On Robustness of Strongly Stable Semigroups with Spectrum on $i\mathbb{R}$

Lassi Paunonen

Abstract We study the robustness properties of strong stability of a strongly continuous semigroup on a Hilbert space. We concentrate on a situation where the generator of the unperturbed semigroup has a finite spectral point on the imaginary axis and the resolvent operator is polynomially bounded elsewhere on the imaginary axis. As our main result we present conditions for preservation of the strong stability of the semigroup under bounded perturbations.

1 Introduction

It is well known that exponential stability of a strongly continuous semigroup $T(t)$ is preserved under all sufficiently small perturbations of its infinitesimal generator A. However, robustness properties of nonexponential stability types are considerably less well-known. In this paper we are interested in strongly stable semigroups, i.e., those satisfying

$$\lim_{t\to\infty} \|T(t)x\| = 0, \qquad \forall x \in X.$$

Unlike exponential stablility, strong stability of a semigroup is in general very sensitive to perturbations, and it may be destroyed even by arbitrarily small perturbations. Recently in [10, 11, 12] conditions for preservation of strong stability were presented for semigroups on Hilbert spaces under suitable assumptions on the behaviour of the resolvent operator of A on the imaginary axis. The purpose of this paper is to extend the perturbation results in [10, 11, 12] to a larger class of strongly stable semigroups. The results have applications in the study of asymp-

Lassi Paunonen
Department of Mathematics, Tampere University of Technology, PO. Box 553, 33101 Tampere, Finland, e-mail: lassi.paunonen@tut.fi

totic behaviour of linear partial differential equations, and in the control of infinite-dimensional linear systems [13].

The references [10, 11] considered a subclass of strongly stable semigroups called the *polynomially stable* semigroups. On a Hilbert space X, the polynomial stability of $T(t)$ is characterized by the property that the resolvent operator $R(\lambda,A)$ exists and is polynomially bounded on the imaginary axis [4, 5]. The key to studying the robustness properties of polynomial stability in [10, 11] was an observation that since the stability is not exponential, the size of the perturbation $A+BC$ should not be measured using the regular operator norms $\|B\|$ and $\|C\|$, but instead with the graph norms $\|(-A)^\beta B\|$ and $\|(-A^*)^\gamma C^*\|$ for suitable exponents β and γ.

In [12] similar methods were used to study the preservation of strong stability for semigroups whose generators have spectrum on the imaginary axis. In particular, it was assumed that for the unperturbed generator A the intersection $\sigma(A)\cap i\mathbb{R} = \{i\omega_k\}_{k=1}^N$ is finite and the norm of the resolvent $R(i\omega,A)$ grows at most polynomially near the points $\{i\omega_k\}_{k=1}^N$. It was further assumed that for large $|\omega|$ the norm of $\|R(i\omega,A)\|$ of the resolvent operator is uniformly bounded. Under these assumptions, it was shown that the strong stability of the semigroup is preserved under a finite rank perturbation $A+BC$ provided that the graph norms

$$\|B\| + \|(i\omega_k - A)^{-\beta_0}B\|, \qquad \text{and} \qquad \|C\| + \|(-i\omega_k - A^*)^{-\gamma_0}C^*\|$$

for suitable exponents $\beta_0,\gamma_0 \geq 0$ are sufficiently small for every k [12, Sec. 2].

In this paper we study a situation that results from combining the assumptions in [12] with those in [10, 11]. In particular, we assume A has spectrum on $i\mathbb{R}$, and for large $|\omega|$ the norm $\|R(i\omega,A)\|$ of the resolvent operator is polynomially bounded. The main result in this paper generalizes the conditions for preservation of strong stability given in [12] by not requiring that the norms $\|R(i\omega,A)\|$ are uniformly bounded for large $|\omega|$. For simplicity, we concentrate on a situation where the operator A has a single spectral point $\sigma(A)\cap i\mathbb{R} = \{0\}$ on the imaginary axis. The standing assumptions on the unperturbed semigroup $T(t)$ are summarized below.

Assumption 1 *Assume A generates a strongly stable semigroup $T(t)$ on a Hilbert space X, $\sigma(A)\cap i\mathbb{R} = \{0\}$, and there exist $\alpha_0,\alpha > 0$, $\omega_0 > 0$, $M_A \geq 1$ such that*

$$\begin{cases} \|R(i\omega,A)\| \leq M_A|\omega|^{-\alpha_0} & 0 < |\omega| \leq 1 \\ \|R(i\omega,A)\| \leq M_A|\omega|^{\alpha} & |\omega| \geq \omega_0. \end{cases}$$

Because $0\in\sigma(A)$, we necessarily have $\alpha_0 \geq 1$ in Assumption 1. Moreover, since the semigroup $T(t)$ is uniformly bounded, the Mean Ergodic Theorem [2, Sec. 4.3] implies that $0 \in \sigma_p(A)\cup\sigma_c(A)$. However, since $0\in\sigma_p(A)$ would contradict the strong stability of $T(t)$, we must have $0\in\sigma_c(A)$.

Semigroups satisfying Assumption 1 were studied recently in [3], where it was shown that the conditions on the growth of the resolvent on $i\mathbb{R}$ are closely related to the nonuniform decay rates of the semigroup $T(t)$. In particular, in [3, Thm. 8.4] it was shown that Assumption 1 is satisfied, then there exists a constant $M \geq 1$ such that

$$\|T(t)(-A)^{\alpha_0}(1-A)^{-(\alpha_0+\alpha)}x\| \le \frac{M}{t}\|x\| \qquad \forall x \in X,\ t>0.$$

The result also has a converse counterpart, see [3, Thm. 8.4] for details.

The main result of this paper introduces conditions for preservation of the stability of $T(t)$ under perturbations of the form $A+BC$ where $B \in \mathscr{L}(Y,X)$, and $C \in \mathscr{L}(X,Y)$ for a separable Hilbert space Y. Since A is injective and $\mathscr{R}(A)$ is dense, the operator $-A$ has a densely defined inverse $(-A)^{-1}$. The operators $-A$ and $-A^*$ sectorial in the sense of [8], and their fractional powers $(-A)^{\beta}$ and $(-A^*)^{\gamma}$ are well-defined for all $\beta,\gamma \in \mathbb{R}$. We also recall that if $(e_k)_{k=1}^{\infty}$ is an orthonormal basis of Y, then $B \in \mathscr{L}(Y,X)$ is said to be a *Hilbert–Schmidt operator* if $(Be_k)_{k=1}^{\infty} \in \ell^2(X)$. We consider perturbations whose components B and C satisfy

$$\mathscr{R}(B) \subset \mathscr{R}((-A)^{\beta_0}) \cap \mathscr{D}((-A)^{\beta}), \quad \mathscr{R}(C^*) \subset \mathscr{R}((-A^*)^{\gamma_0}) \cap \mathscr{D}((-A^*)^{\gamma}) \tag{1}$$

for some $\beta_0,\beta,\gamma_0,\gamma \ge 0$, and for which

$$(-A)^{-\beta_0}B,\ (-A)^{\beta}B,\ (-A^*)^{-\gamma_0}C^*,\ \text{and}\ (-A^*)^{\gamma}C^* \quad \text{are Hilbert–Schmidt.} \tag{2}$$

If Y is finite-dimensional, i.e., if the perturbing operator BC is of finite rank, then the condition (2) follows immediately from (1). The following theorem is the main result of this paper. The proof of Theorem 2 is presented in Section 2.

Theorem 2. *Let Assumption* 1 *be satisfied and let* $\beta_0,\beta,\gamma_0,\gamma \ge 0$ *be such that* $\alpha_0 = \beta_0+\gamma_0$ *and* $\alpha = \beta+\gamma$. *There exists* $\delta>0$ *such that if* $B \in \mathscr{L}(Y,X)$ *and* $C \in \mathscr{L}(X,Y)$ *satisfy* (1) *and* (2) *and*

$$\|B\| + \|(-A)^{-\beta_0}B\| + \|(-A)^{\beta}B\| < \delta, \quad \|C\| + \|(-A^*)^{-\gamma_0}C^*\| + \|(-A^*)^{\gamma}C^*\| < \delta,$$

then $\sigma(A+BC) \subset \mathbb{C}^- \cup \{0\}$ *and* $0 \in \sigma_c(A+BC)$. *Moreover, the semigroup generated by* $A+BC$ *is strongly stable and* $A+BC$ *satisfies the resolvent growth conditions in Assumption* 1.

It should also be noted that if the exponents satisfy $\beta_0,\gamma_0 \ge \alpha_0$ and $\beta,\gamma \ge \alpha$, then the stability of the semigroup is preserved even if the perturbation does not satisfy the condition (2). Indeed, the uniform boundedness of the perturbed semigroup can then be proved similarly as in [10, Proof of Thm. 5].

In this paper we also consider the robustness of stability of $T(t)$ under perturbations $A+B$ where $B \in \mathscr{L}(X)$ commutes with A. In this situation the analysis for preservation of stability becomes particularly simple. The proof of Theorem 3 is presented in Section 3.

Theorem 3. *Let Assumption* 1 *be satisfied. There exists* $\delta>0$ *such that if* $B \in \mathscr{L}(X)$ *commutes with* A *and satisfies* $\mathscr{R}(B) \subset \mathscr{R}((-A)^{\alpha_0}) \cap \mathscr{D}((-A)^{\alpha})$ *and*

$$\|B\| + \|(-A)^{-\alpha_0}B\| + \|(-A)^{\alpha}B\| < \delta,$$

then $\sigma(A+B) \subset \mathbb{C}^- \cup \{0\}$ *and* $0 \in \sigma_c(A+B)$. *Moreover, the semigroup generated by* $A+B$ *is strongly stable and* $A+B$ *satisfies the resolvent growth conditions in Assumption* 1.

If X and Y are Banach spaces and $A : X \to Y$ is a linear operator, we denote by $\mathscr{D}(A)$, $\mathscr{R}(A)$, and $\mathscr{N}(A)$ the domain, the range, and the kernel of A, respectively. The space of bounded linear operators from X to Y is denoted by $\mathscr{L}(X,Y)$. If $A : \mathscr{D}(A) \subset X \to X$, then $\sigma(A)$, $\sigma_p(A)$, $\sigma_c(A)$ and $\rho(A)$ denote the spectrum, the point spectrum, the continuous spectrum and the resolvent set of A, respectively. For $\lambda \in \rho(A)$ the resolvent operator is given by $R(\lambda,A) = (\lambda - A)^{-1}$. The inner product on a Hilbert space is denoted by $\langle \cdot, \cdot \rangle$.

2 Robustness of Stability with Respect to Perturbations $A+BC$

In this section we present the proof of Theorem 2. In the first part, we study the change of the spectrum of A under the perturbation $A+BC$. Subsequently, the preservation of stability is completed by showing that the perturbed semigroup is uniformly bounded.

2.1 The Change of the Spectrum of A

The following result concerns the change of the spectrum of A under perturbations satisfying the assumptions of Theorem 2. However, Theorem 4 does not require $(-A)^{-\beta_0}B$, $(-A)^{\beta}B$, $(-A^*)^{-\gamma_0}C^*$, and $(-A^*)^{\gamma}C^*$ to be Hilbert–Schmidt operators.

Theorem 4. *Assume* Y *is a Banach space, let Assumption* 1 *be satisfied and let* $\beta_0, \beta, \gamma_0, \gamma \geq 0$ *be such that* $\alpha_0 = \beta_0 + \gamma_0$ *and* $\alpha = \beta + \gamma$. *There exists* $\delta > 0$ *such that if* $B \in \mathscr{L}(Y,X)$ *and* $C \in \mathscr{L}(X,Y)$ *satisfy* $\mathscr{R}(B) \subset \mathscr{R}((-A)^{\beta_0}) \cap \mathscr{D}((-A)^{\beta})$ *and* $\mathscr{R}(C^*) \subset \mathscr{R}((-A^*)^{\gamma_0}) \cap \mathscr{D}((-A^*)^{\gamma})$ *and*

$$\|B\| + \|(-A)^{-\beta_0}B\| + \|(-A)^{\beta}B\| < \delta, \quad \|C\| + \|(-A^*)^{-\gamma_0}C^*\| + \|(-A^*)^{\gamma}C^*\| < \delta,$$

then $\sigma(A+BC) \subset \mathbb{C}^- \cup \{0\}$ *and* $0 \notin \sigma_p(A+BC)$. *In particular, under the above conditions we have* $\sup_{\lambda \in \overline{\mathbb{C}^+} \setminus \{0\}} \|(I - CR(\lambda,A)B)^{-1}\| < \infty$.

In the proof of Theorem 4 we use the Sherman–Morrison–Woodbury formula given in the following lemma.

Lemma 1. *Let* $\lambda \in \rho(A)$, $B \in \mathscr{L}(Y,X)$, $C \in \mathscr{L}(X,Y)$. *If* $1 \in \rho(CR(\lambda,A)B)$, *then* $\lambda \in \rho(A+BC)$ *and*

$$R(\lambda, A+BC) = R(\lambda,A) + R(\lambda,A)B(I - CR(\lambda,A)B)^{-1}CR(\lambda,A).$$

Throughout the paper we use the operators

$$\Lambda_0 = (-A)(1-A)^{-1} \in \mathscr{L}(X) \qquad \text{and} \qquad \Lambda_\infty = (1-A)^{-1} \in \mathscr{L}(X).$$

Both Λ_0 and Λ_∞ are sectorial, and for $\beta_0, \beta > 0$ we have $\Lambda_0^{\beta_0} = (-A)^{\beta_0}(1-A)^{-\beta_0}$, and $\Lambda_\infty^\beta = (1-A)^{-\beta}$ [8, Prop. 3.1.9]. We also have $\mathscr{R}(\Lambda_0^{\beta_0}) = \mathscr{R}((-A)^{\beta_0})$ and $\mathscr{R}(\Lambda_\infty^\beta) = \mathscr{D}((-A)^\beta)$, and $\Lambda_0^{\beta_0}$ and Λ_∞^β have inverses $\Lambda_0^{-\beta_0} = (1-A)^{\beta_0}(-A)^{-\beta_0}$ and $\Lambda_\infty^{-\beta} = (1-A)^\beta$ with domains $\mathscr{D}(\Lambda_0^{-\beta_0}) = \mathscr{R}((-A)^{\beta_0})$ and $\mathscr{D}(\Lambda_\infty^{-\beta}) = \mathscr{D}((-A)^\beta)$, respectively. We also define

$$\Lambda_{(\alpha_0,\alpha)} = \Lambda_0^{\alpha_0}\Lambda_\infty^\alpha = (-A)^{\alpha_0}(1-A)^{-(\alpha_0+\alpha)} \in \mathscr{L}(X).$$

The operator $\Lambda_{(\alpha_0,\alpha)}$ is injective, and sectorial by [3, Prop. 3.10] and the identity $\left((-A)^{\frac{\alpha_0}{\alpha_0+\alpha}}(1-A)^{-1}\right)^{\alpha_0+\alpha} = (-A)^{\alpha_0}(1-A)^{-(\alpha_0+\alpha)}$. The most important component in the proof of Theorem 4 is the following fundamental property introduced recently in [3].

Theorem 5. *If Assumption* 1 *is satisfied, then*

$$\sup_{\lambda \in \overline{\mathbb{C}^+}\setminus\{0\}} \|R(\lambda,A)\Lambda_{(\alpha_0,\alpha)}\| < \infty.$$

Proof. See the proof of Theorem 8.4 in [3]. □

Lemma 2. *Let Y be a Banach space and let $\beta_0, \beta, \gamma_0, \gamma \geq 0$. There exists $M_\Lambda \geq 1$ such that*

$$\|\Lambda_0^{-\beta_0}\Lambda_\infty^{-\beta}B\| \leq M_\Lambda \left(\|(-A)^{-\beta_0}B\| + \|(-A)^\beta B\|\right)$$
$$\|(\Lambda_0^{-\gamma_0})^*(\Lambda_\infty^{-\gamma})^*C^*\| \leq M_\Lambda \left(\|(-A^*)^{-\gamma_0}C^*\| + \|(-A^*)^\gamma C^*\|\right)$$

whenever $B \in \mathscr{L}(Y,X)$ and $C \in \mathscr{L}(X,Y)$ satisfy $\mathscr{R}(B) \subset \mathscr{R}((-A)^{\beta_0}) \cap \mathscr{D}((-A)^\beta)$ and $\mathscr{R}(C^*) \subset \mathscr{R}((-A^*)^{\gamma_0}) \cap \mathscr{D}((-A^*)^\gamma)$.

Proof. We begin by proving the first estimate. If $\beta_0 = \beta = 0$, the claim is clearly true. Let $\beta_0, \beta \geq 0$ be such that $\beta_0 + \beta > 0$. We have from [8, Prop. 3.1.9] that $\mathscr{D}((-A)^{\beta_0+\beta}) = \mathscr{D}((1-A)^{\beta_0+\beta})$. The operator $(1-A)^{\beta_0+\beta}$ is a closed operator (its inverse is bounded) from the Banach space $X_A = (\mathscr{D}((-A)^{\beta_0+\beta}), \|\cdot\| + \|(-A)^{\beta_0+\beta}\cdot\|)$ to X. Since $(1-A)^{\beta_0+\beta}$ is defined on all of X_A, we have from the Closed Graph Theorem [6, Thm. B.6] that $(1-A)^{\beta_0+\beta} \in \mathscr{L}(X_A, X)$, which implies that there exists $M' \geq 1$ such that

$$\|(1-A)^{\beta_0+\beta}x\| \leq M'\left(\|x\| + \|(-A)^{\beta_0+\beta}x\|\right), \qquad \forall x \in \mathscr{D}((-A)^{\beta_0+\beta}).$$

If $B \in \mathscr{L}(Y,X)$ is such that $\mathscr{R}(B) \subset \mathscr{R}((-A)^{\beta_0}) \cap \mathscr{D}((-A)^\beta)$, then for every $y \in Y$ we have $(-A)^{-\beta_0}By \in \mathscr{D}((-A)^{\beta_0+\beta})$ and

$$\begin{aligned}\|\Lambda_0^{-\beta_0}\Lambda_\infty^{-\beta}By\| &= \|(1-A)^{\beta_0+\beta}(-A)^{-\beta_0}By\|\\ &\le M'\left(\|(-A)^{-\beta_0}By\| + \|(-A)^{\beta_0+\beta}(-A)^{-\beta_0}By\|\right)\\ &= M'\left(\|(-A)^{-\beta_0}By\| + \|(-A)^{\beta}By\|\right)\end{aligned}$$

Since $y\in Y$ was arbitrary, choosing $M_\Lambda = M'$ concludes the proof of the first estimate. Because $(\Lambda_0^{-\gamma_0})^* = (1-A^*)^{\gamma_0}(-A^*)^{-\gamma_0}$ and $(\Lambda_\infty^{-\gamma})^* = (1-A^*)^{\gamma}$, the second estimate can be proved analogously by replacing A with A^*. □

Lemma 3. *Let Assumption* 1 *be satisfied and let* $\beta_0,\gamma_0\ge 0$ *be such that* $\beta_0+\gamma_0=\alpha_0$. *There exists* $\delta'>0$ *such that if* $B\in\mathscr{L}(Y,X)$ *and* $C\in\mathscr{L}(X,Y)$ *satisfy* $\mathscr{R}(B)\subset\mathscr{R}((-A)^{\beta_0})$ *and* $\mathscr{R}(C^*)\subset\mathscr{R}((-A^*)^{\gamma_0})$ *and*

$$\|B\| + \|(-A)^{-\beta_0}B\| < \delta' \quad \textit{and} \quad \|C^*\| + \|(-A^*)^{-\gamma_0}C^*\| < \delta',$$

then $0\in\sigma(A+BC)\setminus\sigma_p(A+BC)$.

Proof. Choose $0\le\beta_1\le\beta_0$ and $0\le\gamma_1\le\gamma_0$ in such a way that $\beta_1+\gamma_1=1$. Assume $\|(-A)^{-\beta_1}B\|<1$ and $\|(-A^*)^{-\gamma_1}C^*\|<1$. Since $0\le\gamma_1\le 1$, we have $\mathscr{R}(-A)\subset\mathscr{R}((-A)^{\gamma_1})\subset X$, which implies $\overline{\mathscr{D}((-A)^{-\gamma_1})}=X$ due to the fact that $0\in\sigma_c(A)$. Because of this, the operator $C(-A)^{-\gamma_1}$ has a unique bounded extension $C_{\gamma_1}\in\mathscr{L}(X,Y)$ with norm $\|C_{\gamma_1}\| = \|(-A^*)^{-\gamma_1}C^*\|<1$.

Because $\|(-A)^{-\beta_1}BC_{\gamma_1}\|\le\|(-A)^{-\beta_1}B\|\|C_{\gamma_1}\|<1$, the operator $I-(-A)^{-\beta_1}BC_{\gamma_1}$ is boundedly invertible, and

$$(A+BC)x = -(-A)^{\beta_1}(I-(-A)^{-\beta_1}BC_{\gamma_1})(-A)^{\gamma_1}x$$

for all $x\in\mathscr{D}(A+BC)=\mathscr{D}(A)$. Since $(-A)^{\beta_1}$ and $(-A)^{\gamma_1}$ are injective and at least one of them is not surjective, the operator $A+BC$ is injective but not surjective. This implies $0\in\sigma(A+BC)\setminus\sigma_p(A+BC)$.

Finally, The Moment Inequality [8, Prop. 6.6.4] implies that there exists $\delta'>0$ such that $\|(-A)^{-\beta_1}B\|<1$ and $\|(-A^*)^{-\gamma_1}C^*\|<1$ are satisfied whenever $\|B\| + \|(-A)^{-\beta_0}B\|<\delta'$ and $\|C\|+\|(-A^*)^{-\gamma_0}C^*\|<\delta'$. □

Proof of Theorem 4. Let $\beta_0,\beta,\gamma_0,\gamma\ge 0$ be such that $\beta_0+\gamma_0=\alpha_0$ and $\beta+\gamma=\alpha$. By Theorem 5 we can define $M_1>0$ by

$$M_1 = \sup_{\lambda\in\overline{\mathbb{C}^+}\setminus\{0\}} \|R(\lambda,A)\Lambda_{(\alpha_0,\alpha)}\| < \infty.$$

Let $0<c<1$, and let $M_\Lambda\ge 1$ be as in Lemma 2. We choose

$$\delta = \min\left\{\frac{\sqrt{c}}{\sqrt{M_1}M_\Lambda}, \delta'\right\} > 0,$$

where $\delta'>0$ is from Lemma 3. Let $B\in\mathscr{L}(Y,X)$ and $C\in\mathscr{L}(X,Y)$ be such that $\mathscr{R}(B)\subset\mathscr{R}((-A)^{\beta_0})\cap\mathscr{D}((-A)^{\beta})$ and $\mathscr{R}(C^*)\subset\mathscr{R}((-A^*)^{\gamma_0})\cap\mathscr{D}((-A^*)^{\gamma})$, and

$$\|B\| + \|(-A)^{-\beta_0}B\| + \|(-A)^{\beta}B\| < \delta$$
$$\|C^*\| + \|(-A^*)^{-\gamma_0}C^*\| + \|(-A^*)^{\gamma}C^*\| < \delta.$$

Let $x,y \in Y$ be such that $\|x\| = \|y\| = 1$. Then $Bx \in \mathscr{R}((-A)^{\beta_0}) \cap \mathscr{D}((-A)^{\beta})$ and $C^*y \in \mathscr{R}((-A^*)^{\gamma_0}) \cap \mathscr{D}((-A^*)^{\gamma})$, and using $\Lambda_0^{\gamma_0}\Lambda_\infty^{\gamma}\Lambda_0^{\beta_0}\Lambda_\infty^{\beta} = \Lambda_0^{\alpha_0}\Lambda_\infty^{\alpha} = \Lambda_{(\alpha_0,\alpha)}$ we get

$$\begin{aligned}
|\langle CR(\lambda,A)Bx,y\rangle| &= |\langle \Lambda_0^{\gamma_0}\Lambda_\infty^{\gamma}R(\lambda,A)\Lambda_0^{\beta_0}\Lambda_\infty^{\beta}\Lambda_0^{-\beta_0}\Lambda_\infty^{-\beta}Bx, (\Lambda_0^{-\gamma_0})^*(\Lambda_\infty^{-\gamma})^*C^*y\rangle| \\
&= |\langle R(\lambda,A)\Lambda_{(\alpha_0,\alpha)}\Lambda_0^{-\beta_0}\Lambda_\infty^{-\beta}Bx, (\Lambda_0^{-\gamma_0})^*(\Lambda_\infty^{-\gamma})^*C^*y\rangle| \\
&\le \|R(\lambda,A)\Lambda_{(\alpha_0,\alpha)}\|\|\Lambda_0^{-\beta_0}\Lambda_\infty^{-\beta}B\|\|x\|\|(\Lambda_0^{-\gamma_0})^*(\Lambda_\infty^{-\gamma})^*C^*\|\|y\| \\
&\le M_1M_\Lambda^2\left(\|(-A)^{-\beta_0}B\| + \|(-A)^{\beta}B\|\right)\left(\|(-A^*)^{-\gamma_0}C^*\| + \|(-A^*)^{\gamma}C^*\|\right) \\
&\le M_1M_\Lambda^2\delta^2 \le c.
\end{aligned}$$

This shows that $\|CR(\lambda,A)B\| = \sup_{\|x\|=\|y\|=1}|\langle CR(\lambda,A)Bx,y\rangle| \le c < 1$ for all $\lambda \in \overline{\mathbb{C}^+}\setminus\{0\}$. The Sherman–Morrison–Woodbury formula in Lemma 1 therefore concludes that $\sigma(A+BC) \subset \mathbb{C}^- \cup \{0\}$. We also have $0 \in \sigma(A+BC)\setminus\sigma_p(A+BC)$ directly from Lemma 3. Finally, a standard Neumann series argument shows that for every $\lambda \in \overline{\mathbb{C}^+}\setminus\{0\}$ we have $\|(I-CR(\lambda,A)B)^{-1}\| \le 1/(1-c)$, which concludes the final claim of the theorem. □

2.2 Preservation of Uniform Boundedness

To show the preservation of strong stability of $T(t)$, we in particular need to show that the perturbed semigroup is uniformly bounded. For this we use the conditions in the following theorem (the proof can be found in [7, Thm. 2]).

Theorem 6. *Let A generate a semigroup $T(t)$ on a Hilbert space X and let $\sigma(A) \subset \overline{\mathbb{C}^-}$. The semigroup $T(t)$ is uniformly bounded if and only if for all $x,y \in X$ we have*

$$\sup_{\xi>0}\,\xi\int_{-\infty}^{\infty}\left(\|R(\xi+i\eta,A)x\|^2 + \|R(\xi+i\eta,A)^*y\|^2\right)d\eta < \infty.$$

The following two lemmata are used in the proof of Theorem 2.

Lemma 4. *Assume A generates a uniformly bounded semigroup on a Hilbert space X. If Y is a separable Hilbert space and if $\tilde{B} \in \mathscr{L}(Y,X)$ is a Hilbert–Schmidt operator, then*

$$\sup_{\xi>0}\,\xi\int_{-\infty}^{\infty}\|R(\xi+i\eta,A)\tilde{B}\|^2 d\eta < \infty, \qquad \sup_{\xi>0}\,\xi\int_{-\infty}^{\infty}\|R(\xi+i\eta,A)^*\tilde{B}\|^2 d\eta < \infty.$$

Proof. By [14, Rem. 3.2] there exists $M > 0$ such that

$$\sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi+i\eta,A)x\|^2 d\eta \leq M\|x\|^2, \quad \sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi+i\eta,A)^*x\|^2 d\eta \leq M\|x\|^2$$

for all $x \in X$. If Y is a Hilbert space with an orthonormal basis $(e_k)_{k=1}^\infty \subset Y$ and if $(\tilde{B}e_k)_{k=1}^\infty \in \ell^2(X)$, then

$$\sum_{k=1}^\infty \sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi+i\eta,A)\tilde{B}e_k\|^2 d\eta \leq M \sum_{k=1}^\infty \|\tilde{B}e_k\|^2 < \infty.$$

Moreover, for every $R \in \mathscr{L}(X)$ we have $\|R\tilde{B}\|^2 \leq \sum_{k=1}^\infty \|R\tilde{B}e_k\|^2$. Together these properties imply

$$\sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi+i\eta,A)\tilde{B}\|^2 d\eta \leq \sum_{k=1}^\infty \sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|R(\xi+i\eta,A)\tilde{B}e_k\|^2 d\eta < \infty.$$

The second claim can be shown analogously. □

Lemma 5. *Let Assumption 1 be satisfied, let $\beta_0,\beta,\gamma_0,\gamma \geq 0$ satisfy $\alpha_0 = \beta_0+\gamma_0$ and $\alpha = \beta+\gamma$, and let $B \in \mathscr{L}(Y,X)$ and $C \in \mathscr{L}(X,Y)$ be such that $\mathscr{R}(B) \subset \mathscr{R}((-A)^{\beta_0}) \cap \mathscr{D}((-A)^{\beta})$ and $\mathscr{R}(C^*) \subset \mathscr{R}((-A^*)^{\gamma_0}) \cap \mathscr{D}((-A^*)^{\gamma})$. Then there exist constants $M_0, M_\infty \geq 1$ (depending on B and C) such that*

$$\begin{aligned}\|R(\lambda,A)B\|\|CR(\lambda,A)\| \leq M_0 &\|R(\lambda,A)\Lambda_0^{-\beta_0}\Lambda_\infty^{-\beta}B\|^{1-\beta_0/\alpha_0} \\ &\times \|R(\lambda,A)^*(\Lambda_0^{-\gamma_0})^*(\Lambda_\infty^{-\gamma})^*C^*\|^{1-\gamma_0/\alpha_0}\end{aligned}$$

for $\lambda \in \overline{\mathbb{C}^+}\setminus\{0\}$ with $|\mathrm{Im}\,\lambda| \leq 1$, and

$$\begin{aligned}\|R(\lambda,A)B\|\|CR(\lambda,A)\| \leq M_\infty &\|R(\lambda,A)\Lambda_0^{-\beta_0}\Lambda_\infty^{-\beta}B\|^{1-\beta/\alpha} \\ &\times \|R(\lambda,A)^*(\Lambda_0^{-\gamma_0})^*(\Lambda_\infty^{-\gamma})^*C^*\|^{1-\gamma/\alpha}\end{aligned}$$

for $\lambda \in \overline{\mathbb{C}^+}\setminus\{0\}$ with $|\mathrm{Im}\,\lambda| \geq 1$.

Proof. Assume $\beta_0,\beta,\gamma_0,\gamma > 0$. The remaining cases are simpler and can be handled similarly as in [12, Lem. 19]. In the proof of [3, Thm. 8.4] it was shown that

$$\sup_{|\omega|\leq 1} \|R(i\omega,A)\Lambda_0^{\alpha_0}\| < \infty, \qquad \text{and} \qquad \sup_{|\omega|\geq 1} \|R(i\omega,A)\Lambda_\infty^{\alpha}\| < \infty.$$

Since $T(t)$ is uniformly bounded, the Hille–Yosida Theorem shows that there exists $\tilde{M} \geq 1$ such that $|\mathrm{Re}\,\lambda|\|R(\lambda,A)\| \leq \tilde{M}$ for all $\lambda \in \mathbb{C}^+$. Using this and the resolvent identity $R(\lambda,A) = R(i\omega,A) + (i\omega-\lambda)R(\lambda,A)R(i\omega,A)$ we have that for every $\lambda = \xi + i\omega$ with $\xi > 0$ and $|\omega| \geq 1$ we have

$$\begin{aligned}\|R(\lambda,A)\Lambda_\infty^\alpha\| &\leq \|R(i\omega,A)\Lambda_\infty^\alpha\| + |\xi|\|R(\lambda,A)\|\|R(i\omega,A)\Lambda_\infty^\alpha\| \\ &\leq (1+\tilde{M})\|R(i\omega,A)\Lambda_\infty^\alpha\|,\end{aligned}$$

and if $\xi > 0$ and $0 < |\omega| \leq 1$, we have

$$\begin{aligned}\|R(\lambda,A)\Lambda_0^{\alpha_0}\| &\le \|R(i\omega,A)\Lambda_0^{\alpha_0}\| + |\xi|\|R(\lambda,A)\|\|R(i\omega,A)\Lambda_0^{\alpha_0}\|\\ &\le (1+\tilde{M})\|R(i\omega,A)\Lambda_0^{\alpha_0}\|.\end{aligned}$$

Finally, if $\omega = 0$, we can use $\alpha_0 \ge 1$ to estimate

$$\begin{aligned}&\|R(\lambda,A)\Lambda_0^{\alpha_0}\| = \|R(\xi,A)(-A)(1-A)^{-1}\Lambda_0^{\alpha_0-1}\|\\ &\quad\le (1+|\xi|\|R(\xi,A)\|)\|(1-A)^{-1}\|\|\Lambda_0^{\alpha_0-1}\| \le (1+M')\|(1-A)^{-1}\|\|\Lambda_0^{\alpha_0-1}\|.\end{aligned}$$

These estimates conclude that we can define $M_1, M_2 > 0$ by

$$M_1 = \sup_{\substack{\lambda\in\overline{\mathbb{C}^+}\setminus\{0\}\\ |\mathrm{Im}(\lambda)|\le 1}} \|\Lambda_0^{\alpha_0}R(\lambda,A)\| < \infty \qquad \text{and} \qquad M_2 = \sup_{\substack{\lambda\in\overline{\mathbb{C}^+}\setminus\{0\}\\ |\mathrm{Im}(\lambda)|\ge 1}} \|\Lambda_\infty^{\alpha}R(\lambda,A)\| < \infty.$$

Denote $B_{(\beta_0,\beta)} = \Lambda_0^{-\beta_0}\Lambda_\infty^{-\beta}B$ and $\tilde{C}_{(\gamma_0,\gamma)} = (\Lambda_0^{-\gamma_0})^*(\Lambda_\infty^{-\gamma})^*C^*$. For $\lambda \in \overline{\mathbb{C}^+}\setminus\{0\}$ with $|\mathrm{Im}\,\lambda| \le 1$ we can use the Moment Inequality [8, Prop. 6.6.4] to estimate (denoting $R_\lambda = R(\lambda,A)$ for brevity)

$$\begin{aligned}\|R_\lambda B\| &= \|\Lambda_0^{\beta_0}\Lambda_\infty^{\beta}R_\lambda\Lambda_0^{-\beta_0}\Lambda_\infty^{-\beta}B\| \le \|\Lambda_\infty^{\beta}\|\|\Lambda_0^{\beta_0}R_\lambda B_{(\beta_0,\beta)}\|\\ &\le M'\|\Lambda_\infty^{\beta}\|\|R_\lambda B_{(\beta_0,\beta)}\|^{1-\beta_0/\alpha_0}\|\Lambda_0^{\alpha_0}R_\lambda B_{(\beta_0,\beta)}\|^{\beta_0/\alpha_0}\\ &\le M'M_1^{\beta_0/\alpha_0}\|\Lambda_\infty^{\beta}\|\|B_{(\beta_0,\beta)}\|^{\beta_0/\alpha_0}\|R_\lambda B_{(\beta_0,\beta)}\|^{1-\beta_0/\alpha_0}\end{aligned}$$

and using $\|CR_\lambda\| = \|R_\lambda^* C^*\|$ we get

$$\begin{aligned}\|CR_\lambda\| &= \|(\Lambda_0^{\gamma_0})^*(\Lambda_\infty^{\gamma})^*R_\lambda^*(\Lambda_0^{-\gamma_0})^*(\Lambda_\infty^{-\gamma})^*C^*\| \le \|\Lambda_\infty^{\gamma}\|\|(\Lambda_0^*)^{\gamma_0}R_\lambda^*\tilde{C}_{(\gamma_0,\gamma)}\|\\ &\le M''\|\Lambda_\infty^{\gamma}\|\|R_\lambda^*\tilde{C}_{(\gamma_0,\gamma)}\|^{1-\gamma_0/\alpha_0}\|(\Lambda_0^{\alpha_0})^*R_\lambda^*\tilde{C}_{(\gamma_0,\gamma)}\|^{\gamma_0/\alpha_0}\\ &\le M''M_1^{\gamma_0/\alpha_0}\|\Lambda_\infty^{\gamma}\|\|\tilde{C}_{(\gamma_0,\gamma)}\|^{\gamma_0/\alpha_0}\|R_\lambda^*\tilde{C}_{(\gamma_0,\gamma)}\|^{1-\gamma_0/\alpha_0},\end{aligned}$$

where $M', M'' \ge 1$ follow from the Moment Inequality [8, Prop. 6.6.4], and are independent of B, C, and λ. We have $\beta_0 + \gamma_0 = \alpha_0$ by assumption and if we choose

$$M_0 = M'M''M_1\|\Lambda_\infty^{\beta}\|\|B_{(\beta_0,\beta)}\|^{\beta_0/\alpha_0}\|\Lambda_\infty^{\gamma}\|\|\tilde{C}_{(\gamma_0,\gamma)}\|^{\gamma_0/\alpha_0},$$

then the first estimate in the lemma is concluded.

On the other hand, for $\lambda \in \overline{\mathbb{C}^+}\setminus\{0\}$ with $|\mathrm{Im}\,\lambda| \ge 1$ we have

$$\begin{aligned}\|R_\lambda B\| &= \|\Lambda_0^{\beta_0}\Lambda_\infty^{\beta}R_\lambda\Lambda_0^{-\beta_0}\Lambda_\infty^{-\beta}B\| \le \|\Lambda_0^{\beta_0}\|\|\Lambda_\infty^{\beta}R_\lambda B_{(\beta_0,\beta)}\|\\ &\le M'''\|\Lambda_0^{\beta_0}\|\|R_\lambda B_{(\beta_0,\beta)}\|^{1-\beta/\alpha}\|\Lambda_\infty^{\alpha}R_\lambda B_{(\beta_0,\beta)}\|^{\beta/\alpha}\\ &\le M'''M_2^{\beta/\alpha}\|\Lambda_0^{\beta_0}\|\|B_{(\beta_0,\beta)}\|^{\beta/\alpha}\|R_\lambda B_{(\beta_0,\beta)}\|^{1-\beta/\alpha}\end{aligned}$$

and

$$\begin{aligned}\|CR_\lambda\| &= \|(\Lambda_0^{\gamma_0})^*(\Lambda_\infty^{\gamma})^*R_\lambda^*(\Lambda_0^{-\gamma_0})^*(\Lambda_\infty^{-\gamma})^*C^*\| \le \|\Lambda_0^{\gamma_0}\|\|(\Lambda_\infty^*)^\gamma R_\lambda^*\tilde{C}_{(\gamma_0,\gamma)}\| \\ &\le M''''\|\Lambda_0^{\gamma_0}\|\|R_\lambda^*\tilde{C}_{(\gamma_0,\gamma)}\|^{1-\gamma/\alpha}\|(\Lambda_\infty^\alpha)^*R_\lambda^*\tilde{C}_{(\gamma_0,\gamma)}\|^{\gamma/\alpha} \\ &\le M''''M_2^{\gamma/\alpha}\|\Lambda_0^{\gamma_0}\|\|\tilde{C}_{(\gamma_0,\gamma)}\|^{\gamma/\alpha}\|R_\lambda^*\tilde{C}_{(\gamma_0,\gamma)}\|^{1-\gamma/\alpha}\end{aligned}$$

where again $M''', M'''' \ge 1$ follow from the Moment Inequality [8, Prop. 6.6.4], and are independent of B, C, and λ. If we choose (and use $\beta+\gamma=\alpha$)

$$M_0 = M'''M''''M_2\|\Lambda_0^{\beta_0}\|\|B_{(\beta_0,\beta)}\|^{\beta/\alpha}\|\Lambda_0^{\gamma_0}\|\|\tilde{C}_{(\gamma_0,\gamma)}\|^{\gamma/\alpha},$$

we arrive at the second estimate in the lemma. □

Proof of Theorem 2. Let $\delta>0$ be as in Theorem 4. Assume $B\in\mathscr{L}(Y,X)$ and $C\in\mathscr{L}(X,Y)$ satisfy $\mathscr{R}(B)\subset\mathscr{R}((-A)^{\beta_0})\cap\mathscr{D}((-A)^\beta)$, $\mathscr{R}(C^*)\subset\mathscr{R}((-A^*)^{\gamma_0})\cap\mathscr{D}((-A^*)^\gamma)$, $\|B\|+\|(-A)^{-\beta_0}B\|+\|(-A)^\beta B\|<\delta$, and $\|C\|+\|(-A^*)^{-\gamma_0}C^*\|+\|(-A^*)^\gamma C^*\|<\delta$, and assume $(-A)^{-\beta_0}B$, $(-A)^\beta B$, $(-A^*)^{-\gamma_0}C^*$ and $(-A^*)^\gamma C^*$ are Hilbert–Schmidt operators. By Theorem 4 we can choose $M_D\ge 1$ such that $\|(I-CR(\lambda,A)B)^{-1}\|\le M_D$ for all $\lambda\in\overline{\mathbb{C}^+}\setminus\{0\}$. We begin the proof by showing that the semigroup generated by $A+BC$ is uniformly bounded.

Let $x\in X$ and denote $R_\lambda=R(\xi+i\eta,A)$ and $D_\lambda=I-CR(\xi+i\eta,A)B$. Using the Sherman–Morrison–Woodbury formula in Lemma 1 and the scalar inequality $(a+b)^2\le 2(a^2+b^2)$ for $a,b\ge 0$ we get

$$\begin{aligned}\sup_{\xi>0}\xi\int_{-\infty}^{\infty}\|R(\xi+i\eta,A+BC)x\|^2d\eta &= \sup_{\xi>0}\xi\int_{-\infty}^{\infty}\|R_\lambda x+R_\lambda BD_\lambda^{-1}CR_\lambda x\|^2d\eta \\ &\le 2\sup_{\xi>0}\xi\int_{-\infty}^{\infty}\left(\|R_\lambda x\|^2+\|R_\lambda B\|^2\|D_\lambda^{-1}\|^2\|CR_\lambda\|^2\|x\|^2\right)d\eta \\ &\le 2\sup_{\xi>0}\xi\int_{-\infty}^{\infty}\|R_\lambda x\|^2d\eta+2M_D^2\|x\|^2\sup_{\xi>0}\xi\int_{-\infty}^{\infty}\|R_\lambda B\|^2\|CR_\lambda\|^2d\eta.\end{aligned}$$

Similarly, using $\|(R_\lambda BD_\lambda^{-1}CR_\lambda)^*\|=\|R_\lambda BD_\lambda^{-1}CR_\lambda\|\le M_D\|R_\lambda B\|\|CR_\lambda\|$ we get

$$\begin{aligned}\sup_{\xi>0}\xi\int_{-\infty}^{\infty}\|R(\xi+i\eta,A+BC)^*x\|^2d\eta &= \sup_{\xi>0}\xi\int_{-\infty}^{\infty}\|R_\lambda^*x+(R_\lambda BD_\lambda^{-1}CR_\lambda)^*x\|^2d\eta \\ &\le 2\sup_{\xi>0}\xi\int_{-\infty}^{\infty}\|R_\lambda^*x\|^2d\eta+2M_D^2\|x\|^2\sup_{\xi>0}\xi\int_{-\infty}^{\infty}\|R_\lambda B\|^2\|CR_\lambda\|^2d\eta.\end{aligned}$$

In both cases the first supremums are finite by Theorem 6. Because of this, Theorem 6 implies that in order to show that the semigroup generated by $A+BC$ is uniformly bounded, it is sufficient to prove that

$$\sup_{\xi>0}\xi\int_{-\infty}^{\infty}\|R(\xi+i\eta,A)B\|^2\|CR(\xi+i\eta,A)\|^2d\eta<\infty. \tag{3}$$

Let $M_0, M_\infty \geq 1$ be as in Lemma 5 and denote $B_{(\beta_0,\beta)} = \Lambda_0^{-\beta_0}\Lambda_\infty^{-\beta}B$ and $\tilde{C}_{(\gamma_0,\gamma)} = (\Lambda_0^{-\gamma_0})^*(\Lambda_\infty^{-\gamma})^*C^*$. If $(e_k)_{k=1}^\infty$ is an orthonormal basis of Y, then as in the proof of Lemma 2 we can see that

$$\|B_{(\beta_0,\beta)}e_k\|^2 \leq 2M_\Lambda^2\left(\|(-A)^{-\beta_0}Be_k\|^2 + \|(-A)^{\beta}Be_k\|^2\right),$$
$$\|\tilde{C}_{(\gamma_0,\gamma)}e_k\|^2 \leq 2M_\Lambda^2\left(\|(-A^*)^{-\gamma_0}C^*e_k\|^2 + \|(-A^*)^{\gamma}C^*e_k\|^2\right).$$

Since $(-A)^{-\beta_0}B$, $(-A)^{\beta}B$, $(-A^*)^{-\gamma_0}C^*$, and $(-A^*)^{\gamma}C^*$ are Hilbert–Schmidt by assumption, the above estimates imply that also $B_{(\beta_0,\beta)}$ and $\tilde{C}_{(\gamma_0,\gamma)}$ are Hilbert–Schmidt.

Assume $\beta_0, \beta, \gamma_0, \gamma > 0$. The remaining cases are simpler and can be handled similarly as in [12, Lem. 19]. By assumption, we have $1-\beta_0/\alpha_0 + 1 - \gamma_0/\alpha_0 = 2 - (\beta_0+\gamma_0)/\alpha_0 = 1$. If we choose $q = 1/(1-\beta_0/\alpha_0)$ and $r = 1/(1-\gamma_0/\alpha_0)$, then the Hölder inequality and Lemma 4 imply that

$$\begin{aligned}
&\sup_{\xi>0}\ \xi\int_{-1}^{1}\|R(\xi+i\eta,A)B\|^2\|CR(\xi+i\eta,A)\|^2 d\eta\\
&\leq M_0^2\sup_{\xi>0}\ \xi\int_{-1}^{1}\|R(\xi+i\eta,A)B_{(\beta_0,\beta)}\|^{2(1-\frac{\beta_0}{\alpha_0})}\|R(\xi+i\eta,A)^*\tilde{C}_{(\gamma_0,\gamma)}\|^{2(1-\frac{\gamma_0}{\alpha_0})}d\eta\\
&\leq M_0^2\left[\sup_{\xi>0}\xi\int_{-1}^{-1}\|R(\xi+i\eta,A)B_{(\beta_0,\beta)}\|^2 d\eta\right]^q\left[\sup_{\xi>0}\xi\int_{-1}^{-1}\|R(\xi+i\eta,A)^*\tilde{C}_{(\gamma_0,\gamma)}\|^2 d\eta\right]^r
\end{aligned}$$

is finite. Moreover, $1-\beta/\alpha + 1 - \gamma/\alpha = 2-(\beta+\gamma)/\alpha = 1$, and if we choose $q = 1/(1-\beta/\alpha)$ and $r = 1/(1-\gamma/\alpha)$, then by the Hölder inequality and Lemma 4

$$\begin{aligned}
&\sup_{\xi>0}\ \xi\int_{|\eta|\geq 1}\|R(\xi+i\eta,A)B\|^2\|CR(\xi+i\eta,A)\|^2 d\eta\\
&\leq M_\infty^2\sup_{\xi>0}\ \xi\int_{|\eta|\geq 1}\|R(\xi+i\eta,A)B_{(\beta_0,\beta)}\|^{2(1-\frac{\beta}{\alpha})}\|R(\xi+i\eta,A)^*\tilde{C}_{(\gamma_0,\gamma)}\|^{2(1-\frac{\gamma}{\alpha})}d\eta\\
&\leq M_\infty^2\left[\sup_{\xi>0}\xi\int_{|\eta|\geq 1}\|R(\xi+i\eta,A)B_{(\beta_0,\beta)}\|^2 d\eta\right]^q\left[\sup_{\xi>0}\xi\int_{|\eta|\geq 1}\|R(\xi+i\eta,A)^*\tilde{C}_{(\gamma_0,\gamma)}\|^2 d\eta\right]^r
\end{aligned}$$

is finite. Combining the above estimates yields

$$\begin{aligned}
\sup_{\xi>0}\ \xi\int_{-\infty}^{\infty}&\|R(\xi+i\eta,A)B\|^2\|CR(\xi+i\eta,A)\|^2 d\eta\\
&\leq \sup_{\xi>0}\ \xi\int_{-1}^{1}\|R(\xi+i\eta,A)B\|^2\|CR(\xi+i\eta,A)\|^2 d\eta\\
&\quad+\sup_{\xi>0}\ \xi\int_{|\eta|\geq 1}\|R(\xi+i\eta,A)B\|^2\|CR(\xi+i\eta,A)\|^2 d\eta < \infty.
\end{aligned}$$

This concludes (3), and thus the semigroup generated by $A+BC$ is uniformly bounded.

Since the perturbed semigroup is uniformly bounded and X is a Hilbert space, the Mean Ergodic Theorem [2, Sec. 4.3] shows that $\sigma(A+BC)\cap i\mathbb{R}\subset \sigma_p(A+BC)\cup \sigma_c(A+BC)$. In addition, by Theorem 4 we have that $\sigma_p(A+BC)\cap i\mathbb{R}=\varnothing$, $0\in \sigma(A+BC)$, and $i\mathbb{R}\setminus\{0\}\subset \rho(A+BC)$. We must therefore have $\sigma(A+BC)\cap i\mathbb{R} = \sigma_c(A+BC)\cap i\mathbb{R}=\{0\}$. Since the set $\sigma(A+BC)\cap i\mathbb{R}=\{0\}$ finite and since we have $\sigma_p(A+BC)\cap i\mathbb{R}=\varnothing$, the Arent–Batty–Lyubich–Vũ Theorem [1, 9] concludes that the semigroup generated by $A+BC$ is strongly stable.

It remains to show that the resolvent operator $R(\lambda,A+BC)$ satisfies

$$\sup_{0<|\omega|\le 1} |\omega|^{\alpha_0}\|R(i\omega,A+BC)\|<\infty \quad \text{and} \quad \sup_{|\omega|\ge 1} |\omega|^{-\alpha}\|R(i\omega,A+BC)\|<\infty. \tag{4}$$

The Sherman–Morrison–Woodbury formula in Lemma 1 implies that

$$\begin{aligned}\|R(i\omega,A+BC)\| &= \|R(i\omega,A)+R(i\omega,A)B(I-CR(i\omega,A)B)^{-1}CR(i\omega,A)\|\\ &\le \|R(i\omega,A)\|+\|R(i\omega,A)B\|\|(I-CR(i\omega,A)B)^{-1}\|\|CR(i\omega,A)\|\\ &\le \|R(i\omega,A)\|+M_D\|R(i\omega,A)B\|\|CR(i\omega,A)\|\end{aligned}$$

for all $\omega\in\mathbb{R}\setminus\{0\}$. Since $\beta_0/\alpha_0+\gamma_0/\alpha_0=1$, for $0<|\omega|\le 1$ the previous estimate together with Lemma 5 and Assumption 1 imply

$$\begin{aligned}|\omega|^{\alpha_0}\|R(i\omega,A+BC)\| &\le |\omega|^{\alpha_0}\|R(i\omega,A)\|+|\omega|^{\alpha_0}M_D\|R(i\omega,A)B\|\|CR(i\omega,A)\|\\ &\le |\omega|^{\alpha_0}\|R(i\omega,A)\|+M_DM_0\|B_{(\beta_0,\beta)}\|^{\beta_0/\alpha_0}\|\tilde{C}_{(\gamma_0,\gamma)}\|^{\gamma_0/\alpha_0}|\omega|^{\alpha_0}\|R(i\omega,A)\|\\ &\le M_A+M_DM_0\|B_{(\beta_0,\beta)}\|^{\beta_0/\alpha_0}\|\tilde{C}_{(\gamma_0,\gamma)}\|^{\gamma_0/\alpha_0}M_A<\infty.\end{aligned}$$

Since the bound is independent of ω, this concludes the first part of (4). On the other hand, if $|\omega|\ge 1$, then we similarly have

$$\begin{aligned}|\omega|^{-\alpha}\|R(i\omega,A+BC)\| &\le |\omega|^{-\alpha}\|R(i\omega,A)\|+|\omega|^{-\alpha}M_D\|R(i\omega,A)B\|\|CR(i\omega,A)\|\\ &\le \left(1+M_DM_\infty\|B_{(\beta_0,\beta)}\|^{\beta/\alpha}\|\tilde{C}_{(\gamma_0,\gamma)}\|^{\gamma/\alpha}\right)|\omega|^{-\alpha}\|R(i\omega,A)\|\\ &\le \left(1+M_DM_\infty\|B_{(\beta_0,\beta)}\|^{\beta/\alpha}\|\tilde{C}_{(\gamma_0,\gamma)}\|^{\gamma/\alpha}\right)M_A,\end{aligned}$$

since $\beta/\alpha+\gamma/\alpha=1$. This concludes the proof. □

3 Robustness of Stability with Respect to Perturbations Commuting with A

In this section we prove Theorem 3. We begin with an auxiliary lemma.

Lemma 6. *Let Assumption* 1 *be satisfied. There exists* $\delta' > 0$ *such that if* $B \in \mathscr{L}(X)$ *commutes with A, satisfies* $\mathscr{R}(B) \subset \mathscr{R}((-A)^{\alpha_0})$*, and* $\|B\| + \|(-A)^{-\alpha_0}B\| < \delta'$*, then* $0 \in \sigma_c(A+B)$.

Proof. Since $\alpha_0 \geq 1$, we have $\mathscr{R}(B) \subset \mathscr{R}((-A)^{\alpha_0}) \subset \mathscr{R}(A)$. Assume B is such that $\|A^{-1}B\| < 1$. Then the operator $I + A^{-1}B$ is boundedly invertible and $(A+B)x = A(I + A^{-1}B)x$ for all $x \in \mathscr{D}(A+B) = \mathscr{D}(A)$. Since A is injective, we have that $0 \notin \sigma_p(A+B)$. Moreover, $\mathscr{R}(A+B) = \mathscr{R}(A) \neq X$ and $\overline{\mathscr{R}(A+B)} = \overline{\mathscr{R}(A)} = X$. This concludes that $0 \in \sigma_c(A+B)$ if $\|A^{-1}B\| < 1$. If $\alpha_0 = 1$, we can choose $\delta' = 1$ and the proof is concluded. On the other hand, if $\alpha_0 > 1$, the Moment Inequality [8, Prop. 6.6.4] implies that there exists $M' \geq 1$ (independent of B) such that

$$
\begin{aligned}
\|A^{-1}B\| = \|(-A)^{-1}B\| &\leq M'\|B\|^{1-1/\alpha_0}\|(-A)^{-\alpha_0}B\|^{1/\alpha_0} \\
&\leq M'(\max\{\|B\|, \|(-A)^{-\alpha_0}B\|\})^{1-1/\alpha_0}(\max\{\|B\|, \|(-A)^{-\alpha_0}B\|\})^{1/\alpha_0} \\
&\leq M' \max\{\|B\|, \|(-A)^{-\alpha_0}B\|\} \leq M'(\|B\| + \|(-A)^{-\alpha_0}B\|).
\end{aligned}
$$

In this situation we can therefore choose $\delta' = 1/M' > 0$. □

Proof of Theorem 3. We can define $M_1 = \sup_{\lambda \in \overline{\mathbb{C}^+}\setminus\{0\}} \|R(\lambda, A)\Lambda_{(\alpha_0,\alpha)}\| < \infty$ by Theorem 5. Since $\Lambda_{(\alpha_0,\alpha)}^{-1} = \Lambda_0^{-\alpha_0}\Lambda_\infty^{-\alpha}$, by Lemma 2 there exists $M_\Lambda \geq 1$ such that

$$
\|\Lambda_{(\alpha_0,\alpha)}^{-1}B\| \leq M_\Lambda \left(\|(-A)^{-\alpha_0}B\| + \|(-A)^{\alpha}B\|\right)
$$

for all $B \in \mathscr{L}(X)$ satisfying $\mathscr{R}(B) \subset \mathscr{R}((-A)^{\alpha_0}) \cap \mathscr{D}((-A)^{\alpha})$. Let $0 < c < 1$ and choose $\delta = \min\left\{\frac{c}{M_1 M_\Lambda}, \delta'\right\} > 0$, where $\delta' > 0$ is from Lemma 6. Let $B \in \mathscr{L}(X)$ satisfy $\mathscr{R}(B) \subset \mathscr{R}((-A)^{\alpha_0}) \cap \mathscr{D}((-A)^{\alpha})$ and $\|B\| + \|(-A)^{-\alpha_0}B\| + \|(-A)^{\alpha}B\| < \delta$. We then have from Lemma 6 that $0 \in \sigma_c(A+B)$. Moreover, for every $\lambda \in \overline{\mathbb{C}^+} \setminus \{0\}$ we have

$$
\begin{aligned}
\|BR(\lambda, A)\| = \|B\Lambda_{(\alpha_0,\alpha)}^{-1}\Lambda_{(\alpha_0,\alpha)}R(\lambda, A)\| &\leq \|\Lambda_{(\alpha_0,\alpha)}^{-1}B\|\|R(\lambda, A)\Lambda_{(\alpha_0,\alpha)}\| \\
&\leq M_\Lambda \left(\|(-A)^{\alpha}B\| + \|(-A)^{-\alpha_0}B\|\right) M_1 < M_1 M_\Lambda \delta \leq c < 1.
\end{aligned}
$$

Because of this, $I - BR(\lambda, A)$ is invertible for all $\lambda \in \overline{\mathbb{C}^+} \setminus \{0\}$ and a standard Neumann series argument shows that $\|(I - BR(\lambda, A))^{-1}\| \leq 1/(1-c)$. Since

$$
R(\lambda, A+B) = R(\lambda, A)(I - BR(\lambda, A))^{-1} = (I - BR(\lambda, A))^{-1}R(\lambda, A),
$$

this in particular concludes that $\sigma(A+BC) \subset \mathbb{C}^- \cup \{0\}$.

To prove the uniform boundedness of the semigroup generated by $A+B$, let $x \in X$ and denote $D_\lambda = I - BR(\xi + i\eta, A)$ for $\lambda = \xi + i\eta$. As we saw above, there exists $M_D \geq 1$ such that $\|D_\lambda^{-1}\| \leq M_D$ for all $\lambda \in \overline{\mathbb{C}^+} \setminus \{0\}$. Using $R(\lambda, A+B) = D_\lambda^{-1}R(\lambda, A) = R(\lambda, A)D_\lambda^{-1}$ we get

$$\sup_{\xi>0}\ \xi\int_{-\infty}^{\infty}\|R(\xi+i\eta,A+B)x\|^2d\eta=\sup_{\xi>0}\ \xi\int_{-\infty}^{\infty}\|D_\lambda^{-1}R(\xi+i\eta,A)x\|^2d\eta$$
$$\leq M_D^2\sup_{\xi>0}\ \xi\int_{-\infty}^{\infty}\|R(\xi+i\eta,A)x\|^2d\eta<\infty,$$

and

$$\sup_{\xi>0}\ \xi\int_{-\infty}^{\infty}\|R(\xi+i\eta,A+B)^*x\|^2d\eta=\sup_{\xi>0}\ \xi\int_{-\infty}^{\infty}\|D_\lambda^{-*}R(\xi+i\eta,A)^*x\|^2d\eta$$
$$\leq M_D^2\sup_{\xi>0}\ \xi\int_{-\infty}^{\infty}\|R(\xi+i\eta,A)^*x\|^2d\eta<\infty.$$

Theorem 6 concludes that the semigroup generated by $A+B$ is uniformly bounded. Since $\sigma(A+B)\subset\mathbb{C}^-\cup\{0\}$ and $0\in\sigma_c(A+B)$, the Arent–Batty–Lyubich–Vũ Theorem [1, 9] further implies that the semigroup generated by $A+B$ is strongly stable.

It remains to prove that the resolvent operator $R(\lambda,A+B)$ satisfies the conditions in Assumption 1. For all $i\omega\in i\mathbb{R}\setminus\{0\}$ we have an estimate

$$\|R(i\omega,A+B)\|=\|R(i\omega,A)(I-BR(i\omega,A))^{-1}\|\leq M_D\|R(i\omega,A)\|,$$

which together with Assumption 1 immediately implies

$$\sup_{0<|\omega|\leq 1}\ |\omega|^{\alpha_0}\|R(i\omega,A+B)\|\leq M_D\sup_{0<|\omega|\leq 1}\ |\omega|^{\alpha_0}\|R(i\omega,A)\|<\infty$$
$$\sup_{|\omega|\geq 1}|\omega|^{-\alpha}\|R(i\omega,A+B)\|\leq M_D\sup_{|\omega|\geq 1}|\omega|^{-\alpha}\|R(i\omega,A)\|<\infty.$$

This concludes the proof. □

4 A Diagonal Example

In this section we present an example on perturbation of a strongly stable semigroup $T(t)$ generated by a diagonal operator. Let $X=\ell^2(\mathbb{C})$ and define

$$Ax=\sum_{k=-\infty}^{-1}\frac{1}{k}\langle x,e_k\rangle e_k+\sum_{k=1}^{\infty}\big(-\frac{1}{k^\alpha}+ik\big)\langle x,e_k\rangle e_k,$$
$$\mathscr{D}(A)=\Big\{x\in X\ \Big|\ \sum_{k=1}^{\infty}|k|^2|\langle x,e_k\rangle|^2<\infty\Big\}$$

where $\{e_k\}_{k\in\mathbb{Z}}$ are the natural basis vectors of X and $\alpha>0$. The operator A generates a strongly stable semigroup $T(t)$ and satisfies $\sigma(A)\cap i\mathbb{R}=\{0\}\subset\sigma_c(A)$. Since for $0<|\omega|\leq 1$ we have $\|R(i\omega,A)\|=\operatorname{dist}(i\omega,\sigma(A))^{-1}=|\omega|^{-1}$, and $\|R(i\omega,A)\|=$

$\mathscr{O}(|\omega|^\alpha)$ for large $|\omega|$, the conditions of Assumption 1 are satisfied for $\alpha_0 = 1$ and $\alpha > 0$.

For $\beta \geq 0$ fractional domains of $-A$ are given by

$$\mathscr{D}((-A)^\beta) = \left\{ x \in X \;\middle|\; \sum_{k=1}^{\infty} |k|^{2\beta} |\langle x, e_k \rangle|^2 < \infty \right\},$$

and for $x \in \mathscr{D}((-A)^\beta)$ we have an estimate (since $|-1/k^\alpha + ik|^2 = 1/k^{2\alpha} + k^2 \leq 2k^2$)

$$\begin{aligned}
\|(-A)^\beta x\|^2 &= \sum_{k=-\infty}^{-1} \frac{1}{|k|^{2\beta}} |\langle x, e_k \rangle|^2 + \sum_{k=1}^{\infty} \left| -\frac{1}{k^\alpha} + ik \right|^{2\beta} |\langle x, e_k \rangle|^2 \\
&\leq \sum_{k=-\infty}^{-1} |\langle x, e_k \rangle|^2 + 2^\beta \sum_{k=1}^{\infty} k^{2\beta} |\langle x, e_k \rangle|^2.
\end{aligned}$$

On the other hand, for $\beta_0 \geq 0$ we have

$$\mathscr{R}((-A)^{\beta_0}) = \left\{ x \in X \;\middle|\; \sum_{k=-\infty}^{-1} |k|^{2\beta_0} |\langle x, e_k \rangle|^2 < \infty \right\},$$

and for every $x \in \mathscr{R}((-A)^{\beta_0})$ we can estimate

$$\begin{aligned}
\|(-A)^{-\beta_0} x\|^2 &= \sum_{k=-\infty}^{-1} |k|^{2\beta_0} |\langle x, e_k \rangle|^2 + \sum_{k=1}^{\infty} \left| -\frac{1}{k^\alpha} + ik \right|^{-2\beta_0} |\langle x, e_k \rangle|^2 \\
&\leq \sum_{k=-\infty}^{-1} |k|^{2\beta_0} |\langle x, e_k \rangle|^2 + \sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2
\end{aligned}$$

since $|-1/k^\alpha + ik|^2 = 1/k^{2\alpha} + k^2 \geq 1$ for all $k \in \mathbb{N}$. Because

$$A^* x = \sum_{k=-\infty}^{-1} \frac{1}{k} \langle x, e_k \rangle e_k + \sum_{k=1}^{\infty} \big(-\frac{1}{k^\alpha} - ik \big) \langle x, e_k \rangle e_k$$

with domain $\mathscr{D}(A^*) = \mathscr{D}(A)$, we similarly have

$$\begin{aligned}
\|(-A^*)^\gamma x\|^2 &\leq \sum_{k=-\infty}^{-1} |\langle x, e_k \rangle|^2 + 2^\gamma \sum_{k=1}^{\infty} k^{2\gamma} |\langle x, e_k \rangle|^2 \\
\|(-A^*)^{-\gamma_0} x\|^2 &\leq \sum_{k=-\infty}^{-1} |k|^{2\gamma_0} |\langle x, e_k \rangle|^2 + \sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2.
\end{aligned}$$

We consider the preservation of the strong stability of $T(t)$ under a rank one perturbations $A + \langle \cdot, c \rangle b$ with $b, c \in X$. Theorem 2 together with the earlier estimates implies that the semigroup generated by the $A + \langle \cdot, c \rangle b$ is strongly stable if for some $\beta_0, \beta, \gamma_0, \gamma \geq 0$ satisfying $\beta_0 + \gamma_0 = 1$ and $\beta + \gamma = \alpha$ the weighted norms

$$\sum_{k=-\infty}^{-1} |k|^{2\beta_0} |\langle b, e_k\rangle|^2 + \sum_{k=1}^{\infty} k^{2\beta} |\langle b, e_k\rangle|^2 \quad \text{and} \quad \sum_{k=-\infty}^{-1} |k|^{2\gamma_0} |\langle c, e_k\rangle|^2 + \sum_{k=1}^{\infty} k^{2\gamma} |\langle c, e_k\rangle|^2$$

are finite and sufficiently small.

5 Conclusions

In this paper we have studied the preservation of strong stability of a semigroup $T(t)$ under perturbations of its generator A. The results have applications in the study of the asymptotic behaviour of linear partial differential equations. We have limited our attention to a situation where the generator A has a single finite spectral point on the imaginary axis. However, the techniques in [12] can be used to extend the results in this paper to the case where A has a finite number of spectral points on $i\mathbb{R}$, and the resolvent operator $\|R(i\omega,A)\|$ is polynomially bounded for large $|\omega|$.

Acknowledgement: The author is grateful to Yuri Tomilov for suggesting the extension of Theorem 2 from finite rank perturbations to perturbations where B and C are Hilbert–Schmidt operators.

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