POLYNOMIAL STABILITY OF NON-LINEARLY DAMPED CONTRACTION SEMIGROUPS

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Abstract. We investigate the stability properties of an abstract class of semi-linear systems. Our main result establishes rational rates of decay for classical solutions assuming a certain non-uniform observability estimate for the linear part and suitable conditions on the non-linearity. We illustrate the strength of our abstract results by applying them to a one-dimensional wave equation with weak non-linear damping and to an Euler–Bernoulli beam with a tip mass subject to non-linear damping.

1. Introduction

We consider semi-linear systems of the form

$$\dot{x}(t) = Ax(t) - B\phi(B^*x(t)), \qquad t \geq 0, \tag{1.1}$$

to be solved subject to the initial condition $x(0) = x_0$. Here A is assumed to be the generator of a contraction semigroup $(T(t))_{t\geq 0}$ on a Hilbert space X, $x_0 \in X$ and B is a bounded linear operator from another Hilbert space U into X. Furthermore, $\phi\colon U \to U$ is a potentially non-linear map which satisfies $\phi(0) = 0$ and is *monotone* in the sense that

$$\mathrm{Re}\langle \phi(u_1) - \phi(u_2), u_1 - u_2\rangle \geq 0, \qquad u_1, u_2 \in U.$$

These conditions on ϕ allow us to interpret the second summand on the right-hand side of (1.1) as a non-linear *damping* term.

We investigate the stability properties of the equation (1.1). Our main result in Theorem 2.1 establishes a rational decay rate for classical solutions of (1.1) based on a generalised observability-type condition on the operators A and B and the additional assumption that

$$\mathrm{Re}\langle \phi(u), u\rangle \gtrsim \begin{cases} \|u\|^2, & \|u\| \leq \delta, \\ 1, & \|u\| > \delta, \end{cases} \tag{1.2}$$

for some $\delta > 0$. We illustrate our theoretical results in Section 3 by applying them to a one-dimensional wave equation with weak non-linear damping, and to an Euler–Bernoulli beam with a tip mass subject to non-linear damping.

Our results augment earlier studies on asymptotic stability with non-linear damping in [14, 15, 22, 24, 30] by providing rates of decay. They

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in particular generalise analogous results obtained previously in the linear case [7]. The results in [7] have previously been extended to abstract wave-type equations with non-linear viscous damping in [3, 4, 5, 29]. Rational and generalised decay rates for wave equations with non-linear viscous damping have also been studied in [2, 12, 21], the case of boundary damping in [1, 23, 35, 36, 37]. Exponential stability of wave equations and semigroups with non-linear damping has been investigated for instance in in [13, 19, 25, 34]. Our results differ from existing work in that we establish rational decay rates for general abstract systems with non-linear damping based on a generalised observability property of the linear system. In particular, our class of operators and the type of damping differ from those considered in the most closely related references [3, 4, 5, 29].

A function $x(\cdot) \in C^1(\mathbb{R}_+; X)$ is said to be a *classical solution* of (1.1) if $x(t) \in D(A)$ for all $t \geq 0$ and (1.1) holds for all $t \geq 0$. A function $x(\cdot) \in C(\mathbb{R}_+; X)$ is said to be a *generalised solution* of (1.1) if there exists a sequence $(x_n(\cdot))_{n\geq 1}$ of classical solutions of (1.1) such that $\sup_{t\in[0,\tau]} \|x(t)-x_n(t)\| \to 0$ as $n \to \infty$ for all $\tau > 0$. Note that if ϕ is a linear map then this notion of a classical solution is consistent with the terminology used for linear abstract Cauchy problems, while the notion of a generalised solution describes what would typically be referred to as a *mild* solution in the linear setting. We recall, moreover, that in the linear case the mild (or generalised) solution is given by the semigroup orbit, and that it is a classical solution precisely when $x_0 \in D(A)$.

If x is a classical solution of (1.1), then dissipativity of A gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2 = 2\operatorname{Re}\langle Ax(t) - B\phi(B^*x(t)), x(t)\rangle \leq -2\operatorname{Re}\langle \phi(B^*x(t)), B^*x(t)\rangle.$$

It follows that

$$\|x(t)\|^2 + 2\int_0^t \operatorname{Re}\langle \phi(B^*x(s)), B^*x(s)\rangle\,\mathrm{d}s \leq \|x(0)\|^2, \qquad t \geq 0, \tag{1.3}$$

and in particular monotonicity of ϕ implies that any classical solution has non-increasing norm. Thus [16, Thm. 11.1.5(b)] shows that (1.1) has a unique classical solution whenever ϕ is locally Lipschitz continuous and $x_0 \in D(A)$. In fact, by monotonicity of ϕ any two classical solutions $x_1(\cdot), x_2(\cdot)$ satisfy

$$\|x_2(t) - x_1(t)\| \leq \|x_2(0) - x_1(0)\|, \qquad t \geq 0,$$

and by [26, Cor. 3.7] the function $\|\dot{x}(\cdot)\|$, too, is non-increasing.

Given (complex) Hilbert spaces X and Y, we write $\mathcal{B}(X, Y)$ for the space of bounded linear operators from X to Y, and we write $\mathcal{B}(X)$ for $\mathcal{B}(X, X)$. We denote the domain, kernel, spectrum and resolvent set of a linear operator A by $D(A)$, $\operatorname{Ker} A$, $\sigma(A)$, and $\rho(A)$, respectively. If p and q are two real-valued quantities we write $p \lesssim q$ to express that $p \leq Cq$ for some constant $C > 0$ which is independent of all parameters that are free to vary in a given situation. We shall also make use of standard 'big-O' and 'little-o' notation.

2. Polynomial stability of abstract semi-linear systems

We now come to our main result. It provides polynomial decay rates for classical solutions of (1.1) under suitable assumptions on the linear counterpart with ϕ equal to the identity function. We write A_B for the corresponding infinitesimal generator $A - BB^*$ with domain $D(A_B) = D(A)$. The second part of the result shows that if ϕ is sufficiently close to linear near the origin, then the decay rate of classical solutions is determined by the rate of resolvent growth in the corresponding linear equation. In this case, even a suboptimal observability inequality (2.1) will, when combined with a sharp resolvent estimate, lead to the optimal decay rate.

Theorem 2.1. *Suppose that $\phi : U \to U$ is monotone, locally Lipschitz continuous and satisfies $\phi(0) = 0$, and that (1.2) holds for some $\delta > 0$. Suppose further that there exist $\beta, \tau, c_\tau > 0$ such that*

$$c_\tau \|(I - A)^{-\beta} x_0\|^2 \leq \int_0^\tau \|B^* T(t) x_0\|^2 \,\mathrm{d}t, \qquad x_0 \in X. \tag{2.1}$$

Then $i\mathbb{R} \subseteq \rho(A_B)$. Furthermore, all generalised solutions of (1.1) satisfy $\|x(t)\| \to 0$ as $t \to \infty$, and all classical solutions of (1.1) satisfy $\|x(t)\| = O(t^{-1/(2\beta)})$ as $t \to \infty$.

If, in addition, $\|(is - A_B)^{-1}\| \lesssim 1 + |s|^\alpha$ for some $\alpha > 0$ and all $s \in \mathbb{R}$, and if there exist $\gamma > \alpha/2 + 1$ and $\kappa, \varepsilon > 0$ such that

$$\|\phi(u) - \kappa u\| \lesssim \|u\|^\gamma, \qquad \|u\| \leq \varepsilon, \tag{2.2}$$

then $\|x(t)\| = O(t^{-1/\alpha})$ as $t \to \infty$ for any classical solution of (1.1).

Proof. Let $x(\cdot)$ be a classical solution of (1.1), and suppose there exists $t_0 \geq 0$ such that $\|B^* x(t)\| \leq \delta$ for all $t \geq t_0$. Furthermore, let $L_\delta > 0$ be such that $\|\phi(u)\| \leq L_\delta \|u\|$ for all $u \in U$ with $\|u\| \leq \delta$. Let $\tau > 0$. For $u \in L^2(0, \tau; U)$ we define

$$(\mathbb{F}_\tau u)(t) = \int_0^t B^* T(t-s) B u(s) \,\mathrm{d}s, \qquad 0 \leq t \leq \tau.$$

Then $\mathbb{F}_\tau \in \mathcal{B}(L^2(0, \tau; U))$ and

$$B^* T(s) x(t) = B^* x(t+s) + (\mathbb{F}_\tau \phi(B^* x(t + \cdot)))(s), \qquad 0 \leq s \leq \tau,\ t \geq 0,$$

by the variation of parameters formula, and hence

$$\|B^* T(\cdot) x(t)\|_{L^2(0,\tau;U)} \lesssim \|B^* x(t + \cdot)\|_{L^2(0,\tau;U)} + \|\phi(B^* x(t + \cdot))\|_{L^2(0,\tau;U)}$$

for all $t \geq 0$. For $t \geq t_0$ it follows that

$$\|B^* T(\cdot) x(t)\|_{L^2(0,\tau;U)} \lesssim \|B^* x(t + \cdot)\|_{L^2(0,\tau;U)},$$

and combining this with (2.1) gives

$$\|(I - A)^{-\beta} x(t)\| \lesssim \|B^* x(t + \cdot)\|_{L^2(0,\tau;U)}, \qquad t \geq t_0.$$

Now let $k \in \mathbb{N}$ be such that $k\tau \geq t_0$. Using (1.3) and (1.2) it follows that

$$\begin{aligned}\|x(k\tau)\|^2 - \|x((k+1)\tau)\|^2 &= 2 \int_0^\tau \mathrm{Re}\langle \phi(B^* x(k\tau + s)), B^* x(k\tau + s)\rangle \,\mathrm{d}s \\ &\gtrsim \|B^* x(k\tau + \cdot))\|^2_{L^2(0,\tau;U)} \gtrsim \|(I - A)^{-\beta} x(k\tau)\|^2,\end{aligned}$$

where the implicit constants are independent of k. Noting that $x(k\tau) \in D(A)$, we may apply the moment inequality [17, Thm. II.5.34] to obtain

$$\|x(k\tau)\| \le \|(I-A)x(k\tau)\|^{\frac{\beta}{1+\beta}} \|(I-A)^{-\beta}x(k\tau)\|^{\frac{1}{1+\beta}}.$$

Thus

$$\|x(k\tau)\|^2 - \|x((k+1)\tau)\|^2 \gtrsim \frac{\|x((k+1)\tau)\|^{2(1+\beta)}}{\|(I-A)x(k\tau)\|^{2\beta}},$$

where we have used the fact that $\|x((k+1)\tau)\| \le \|x(k\tau)\|$. Now

$$\begin{aligned}\|(I-A)x(k\tau)\| &\le \|x(k\tau)\| + \|Ax(k\tau) - B\phi(B^*x(k\tau))\| + \|B\|\|\phi(B^*x(k\tau))\| \\ &\le (1+L_\delta\|B\|^2)\|x(k\tau)\| + \|\dot{x}(k\tau)\| \\ &\le (1+L_\delta\|B\|^2)\|x(t_0)\| + \|\dot{x}(t_0)\| \\ &= (1+L_\delta\|B\|^2)\|x(t_0)\| + \|Ax(t_0) - B\phi(B^*x(t_0))\| \\ &\le 2(1+L_\delta\|B\|^2)\|x(t_0)\| + \|(I-A)x(t_0)\| \\ &\lesssim \|(I-A)x(t_0)\|,\end{aligned}$$

where we have used monotonicity of the functions $\|x(\cdot)\|$ and $\|\dot{x}(\cdot)\|$. It follows that

$$\frac{\|x((k+1)\tau)\|^2}{\|(I-A)x(t_0)\|^2} \le \frac{\|x(k\tau)\|^2}{\|(I-A)x(t_0)\|^2} - c\left(\frac{\|x((k+1)\tau)\|^2}{\|(I-A)x(t_0)\|^2}\right)^{1+\beta}$$

for some constant $c>0$ and all sufficiently large $k \ge 1$, and hence

$$\|x(k\tau)\| \lesssim \frac{\|(I-A)x(t_0)\|}{(k\tau+1)^{1/(2\beta)}}$$

for all sufficiently large $k \ge 1$ by [6, Lem. 1.3.4]. By monotonicity of $\|x(\cdot)\|$ we deduce that

$$\|x(t)\| \lesssim \frac{\|(I-A)x(t_0)\|}{(t+1)^{1/(2\beta)}}$$

for all sufficiently large $t \ge 0$. Next we observe that, for any $\delta \ge \|B^*\|\|x_0\|$, the identity function on U defined by $\phi(u) = u$ for all $u \in U$ satisfies our assumptions with $t_0 = 0$ and $L_\delta = 1$. We deduce that $\|T_B(t)(I-A_B)^{-1}\| = O(t^{-1/(2\beta)})$ as $t \to \infty$, where $(T_B(t))_{t\ge0}$ is the C_0-semigroup of contractions generated by A_B. It follows from [9, Thm. 1.1] that $i\mathbb{R} \subseteq \rho(A_B)$. Moreover, $(T_B(t))_{t\ge0}$ is strongly stable in the sense that $\|T_B(t)x\| \to 0$ as $t \to \infty$ for all $x \in X$, by a standard density argument. It then follows from [15, Thm. 2.2] (noting that the result carries over, with the appropriate modifications, to the setting of complex Hilbert spaces, and that the assumptions of compact resolvent and approximate observability can be replaced by strong stability of $(T_B(t))_{t\ge0}$) that whenever ϕ satisfies the conditions of our theorem we have $\|x(t)\| \to 0$ as $t \to \infty$ for all generalised solutions of (1.1). Hence the first part of the proof shows that $\|x(t)\| = O(t^{-1/(2\beta)})$ as $t \to \infty$ for all classical solutions of (1.1), which completes the proof of the first part of the result.

Now assume, in addition, that $\|(is - A_B)^{-1}\| \lesssim 1 + |s|^\alpha$ for some $\alpha > 0$ and all $s \in \mathbb{R}$, and that there exist $\gamma > \alpha/2 + 1$ and $\kappa, \varepsilon > 0$ such that (2.2) holds. If $\alpha \ge 2\beta$ then the result already follows without any of the additional assumptions from what has already been proved, so we may assume that

$\alpha < 2\beta$. Recall from the first part of the proof that $\|x(t)\| \to 0$ as $t \to \infty$ for all generalised solutions of (1.1), and that $\|x(t)\| = O(t^{-1/(2\beta)})$ as $t \to \infty$ for all classical solutions of (1.1). Note also that by [11, Lem. 2.11(c)] the operator $A_\kappa = A - \kappa BB^*$ satisfies $i\mathbb{R} \subseteq \rho(A_\kappa)$ and $\|T_\kappa(t)A_\kappa^{-1}\| = O(t^{-1/\alpha})$ as $t \to \infty$, where $(T_\kappa(t))_{t\geq 0}$ is the contraction semigroup generated by A_κ. Let $x(\cdot)$ be a classical solution of (1.1), and let $t_0 \geq 0$ be such that $\|B^*x(t)\| \leq \varepsilon$ for all $t \geq t_0$. We have

$$\dot{x}(t) = (A - \kappa BB^*)x(t) + By(t), \qquad t \geq 0,$$

where $y(t) = \kappa B^*x(t) - \phi(B^*x(t))$ for all $t \geq 0$. Note that, by (2.2) and boundedness of B, we have $\|y(t)\| \lesssim \|x(t)\|^\gamma$ for $t \geq t_0$. Given $t \geq 0$ let $t_\theta = \max\{t_0, t - t^\theta/2\}$, where $\theta \in (0, 1]$ is to be chosen later, noting that $t_\theta \to \infty$ and $t - t_\theta \to \infty$ as $t \to \infty$. The variation of parameters formula gives

$$x(t) = T_\kappa(t - t_0)x(t_0) + T_\kappa(t - t_\theta)\int_{t_0}^{t_\theta} T_\kappa(t_\theta - s)By(s)\,\mathrm{d}s + \int_{t_\theta}^{t} T_\kappa(t - s)By(s)\,\mathrm{d}s \tag{2.3}$$

for all $t \geq t_0$. Since $x(t) \in D(A)$ for all $t \geq 0$, we have $\|T_\kappa(t - t_0)x(t_0)\| = O(t^{-1/\alpha})$ as $t \to \infty$. Moreover, since ϕ is locally Lipschitz continuous and $Bx(\cdot)$ is continuously differentiable, we have $\int_{t_0}^{t_\theta} T_\kappa(t_\theta - s)By(s)\,\mathrm{d}s \in D(A)$ for all $t \geq t_0$ by [8, Cor. 3.1.17]. The second term in (2.3) therefore satisfies

$$\left\|T_\kappa(t - t_\theta)\int_{t_0}^{t_\theta} T_\kappa(t_\theta - s)By(s)\,\mathrm{d}s\right\| \leq \|T_\kappa(t - t_\theta)A_\kappa^{-1}\|\left\|A_\kappa\int_{t_0}^{t_\theta} T_\kappa(t_\theta - s)By(s)\,\mathrm{d}s\right\|$$

for all $t \geq t_\theta$. Now $\|T_\kappa(t - t_\theta)A_\kappa^{-1}\| = O(t^{-\theta/\alpha})$ as $t \to \infty$. On the other hand, we have

$$\int_{t_0}^{t_\theta} T_\kappa(t - s)By(s)\,\mathrm{d}s = x(t_\theta) - T_\kappa(t_\theta - t_0)x(t_0), \qquad t \geq t_0,$$

by the variation of parameters formula. Thus

$$\begin{aligned}\left\|A_\kappa\int_{t_0}^{t_\theta} T_\kappa(t_\theta - s)By(s)\,\mathrm{d}s\right\| &\leq \|A_\kappa x(t_\theta)\| + \|A_\kappa T_\kappa(t_\theta - t_0)x(t_0)\| \\ &\leq \|Ax(t_\theta) - B\phi(B^*x(t_\theta))\| + \|By(t_\theta)\| + \|T_\kappa(t_\theta - t_0)A_\kappa x(t_0)\| \\ &\lesssim \|\dot{x}(t_0)\| + \|x(t_0)\|^\gamma + \|A_\kappa x(t_0)\|\end{aligned}$$

for all $t \geq t_0$, where we have used contractivity of $(T_\kappa(t))_{t\geq 0}$ and the fact that both $\|x(\cdot)\|$ and $\|\dot{x}(\cdot)\|$ are non-increasing functions. Thus the second term in (2.3) satisfies

$$\left\|T_\kappa(t - t_\theta)\int_{t_0}^{t_\theta} T_\kappa(t_\theta - s)By(s)\,\mathrm{d}s\right\| = O(t^{-\theta/\alpha}), \qquad t \to \infty.$$

The third and final term in (2.3) satisfies

$$\int_{t_\theta}^t T_\kappa(t-s)By(s)\,\mathrm{d}s = \int_0^t T_\kappa(t-s)By_\theta(s)\,\mathrm{d}s,$$

where $y_\theta(t)=0$ for $t\in[0,t_\theta)$ and $y_\theta(t)=y(t)$ for $t\geq t_\theta$. By [27, Lem. 2.2.6] the input maps $\Phi_t\in\mathcal{B}(L^2(0,t;U),X)$ of the system $(A-\kappa BB^*,B,B^*)$, defined by $\Phi_t u=\int_0^t T_\kappa(t-s)Bu(s)\,\mathrm{d}s$ for $t\geq 0$ and $u\in L^2(0,t;U)$, have uniformly bounded operator norms; see also [16, Thm. 6.5.6], [31, Cor. 6.1]. Since $\|y(t)\|\lesssim\|x(t)\|^\gamma$ for all $t\geq t_0$, and since $\|x(t)\|=O(t^{-1/(2\beta)})$ as $t\to\infty$ by the first part of the result, we deduce that

$$\left\|\int_{t_\theta}^t T_\kappa(t-s)By(s)\,\mathrm{d}s\right\|^2\lesssim\int_{t_\theta}^t\|y(s)\|^2\,\mathrm{d}s\lesssim\int_{t_\theta}^t\frac{\mathrm{d}s}{(1+s)^{\gamma/\beta}}\leq\frac{t-t_\theta}{(1+t_\theta)^{\gamma/\beta}}$$

for all $t\geq t_0$. It follows that the three terms in (2.3) are of order $O(t^{-1/\alpha})$, $O(t^{-\theta/\alpha})$ and $O(t^{-\mu})$, respectively, as $t\to\infty$, where $\mu=\gamma/(2\beta)-\theta/2$. If $\beta\leq\alpha\gamma/(\alpha+2)$ we set $\theta=1$. Then $\mu\geq 1/\alpha$, so the third term decays at least as fast as the first two, and we obtain $\|x(t)\|=O(t^{-1/\alpha})$ as $t\to\infty$, which is what we wanted to prove. On the other hand, if $\beta>\alpha\gamma/(\alpha+2)$ we set $\theta=\alpha\gamma/(\beta(\alpha+2))$. Then $\theta\in(0,1)$ and the rate is determined by the second and third terms, which decay at the same speed, giving $\|x(t)\|=O(t^{-1/(2\sigma\beta)})$ as $t\to\infty$, where $\sigma=(\alpha+2)/(2\gamma)$. The latter estimate is strictly worse than the one we wish to prove. On the other hand, since $\gamma>\alpha/2+1$ by assumption, we have $\sigma\in(0,1)$ and hence this decay rate is strictly *better* than the estimate $\|x(t)\|=O(t^{-1/(2\beta)})$ as $t\to\infty$ coming from the first part of the result. In this second case, we may therefore repeat the argument with β replaced by $\sigma\beta$. As above, we find that if $\sigma\beta\leq\alpha\gamma/(\alpha+2)$ then $\|x(t)\|=O(t^{-1/\alpha})$ as $t\to\infty$ and the proof is complete, while if $\sigma\beta>\alpha\gamma/(\alpha+2)$ then $\|x(t)\|=O(t^{-1/(2\sigma^2\beta)})$ as $t\to\infty$. If necessary we may now iterate this process, terminating after $k\geq 0$ repetitions if $\sigma^k\beta\leq\alpha\gamma/(\alpha+2)$, giving the desired the rate $\|x(t)\|=O(t^{-1/\alpha})$ as $t\to\infty$, and otherwise replacing $\sigma^k\beta$ by $\sigma^{k+1}\beta$. Since $\sigma^k\beta\leq\alpha\gamma/(\alpha+2)$ for all sufficiently large $k\geq 0$ by virtue of the fact that $\sigma\in(0,1)$, the process must eventually terminate yielding $\|x(t)\|=O(t^{-1/\alpha})$ as $t\to\infty$, as required. □

Remarks 2.2. (a) If (2.2) holds then necessarily $\phi(0)=0$, so this condition can be omitted whenever we assume (2.2). It moreover follows from (2.2) that ϕ is differentiable at zero, with derivative $D\phi(0)=\kappa I$, and that $\mathrm{Re}\langle\phi(u),u\rangle\gtrsim\|u\|^2$ for all $u\in U$ of sufficiently small norm. The latter implies that whenever we are assuming (2.2) we need only verify the second part of (1.2).

(b) The class of functions $\phi\colon U\to U$ to which the Theorem 2.1 can be applied includes *radial* functions defined by $\phi(0)=0$ and

$$\phi(u)=\psi(\|u\|)\frac{u}{\|u\|},\qquad u\in U\setminus\{0\},$$

where $\psi:\mathbb{R}_+\to\mathbb{R}_+$ is a non-decreasing and locally Lipschitz continuous function satisfying $\psi(0)=0$. Then ϕ is monotone and locally Lipschitz continuous, and (1.2) holds provided there exists $\delta>0$ such that $\psi(r)\gtrsim r$ for $0\leq r\leq\delta$. Furthermore, if ψ is twice continuously differentiable with $\psi'(0) > 0$ a simple estimate using Taylor's theorem shows that (2.2) holds with $\kappa = \psi'(0)$ for some $\varepsilon > 0$ and some $\gamma \geq 2$. A concrete example is the function ψ defined by $\psi(r) = \tanh(r)$ for $r \geq 0$. In this case, there exists $\varepsilon > 0$ such that (2.2) holds for $\kappa = 1$ and $\gamma = 3$.

(c) If in addition to the assumptions made in the first part of Theorem 2.1 we assume that ϕ is *globally* Lipschitz continuous and satisfies $\mathrm{Re}\langle\phi(u), u\rangle \gtrsim \|u\|^2$ for *all* $u \in U$, then a slight adaptation of the proof yields *uniform* polynomial decay of the form

$$\|x(t)\| \lesssim \frac{\|x(0)\|_{D(A)}}{(1+t)^{1/(2\beta)}}, \qquad t \geq 0,$$

for all classical solutions of (1.1), where $\|\cdot\|_{D(A)}$ denotes the graph norm of A. This conclusion is analogous to earlier results showing that exponential stability may be preserved under global growth conditions on the non-linear damping; see for instance [23].

As a simple consequence of Theorem 2.1 we obtain the following improvement of [11, Thm. 4.4] and [28, Thm. 4.2] in the linear case. Both of these earlier results include the assumption that $\beta \in (0, 1]$, and in [11, Thm. 4.4] the further assumption that $D(A) = D(A^*)$ is added. Our present approach requires neither. As before, we assume that A is the generator of a contraction semigroup $(T(t))_{t\geq 0}$ on a Hilbert space X, and that $B \in \mathcal{B}(U, X)$ for some other Hilbert space U. Moreover, we let $A_B = A - BB^*$ with domain $D(A_B) = D(A)$, and we let $(T_B(t))_{t\geq 0}$ denote the contraction semigroup on X generated by A_B.

Corollary 2.3. *Suppose that there exist $\beta, \tau, c_\tau > 0$ such that*

$$c_\tau\|(I - A)^{-\beta}x_0\|^2 \leq \int_0^\tau \|B^*T(t)x_0\|^2 \,\mathrm{d}t, \qquad x_0 \in X.$$

Then $i\mathbb{R} \subseteq \rho(A_B)$, $\|T_B(t)x\| \to 0$ as $t \to \infty$ for all $x \in X$, and $\|T_B(t)x\| = o(t^{-1/(2\beta)})$ as $t \to \infty$ for all $x \in D(A)$.

Proof. Applying Theorem 2.1 with $\phi\colon U \to U$ taken to be the identity map we obtain $i\mathbb{R} \subseteq \rho(A_B)$ and $\|T_B(t)x_0\| = O(t^{-1/(2\beta)})$ as $t \to \infty$ for all $x_0 \in D(A)$. Since the semigroup $(T_B(t))_{t\geq 0}$ is contractive and $D(A)$ is dense in X a simple approximation argument shows that $\|T_B(t)x_0\| \to 0$ as $t \to \infty$ for all $x_0 \in X$. Moreover, an application of the uniform boundedness theorem yields $\|T_B(t)A_B^{-1}\| = O(t^{-1/(2\beta)})$ as $t \to \infty$, and hence $\|T_B(t)x_0\| = o(t^{-1/(2\beta)})$ as $t \to \infty$ for all $x_0 \in D(A)$ by [10, Thm. 2.4]. □

We conclude this section by showing that resolvent estimates for A along the imaginary axis provide a sufficient condition for the observability estimate (2.1) to hold, at least in the special case when A is skew-adjoint and has uniform spectral gap.

Proposition 2.4. *Suppose that A is skew-adjoint and that $\sigma(A)$ consists of simple and uniformly separated eigenvalues. Suppose further that $i\mathbb{R} \subseteq \rho(A_B)$ and that $\|(is - A_B)^{-1}\| \lesssim 1 + |s|^\alpha$ for some $\alpha > 0$ and all $s \in \mathbb{R}$. Then there exist $\tau, c_\tau > 0$ such that (2.1) holds with $\beta = \alpha/2$.*

Proof. Let us denote the eigenvalues of A by is_k, $k \geq 1$, where $s_k \in \mathbb{R}$ for all k. By assumption there exists $\delta > 0$ such that $|s_k - s_j| \geq \delta$ for all $j, k \geq 1$ with $j \neq k$. Let $\beta = \alpha/2$. By [11, Prop. 5.1] we have

$$\|B^* x\| \gtrsim \frac{\|x\|}{1 + |s_k|^\beta} \tag{2.4}$$

for all $k \geq 1$ and $x \in \operatorname{Ker}(is_k - A)$. For $k \geq 1$, let e_k be a normalised eigenvector corresponding to the eigenvalue is_k. Then $\{e_k : k \geq 1\}$ is an orthonormal basis for X, and it follows from (2.4) that $(1+|s_k|^2)^\beta \|B^* e_k\|^2 \gtrsim 1$ for all $k \geq 1$. Let $\tau > 2\pi/\delta$. It follows from Ingham's inequality [6, Prop. 1.5.2] that

$$\begin{aligned}
\int_0^\tau \|B^* T(t) x_0\|^2 \,\mathrm{d}t &= \int_0^\tau \left\| \sum_{k=1}^\infty e^{is_k t} \langle x_0, e_k \rangle B^* e_k \right\|^2 \mathrm{d}t \\
&\gtrsim \sum_{k=1}^\infty |\langle x_0, e_k \rangle|^2 \|B^* e_k\|^2 \\
&= \sum_{k=1}^\infty |\langle (I - A)^{-\beta} x_0, e_k \rangle|^2 (1 + |s_k|^2)^\beta \|B^* e_k\|^2 \\
&\gtrsim \sum_{k=1}^\infty |\langle (I - A)^{-\beta} x_0, e_k \rangle|^2 = \|(I - A)^{-\beta} x_0\|^2
\end{aligned}$$

for all $x_0 \in X$, and the proof is complete. ☐

Remarks 2.5. (a) The estimate in (2.4) is sometimes referred to as a *wavepacket condition*. The above proof shows that, under a uniform spectral gap condition, we may pass from a resolvent estimate for A_B to a wavepacket condition and thence to an observability estimate. In [11, Sect. 3] it is shown, conversely, that even without the spectral gap condition general wavepacket conditions of this kind can be used to obtain resolvent estimates for A_B, which in turn imply decay rates for classical orbits of the damped semigroup $(T_B(t))_{t \geq 0}$.

(b) By applying more sophisticated versions of Ingham's inequality such as [6, Cor. 1.5.4] we may weaken the uniform gap condition to allow for a degree of repetition and clustering in the eigenvalues of A.

Combining Proposition 2.4 with Theorem 2.1 gives the following result.

Corollary 2.6. *Suppose that A is skew-adjoint, $\sigma(A)$ consists of simple and uniformly separated eigenvalues, and that $i\mathbb{R} \subseteq \rho(A_B)$ and $\|(is - A_B)^{-1}\| \lesssim 1 + |s|^\alpha$ for some $\alpha > 0$ and all $s \in \mathbb{R}$. Suppose further that $\phi : U \to U$ is monotone and locally Lipschitz continuous and that (1.2) holds for some $\delta > 0$. Then all generalised solutions of (1.1) satisfy $\|x(t)\| \to 0$ as $t \to \infty$, and all classical solutions of (1.1) satisfy $\|x(t)\| = O(t^{-1/\alpha})$ as $t \to \infty$.*

3. Applications to non-linearly damped evolution equations

In this section we illustrate the strength of our main result in two concrete applications, namely a one-dimensional wave equation with weak non-linear damping and an Euler–Bernoulli beam with a tip mass subject to non-linear damping. In both cases the eigenvalues are simple and uniformly separated, so we may apply Corollary 2.6. We emphasise, however, that our main result, Theorem 2.1, is applicable much more generally provided one is able to obtain a non-uniform observability estimate as in (2.1). For an interesting potential application in a setting where there is no uniform spectral gap we refer the reader to the system studied in [32], which models the dynamics of small-amplitude water waves.

3.1. **The wave equation with weak non-linear damping.** Consider the wave equation on the unit interval subject to weak non-linear damping, namely

$$u_{tt}(x,t) = u_{xx}(x,t) + b(x)\phi\left(\int_0^1 b(s)u_t(s,t)\,\mathrm{d}s\right), \qquad x \in (0,1),\ t > 0,$$

to be solved subject to the boundary conditions $u(0,t) = u(1,t) = 0$ for all $t > 0$ and the initial conditions $u(\cdot,0) \in H_0^1(0,1)$, $u_t(\cdot,0) \in L^2(0,1)$. Here the function $b \in L^2(0,1;\mathbb{R})$ models the presence of weak (distributed) damping and the function $\phi : \mathbb{C} \to \mathbb{C}$ is potentially non-linear. We may formulate the problem in the form of (1.1) for the state variable $x(t) = (u(\cdot,t), u_t(\cdot,t))$, $t \geq 0$, by setting $X = H_0^1(0,1) \times L^2(0,1)$, $U = \mathbb{C}$, choosing A to be the operator defined by $A(u,v) = (v,u'')$ for all (u,v) in the domain $D(A) = (H^2(0,1) \cap H_0^1(0,1)) \times H_0^1(0,1)$ and defining $B \in \mathcal{B}(U,X)$ by $Bz = (0, b(\cdot)z)$ for all $z \in \mathbb{C}$. We denote the (rescaled) Fourier sine series coefficients of b by

$$b_n = \int_0^1 b(x)\sin(n\pi x)\,\mathrm{d}x, \qquad n \geq 1.$$

Proposition 3.1. *Consider the system (1.1) for the weakly damped wave equation as above, and suppose there exists $\beta > 0$ such that $|b_n| \gtrsim n^{-\beta}$ for all $n \geq 1$. Suppose further that $\phi : U \to U$ is monotone and locally Lipschitz continuous, and that (1.2) holds for some $\delta > 0$. Then all generalised solutions of (1.1) satisfy $\|x(t)\| \to 0$ as $t \to \infty$, and all classical solutions satisfy $\|x(t)\| = O(t^{-1/(2\beta)})$ as $t \to \infty$.*

Proof. The operator A is skew-adjoint, and its spectrum consists of the simple eigenvalues $i\pi n$ for $n \in \mathbb{Z}\setminus\{0\}$, which are uniformly separated. As in the proof of [11, Cor. 6.3], our assumption on the decay of the Fourier sine series coefficients of b implies that $i\mathbb{R} \subseteq \rho(A_B)$ and $\|(is - A_B)^{-1}\| \lesssim 1 + |s|^{2\beta}$ for all $s \in \mathbb{R}$. The result now follows from Corollary 2.6. □

Remarks 3.2. (a) Note that in the particular case where ϕ is the identity map on $\mathbb{C}$, so that (1.2) holds for $\delta = 1$, Proposition 3.1 is sharp in the sense that if $\limsup_{n\to\infty} n^\beta |b_n| > 0$ then for any function $r : \mathbb{R}_+ \to (0,\infty)$ such that $r(t) = o(t^{-1/(2\beta)})$ as $t \to \infty$ there exists a classical solution such that $\|x(t)\| \neq O(r(t))$ as $t \to \infty$. Indeed, if this is not the case then an application of the uniform boundedness principle gives $\|T_B(t)A_B^{-1}\| = O(r(t))$ as $t \to \infty$. On the other hand, since $\limsup_{n\to\infty} n^\beta |b_n| > 0$ it follows from [11, Prop. 5.1] that

$$\limsup_{|s|\to\infty} \frac{\|(is - A_B)^{-1}\|}{|s|^{2\beta}} > 0.$$

Now [11, Prop. 5.3] implies that $\limsup_{t\to\infty} t^{1/(2\beta)}\|T_B(t)A_B^{-1}\| > 0$, yielding the required contradiction.

(b) Note that since $b \in L^2(0,1;\mathbb{R})$ we must have $(b_n)_{n\geq 1} \in \ell^2$, so necessarily $\beta > 1/2$ in Proposition 3.1. On the other hand, every $\beta > 1/2$ can be achieved, for instance by considering the function b whose Fourier sine series coefficients are $b_n = n^{-\beta}$ for $n \geq 1$. We refer the interested reader to [11, Rem. 6.4] for a discussion on the possibility of achieving these decay rates by means of functions b that possess additional regularity.

3.2. **The SCOLE model with non-linear damping.** In this section we analyse the stability of the SCOLE model [20, 33], which consists of an Euler–Bernoulli beam equation coupled with an ODE modelling the dynamics of a *tip mass*. The system has the form

$$\begin{cases} \rho(x)u_{tt}(x,t) = -(EI(x)u_{xx}(x,t))_{xx}, & x \in (0,1),\ t > 0,\\ mu_{tt}(1,t) - (EIu_{xx})_x(1,t) = -\phi_1(u_t(1,t),u_{xt}(1,t)), & t > 0,\\ Ju_{xtt}(1,t) + EI(1)u_{xx}(1,t) = -\phi_2(u_t(1,t),u_{xt}(1,t)), & t > 0, \end{cases}$$

to be solved subject to the boundary conditions $u(0,t) = u_x(1,t) = 0$ for all $t > 0$ and the initial conditions $u(\cdot,0) \in H_L^2(0,1)$, $u_t(\cdot,0) \in L^2(0,1)$, where $H_L^2(0,1) = \{u \in H^2(0,1) : u(0) = u'(0) = 0\}$. In this model, $w(x,t)$ is the deflection of the beam at $x \in [0,1]$ and time $t \geq 0$, $EI \in C^4([0,1])$ and $\rho \in C^4([0,1])$ are the (uniformly positive) flexural rigidity and mass density of the beam, respectively, and $m, J > 0$ are, respectively, the mass and moment of inertia of the tip mass. Finally, the two functions $\phi_1, \phi_2 \colon \mathbb{C}^2 \to \mathbb{C}$ describe the non-linear effects of the boundary condition at $x = 1$. We assume that there exist $a, b > 0$ and $\zeta \in C^2([0,1])$ such that $\zeta(0) = 0$ and

$$\begin{aligned} 2(1-a)\rho(x) - (\rho(x)\zeta(x))' &< -b,\\ EI(x)(1-a-2\zeta'(x)) + \frac{1}{2}(EI(x)\zeta(x))' &< -b \end{aligned}$$

for all $x \in [0,1]$. Note that these conditions are in particular satisfied if EI and ρ are constant functions, in which case we may take $\zeta(x) = 2x$ for $0 \leq x \leq 1$.

We may formulate the problem in the form of (1.1) for the state variable $x(t) = (u(\cdot,t), u_t(\cdot,t), u_t(1,t), u_{xt}(1,t))$, $t \geq 0$, by setting $X = H_L^2(0,1) \times L^2(0,1) \times \mathbb{C}^2$, $U = \mathbb{C}^2$, defining the operator A by

$$A(u,v,\lambda,\mu) = \big(v, -\rho^{-1}(EIu'')'', m^{-1}(EIu'')'(1), -J^{-1}EI(1)u''(1)\big)$$

for all (u,v,λ,μ) in the domain

$$D(A) = \big\{(u,v,\lambda,\mu) \in H^4(0,1) \times H_L^2(0,1) \times \mathbb{C}^2 : v(1) = \lambda,\ v'(1) = \mu\big\},$$

the operator $B \in \mathcal{B}(U,X)$ by $B(\lambda,\mu) = (0,0,\lambda,\mu)$ for all $(\lambda,\mu) \in U$, and the map $\phi\colon U \to U$ by $\phi(\lambda,\mu) = (m^{-1}\phi_1(\lambda,\mu), J^{-1}\phi_2(\lambda,\mu))$ for all $(\lambda,\mu) \in U$. Note that ϕ is locally Lipschitz continuous if and only if both ϕ_1 and ϕ_2 are locally Lipschitz continuous. The function ϕ is monotone for instance if ϕ_1 is independent of the second variable, ϕ_2 is independent of the first variable and both of the maps $\lambda \mapsto \phi_1(\lambda,0)$ and $\mu \mapsto \phi_2(0,\mu)$ are monotone. We obtain the following polynomial stability result.

Proposition 3.3. *Consider the system* (1.1) *for the SCOLE model with non-linear damping as above. Suppose that* $\phi\colon U \to U$ *is monotone and locally Lipschitz continuous, and that* (1.2) *holds for some* $\delta > 0$. *Then all generalised solutions of* (1.1) *satisfy* $\|x(t)\| \to 0$ *as* $t \to \infty$, *and all classical solutions satisfy* $\|x(t)\| = O(t^{-1/2})$ *as* $t \to \infty$.

Proof. By [20, Prop. 1.1] the operator A is skew-adjoint, has compact resolvent, and its eigenvalues are simple. In addition, it follows from [20, Prop. 1.2] that the eigenvalues of A are uniformly separated. Finally, $i\mathbb{R} \subseteq \rho(A_B)$ and $\|(is - A_B)^{-1}\| \lesssim 1 + s^2$ for $s \in \mathbb{R}$ by [18, Thm. 3.1] and [9, Prop. 1.3], so result follows from Corollary 2.6. □

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