(0) = u_3$ and $y_3 = \beta_2 w_2'(1) - \beta_3 w_3'(0)$. Since these interconnections fit into the framework of [1], the properties in Lemma 3.1 and [1, Thm. 3.3] together with [24, Rem. 2.2] imply that (G_2, L_2, K_2) is an impedance passive boundary node on $(\mathbb{C}^3, X_2, \mathbb{C}^3)$. Furthermore, using the diagonal structure of the restriction A_2 of L_2 to $\operatorname{Ker} G_2$ it is easy to see that the semigroup generated by A_2 is uniformly exponentially stable.

It remains to establish bounds for the transfer function P_2 of the boundary node (G_2, L_2, K_2). Let $\lambda \in \rho(A_2)$ and $u \in \mathbb{C}^3$. From the definition of the transfer function we have $P_2(\lambda)u = K_2 x$, where $x \in D(L_2)$ is the unique solution of the abstract elliptic problem $(\lambda - L_2)x = 0$ subject to the boundary condition $G_2 x = u$. From the properties of the interconnections described in the previous paragraph it is clear that x solves this problem if and only if it is the unique element of $D(L)$ solving the problem $(\lambda - L)x = 0$ subject to the boundary condition $Gx = R^\top u$, where

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

Since $K_2 x = RKx$, we deduce that $P_2(\lambda)u = RP(\lambda)R^\top u$, and hence $P_2(\lambda) = RP(\lambda)R^\top$. Now from Lemma 3.1 and the diagonal structure of L it is clear that $P(\lambda) = \operatorname{diag}(P^1(\lambda), P^2(\lambda), P^3(\lambda))$ for $\lambda \in \rho(A_2)$, where

$$P^k(\lambda) = \begin{pmatrix} p_1^k(\lambda) & p_2^k(\lambda) \\ p_2^k(\lambda) & p_1^k(\lambda) \end{pmatrix}, \qquad \lambda \in \overline{\mathbb{C}_+},$$

with

$$p_1^k(\lambda) = \frac{\sqrt{\lambda\beta_k}}{\tanh\sqrt{\lambda/\beta_k}}, \qquad p_2^k(\lambda) = -\frac{\sqrt{\lambda\beta_k}}{\sinh\sqrt{\lambda/\beta_k}}$$

for $\lambda \in \overline{\mathbb{C}_+} \setminus \{0\}$ and $p_1^k(0) = -p_2^k(0) = \beta_k$ for $k = 1, 2, 3$. A straightforward computation now gives

$$P_2(\lambda) = \begin{pmatrix} p_1^1(\lambda) + p_1^3(\lambda) & p_2^1(\lambda) & p_2^3(\lambda) \\ p_2^1(\lambda) & p_1^1(\lambda) + p_1^2(\lambda) & p_2^2(\lambda) \\ p_2^3(\lambda) & p_2^2(\lambda) & p_1^2(\lambda) + p_1^3(\lambda) \end{pmatrix}, \qquad \lambda \in \overline{\mathbb{C}_+}.$$

From the formulas for p_1^k and p_2^k for $k = 1, 2, 3$ it follows easily that $\|P_2(1 + is)\| \lesssim 1 + |s|^{1/2}$ for $s \in \mathbb{R}$. Finally we establish a lower bound for $\operatorname{Re} P_2(is)$ when $s \in \mathbb{R}\setminus\{0\}$. To this end, let us write $q_j^k(s)$ for $\operatorname{Re} p_j^k(is)$, where $j = 1, 2$, $k = 1, 2, 3$ and $s \in \mathbb{R}$, so that

$$\operatorname{Re} P_2(is) = \begin{pmatrix} q_1^1(s) + q_1^3(s) & q_2^1(s) & q_2^3(s) \\ q_2^1(s) & q_1^1(s) + q_1^2(s) & q_2^2(s) \\ q_2^3(s) & q_2^2(s) & q_1^2(s) + q_1^3(s) \end{pmatrix}, \qquad s \in \mathbb{R}.$$

Let $a_k(s) = (|s|/(2\beta_k))^{1/2}$ for $k = 1, 2, 3$ and $s \in \mathbb{R}$. Then

$$q_1^k(s) = \beta_k a_k(s) \frac{\cosh a_k(s) \sinh a_k(s) + \cos a_k(s) \sin a_k(s)}{\sinh^2 a_k(s) + \sin^2 a_k(s)},$$

$$q_2^k(s) = -\beta_k a_k(s) \frac{\cos a_k(s) \sinh a_k(s) + \cosh a_k(s) \sin a_k(s)}{\sinh^2 a_k(s) + \sin^2 a_k(s)}$$

for $k = 1, 2, 3$ and $s \in \mathbb{R}\setminus\{0\}$. We thus see that $q_1^k(s) > 0$ for $k = 1, 2, 3$ and $s \in \mathbb{R} \setminus \{0\}$. A standard calculation shows that

$$q_1^k(s)^2 - q_2^k(s)^2 = \frac{\beta_k|s|}{2} \frac{\sinh^2 a_k(s) - \sin^2 a_k(s)}{\sinh^2 a_k(s) + \sin^2 a_k(s)},$$

and in particular $|q_1^k(s)| > |q_2^k(s)|$ for $k = 1, 2, 3$ and $s \in \mathbb{R} \setminus \{0\}$. Hence the matrix $\operatorname{Re} P_2(is)$ is strictly diagonally dominant, and in particular non-singular, for all $s \in \mathbb{R} \setminus \{0\}$. It follows that there exists a continuous map $\eta \colon \mathbb{R} \setminus \{0\} \to (0, \infty)$ such that $\operatorname{Re} P_2(is) \geq \eta(s)I$ for all $s \in \mathbb{R} \setminus \{0\}$. Since $q_1^k(s) \gtrsim |s|^{1/2}$ as $|s| \to \infty$ while $|q_2^k(s)| \to 0$ exponentially fast as $|s| \to \infty$ for $k = 1, 2, 3$, we obtain the lower bound in (3.2) for arbitrary $\varepsilon > 0$. □

4. Energy decay

In this final section we combine the results presented in Sections 2 and 3 in order to prove our main result.

Proof of Theorem 1.1. We consider the abstract Cauchy problem corresponding to our wave-heat system as explained in Section 3. The observations made in Section 3.1 together with Proposition 3.2 imply that the boundary nodes (G_1, L_1, K_1) and (G_2, L_2, K_2) satisfy the conditions of Theorem 2.1 with $E = \mathbb{R}\setminus(-\varepsilon, \varepsilon)$ for any $\varepsilon > 0$. Therefore the operator A defined in Section 3 generates a contraction semigroup $(T(t))_{t\geq 0}$ on the space $X = X_1 \times X_2$. Proposition 3.2 also tells us that $\|P_2(1 + is)\|^2 \lesssim 1 + |s|$ for $s \in \mathbb{R}$ and that, given any $\varepsilon > 0$, there exists c_ε such that $\operatorname{Re} P_2(is) \geq c_\varepsilon(1 + |s|^{1/2})I$ for all $s \in \mathbb{R}\setminus(-\varepsilon, \varepsilon)$. It follows from Theorem 2.1 that $\{is : s \in \mathbb{R}, |s| > \varepsilon\} \subseteq \rho(A)$

and $\|(is-A)^{-1}\| \lesssim 1+|s|^{1/2}$ for $|s|>\varepsilon$, for any given $\varepsilon>0$. Next we establish that $0\in\rho(A)$. We first note that since both A_1 and A_2 have compact resolvent, it follows from [24, Thm. 3.7] that A, too, has compact resolvent. It therefore suffices to show that A is injective. To this end, suppose that $x=(u_1,v_1,u_2,v_2,w_1,w_2,w_3)^\top \in \operatorname{Ker} A$. Then the functions u_k, v_k are constant for $k=1,2$, and the functions w_k are affine for $k=1,2,3$. Moreover, the coupling conditions in $D(A)$ give

$$\begin{aligned} v_1(0)=0, \quad v_1(1)=w_1(0)&=w_3(1), \quad u_1(1)-\beta_1 w_1'(0)+\beta_3 w_3'(1)=0,\\ w_1(1)=w_2(0)&=v_2(0), \quad \beta_1 w_1'(1)-\beta_2 w_2'(0)-u_2(0)=0,\\ w_2(1)=w_3(0)&=v_2(1), \quad \beta_2 w_2'(1)-\beta_3 w_3'(0)+u_2(1)=0. \end{aligned}$$

Since $\beta_1,\beta_2,\beta_3>0$, it follows easily that $x=0$. Hence A is injective, and therefore invertible. By choosing $\varepsilon>0$ in the previous part of the argument to be sufficiently small we may therefore deduce that $i\mathbb{R}\subseteq\rho(A)$ with $\|(is-A)^{-1}\|\lesssim 1+|s|^{1/2}$ for $s\in\mathbb{R}$. Thus the semigroup $(T(t))_{t\ge 0}$ is strongly stable by an application of the theorem of Arendt, Batty, Lyubich and Vũ [14, Thm. V.2.21], and by [8, Thm. 2.4] we obtain $\|T(t)z_0\|=o(t^{-2})$ as $t\to\infty$ for all $z_0\in D(A)$. Since the energy of a solution with initial data $z_0\in X$ is given by $E(t)=\frac{1}{2}\|T(t)z_0\|^2$ for all $t\ge 0$, the conclusions of Theorem 1.1 follow immediately. □

Remark 4.1. We may alternatively consider the wave-heat system with the Neumann boundary condition $y_x^1(t,0)=0$ for $t>0$ at the left end of the first wave equation in Figure 1. The analysis is largely the same as in our case, with the important difference that the point 0 becomes an eigenvalue of the operator A.

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