Output Regulation for General Infinite-Dimensional Exosystems

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Abstract: In this paper we study output regulation of distributed parameter systems with infinite-dimensional exosystems. The purpose of the paper is to find simple and minimal conditions on the signal generator under which the solvability of the output regulation problem can be characterized by the solvability of the regulator equations. We also study the properties of the dynamic steady state of the closed-loop system and the uniqueness of the solution of Sylvester operator equations. The presented results have applications in robust regulation of infinite-dimensional systems.

Keywords: Robust output regulation, distributed parameter systems, infinite-dimensional exosystems.

1. INTRODUCTION

The topic of this paper is the output regulation theory for infinite-dimensional linear systems, see Byrnes et al. [2000], Hämäläinen and Pohjolainen [2010], Paunonen and Pohjolainen [2010]. In particular we are interested in extending the theory and the existing main results for more general exosystems. This generalization allows us to consider a larger class of reference and disturbance signals in connection with the problems of output tracking and disturbance rejection. In many applications, such as in control of robot arms, disk drive systems, and magnetic power supplies for proton synchrotrons [Yamamoto, 1993, and references therein], it is necessary to track nonsmooth functions with high accuracy. In order to generate such reference signals, we must in particular consider infinite-dimensional exosystems.

The novelty of this paper is that — instead of considering infinite-dimensional exosystems of certain types — we impose minimal conditions on the signal generator and study the consequences of such assumptions. In particular we are interested in the well-known result that for finite-dimensional and for certain classes of infinite-dimensional exosystems the solvability of the output regulation problem can be characterized using the solvability of the regulator equations of the form, see Byrnes et al. [2000]

$$\Sigma S = A_e \Sigma + B_e \tag{1a}$$
$$0 = C_e \Sigma + D_e. \tag{1b}$$

Here the operators A_e, B_e, C_e, and D_e are parameters of the closed-loop system consisting of the plant and the controller. The operator S is the system operator of the exosystem generating the reference and disturbance signals, i.e.,

$$\dot{v}(t) = Sv(t), \qquad v(0) = v_0 \in W \tag{2a}$$
$$y_{ref}(t) = Fv(t). \tag{2b}$$

In this paper we consider a very general situation in which the exosystem is an infinite-dimensional linear system on a Banach space W and where S generates a strongly continuous semigroup $T_S(t)$.

The main motivation for this study is that the regulator equations (1) are an important theoretical tool in the study of the robust output regulation problem, where the control law is required to achieve output tracking and disturbance rejection despite uncertainties and perturbations in the parameters of the plant. Recently, in Paunonen and Pohjolainen [2010] the internal model principle of Francis and Wonham was generalized to distributed parameter systems and infinite-dimensional block-diagonal exosystems. The results presented in the current paper are a crucial step in further extending the internal model principle for more general classes of reference and disturbance signals.

In most of the theory concerning control of distributed parameter systems, the exosystem (2) is assumed to be a finite-dimensional system on $W = \mathbb{C}^q$. More recently in Immonen [2007], Hämäläinen and Pohjolainen [2010], Paunonen and Pohjolainen [2010] the output regulation problem has been studied for certain types of infinite-dimensional signal generators on separable Hilbert spaces and more general Banach spaces. The main purpose of this paper is to study the problem of output tracking and disturbance rejection of a distributed parameter system in the case where we only assume that the signals generated by the exosystem (2) do not decay asymptotically.

We also study the implications of such *nondecay conditions* on the results concerning output regulation and on the regulator equations (1). In particular we show that if the semigroup generated by A_e in (1a) is strongly stable, then a suitable nondecay condition on the semigroup $T_S(t)$ can be used to show that the Sylvester equation (1a) can have at most one bounded solution Σ.

In Hämäläinen and Pohjolainen [2010], Paunonen and Pohjolainen [2010] the development of the theory of robust output regulation made use of the *dynamic steady state* of the closed-loop system. More precisely, it was shown that the asymptotic behavior of the state $x_e(t)$ of a stable closed-loop system satisfies

$$x_e(t) \sim \Sigma v(t),$$

for large t. Here Σ is the solution of the Sylvester equation (1a) and $v(t)$ is the state of the exosystem (2). We will show that this concept of a dynamic steady state is indeed well-defined by showing that even if the solution of the Sylvester equation (1a) might be nonunique, the different solutions still lead to the same dynamic steady state.

In Section 7 we present some sufficient conditions for assumptions on the exosystem. In particular we show that any exosystem generating almost periodic signals and the exosystem studied in Paunonen and Pohjolainen [2010, Submitted] satisfy the nondecay conditions.

2. MATHEMATICAL PRELIMINARIES

If X and Y are Banach spaces and $A : X \to Y$ is a linear operator, we denote by $\mathcal{D}(A)$ the domain of A. The space of bounded linear operators from X to Y is denoted by $\mathcal{L}(X,Y)$. If $A : X \to X$, then $\sigma(A)$, $\sigma_p(A)$, $\sigma_c(A)$ and $\rho(A)$ denote the spectrum, the point spectrum, the continuous spectrum and the resolvent set of A, respectively. For $\lambda \in \rho(A)$ the resolvent operator is given by $R(\lambda, A) = (\lambda I - A)^{-1}$. The dual pair on a Banach space and the inner product on a Hilbert space are both denoted by $\langle \cdot, \cdot \rangle$.

In this paper we consider a linear system

$$\dot{x}(t) = Ax(t) + Bu(t) + w(t), \qquad x(0) = x_0 \in X$$
$$y(t) = Cx(t) + Du(t)$$

where $x(t) \in X$ is the state of the system, $y(t) \in Y$ is the output, and $u(t) \in U$ the input. The spaces X, U and Y are general Banach spaces. Here $w(t) \in X$ denotes the disturbance signal to the state of the plant. We assume that $A : \mathcal{D}(A) \subset X \to X$ generates a C_0-semigroup on X and the other operators are bounded, $B \in \mathcal{L}(U,X)$, $C \in \mathcal{L}(X,Y)$, $D \in \mathcal{L}(U,Y)$.

The reference signal $y_{ref}(t)$ to be tracked and disturbance signal $w(t)$ are generated using a possibly infinite-dimensional exosystem

$$\dot{v}(t) = Sv(t) \qquad v(0) = v_0 \in W \tag{3a}$$
$$y_{ref}(t) = Fv(t) \tag{3b}$$
$$w(t) = Ev(t) \tag{3c}$$

on a Banach space W. We assume S generates a strongly continuous semigroup $T_S(t)$ on W and that $E \in \mathcal{L}(W,X)$ and $F \in \mathcal{L}(W,Y)$.

The dynamic feedback controller on a Banach space Z is of the form

$$\dot{z}(t) = \mathcal{G}_1 z(t) + \mathcal{G}_2 e(t), \qquad z(0) = z_0 \in Z$$
$$u(t) = Kz(t)$$

where $e(t) = y(t) - y_{ref}(t)$ is the *regulation error*, the operator $\mathcal{G}_1 : \mathcal{D}(\mathcal{G}_1) \subset Z \to Z$ generates a C_0-semigroup on Z, $\mathcal{G}_2 \in \mathcal{L}(Y,Z)$ and $K \in \mathcal{L}(Z,U)$.

The closed-loop system consisting of the plant and the controller is a linear system on the space $X_e = X \times Z$ with state $x_e(t) = (x(t), z(t))^T$. This system is described by equations

$$\dot{x}_e(t) = A_e x_e(t) + B_e v(t), \qquad x_e(0) = x_{e0} = \begin{bmatrix} x_0 \\ z_0 \end{bmatrix} \tag{4a}$$
$$e(t) = C_e x_e(t) + D_e v(t), \tag{4b}$$

where $C_e = [C \;\; DK]$, $D_e = -F$,

$$A_e = \begin{bmatrix} A & BK \\ \mathcal{G}_2 C & \mathcal{G}_1 + \mathcal{G}_2 DK \end{bmatrix} \quad \text{and} \quad B_e = \begin{bmatrix} E \\ -\mathcal{G}_2 F \end{bmatrix}$$

and the operator $A_e : \mathcal{D}(A) \times \mathcal{D}(\mathcal{G}_1) \subset X_e \to X_e$ generates a C_0-semigroup $T_e(t)$ on X_e.

We remark that the results studied in this paper are represented using only the parameters of the closed-loop system. Therefore they continue to hold for any type of controller which can be written with the plant in the closed-loop form similar to (4). These include, for example, control laws incorporating state feedback.

3. THE MOTIVATING PROBLEM: OUTPUT REGULATION

The main control problem studied in this paper is stated as follows.

The Output Regulation Problem. Choose the parameters $(\mathcal{G}_1, \mathcal{G}_2, K)$ of the error feedback controller in such a way that the following are satisfied:

(a) The closed-loop system operator A_e generates a strongly stable semigroup on X_e;
(b) For all initial states $v_0 \in W$ and $x_{e0} \in X_e$ the output of the plant tracks the reference signal $y_{ref}(t)$ asymptotically, i.e.,

$$\lim_{t\to\infty} \|y(t) - y_{ref}(t)\| = 0.$$

As we already discussed in the introduction, it is well-known that in the case of finite-dimensional and certain infinite-dimensional exosystems the solvability of the output regulation problem can be characterized using the solvability of the so-called *regulator equations*, see Byrnes et al. [2000], Hämäläinen and Pohjolainen [2010], Paunonen and Pohjolainen [2010]. In this paper we study the same result for more general infinite-dimensional exosystems. We now state the basic form of this theorem, and in the next section we discuss the minimal conditions under which the theorem remains valid.

Theorem 1. Assume that the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ is such that A_e generates a strongly stable C_0-semigroup on X_e and that the Sylvester equation

$$\Sigma S = A_e \Sigma + B_e \tag{5}$$

has a solution $\Sigma \in \mathcal{L}(W, X_e)$ satisfying $\Sigma(\mathcal{D}(S)) \subset \mathcal{D}(A_e)$. Then the following are equivalent:

(a) The controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ solves the output regulation problem.
(b) The solution Σ of the Sylvester equation (5) satisfies

$$C_e \Sigma + D_e = 0. \tag{6}$$

Together the operators equations (5) and (6) are commonly known as the regulator equations.

4. A NONDECAY CONDITION FOR THE EXOSYSTEM

In this section we present general conditions under which the characterization of the solvability of the output regulation problem presented in Theorem 1 remains valid. It turns out that for a general exosystem on a Banach space W, it is sufficient to assume that regardless of the choice of the operator F, the exosystem (3) may not generate reference signals that decay to zero asymptotically. This is precisely the content of the next definition.

Definition 2. The exosystem (3) is said to *satisfy the nondecay condition* if for all $Q \in \mathcal{L}(W, Y)$ and all $v_0 \in W$ we have

$$QT_S(t)v_0 \overset{t\to\infty}{\longrightarrow} 0 \quad \Rightarrow \quad QT_S(t_0)v_0 = 0 \;\; \forall t_0 \geq 0. \tag{7}$$

We will now show that Theorem 1 is true for exosystems satisfying the nondecay condition in Definition 2. For this we follow the treatment presented in Hämäläinen and Pohjolainen [2010], Paunonen and Pohjolainen [2010], and use the concept of the dynamic steady state to study the asymptotic behaviours of the state of the closed-loop system and of the regulation error.

4.1 The Dynamic Steady State

In this section we show that the solution of the Sylvester equation (5) can be used to express the state of the closed-loop system and the corresponding regulation error $e(t) = y(t) - y_{ref}(t)$. These formulas can be used to study the asymptotic behaviours of the state of the closed-loop system and of the regulation error and to ultimately prove Theorem 1 at the end of the section.

Theorem 3. Let $\Sigma \in \mathcal{L}(W, X_e)$ be a solution of the Sylvester equation (5). Then for all initial states $x_{e0} \in X_e$ and $v_0 \in W$ and for all $t \geq 0$ the state of the closed-loop system and the regulation error can be written as

$$x_e(t) = T_e(t)(x_{e0} - \Sigma v_0) + \Sigma v(t) \tag{8}$$

$$e(t) = C_e T_e(t)(x_{e0} - \Sigma v_0) + (C_e\Sigma + D_e)v(t). \tag{9}$$

Proof. If $\Sigma(\mathcal{D}(S)) \subset \mathcal{D}(A_e)$ and if the Sylvester equation (5) is satisfied, we have for any $v \in \mathcal{D}(S)$ and for all $t > s$

$$\begin{aligned} &T_e(t-s)B_eT_S(s)v = T_e(t-s)(\Sigma S - A_e\Sigma)T_S(s)v \\ = \;& -T_e(t-s)A_e\Sigma T_S(s)v + T_e(t-s)\Sigma S T_S(s)v \\ = \;& \frac{d}{ds}\left(T_e(t-s)\Sigma T_S(s)v\right). \end{aligned}$$

Integrating both sides of this equation from 0 to an arbitrary $t > 0$ gives

$$\int_0^t T_e(t-s)B_eT_S(s)v\,ds = \Sigma T_S(t)v - T_e(t)\Sigma v. \tag{10}$$

Since the operators on both sides of this equation are in $\mathcal{L}(W, X_e)$ and since $\mathcal{D}(S)$ is dense in W, we have that (10) holds for all $v \in W$ and $t > 0$.

For all $x_{e0} \in X_e$ and $v_0 \in W$ the mild state of the closed-loop system is given by

$$x_e(t) = T_e(t)x_{e0} + \int_0^t T_e(t-s)B_eT_S(s)v_0\,ds.$$

We can now use (10) to conclude that

$$\begin{aligned} x_e(t) &= T_e(t)x_{e0} + \Sigma T_S(t)v_0 - T_e(t)\Sigma v_0 \\ &= T_e(t)(x_{e0} - \Sigma v_0) + \Sigma T_S(t)v_0. \end{aligned}$$

The regulation error is therefore given by

$$\begin{aligned} e(t) &= C_e x_e(t) + D_e v(t) \\ &= C_e T_e(t)(x_{e0} - \Sigma v_0) + (C_e\Sigma + D_e)v(t). \end{aligned}$$

□

The formula (8) also allows us to describe the asymptotic behavior of a stable closed-loop system. Indeed, if the closed-loop system is strongly stable, then the first term on the right-hand side of (8) decays to zero for all initial states of the closed-loop system and the signal generator. Therefore the state of a strongly stable closed-loop system behaves asymptotically as

$$x_e(t) \sim \Sigma v(t).$$

For this reason, we call the mapping $t \mapsto \Sigma v(t)$ a *dynamic steady state* of the closed-loop system. Likewise, for a strongly stable closed-loop system the formula (9) analogously shows that the asymptotic behavior of the regulation error is given by

$$e(t) \sim (C_e\Sigma + D_e)v(t). \tag{11}$$

This expression also shows very clearly that the role of the regulation constraint (6) in Theorem 1 is to force the asymptotic behaviour of the regulation error to be zero.

The above properties of the dynamic steady state of the closed-loop system are summarized in the following corollary.

Corollary 4. Assume that the closed-loop system is strongly stable and that $\Sigma \in \mathcal{L}(W, X_e)$ is a solution of the Sylvester equation (5). Then for all initial states $x_{e0} \in X_e$ and $v_0 \in W$ of the closed-loop system and the signal generator the state of the closed-loop system and the regulation error satisfy

$$\lim_{t\to\infty} \|x_e(t) - \Sigma v(t)\| = 0 \tag{12a}$$

$$\lim_{t\to\infty} \|e(t) - (C_e\Sigma + D_e)v(t)\| = 0. \tag{12b}$$

So far we have not needed any assumptions on the exosystem generating the reference and disturbance signals. These will become necessary in the last step of the proof of Theorem 1. To prove Theorem 1 it only remains to verify that the asymptotic behavior (11) of the regulation error is equal to zero if and only if the regulation constraint is satisfied. The next lemma shows that this is true for all exosystems satisfying the nondecay condition in Definition 2.

Lemma 5. Assume the exosystem (3) satisfies the nondecay condition in Definition 2 and let $\Sigma \in \mathcal{L}(W, X_e)$. Then

$$\lim_{t\to\infty} (C_e\Sigma + D_e)T_S(t)v_0 = 0, \qquad \forall v_0 \in W$$

if and only if $C_e\Sigma + D_e = 0$.

Proof. It is clearly sufficient to prove the "only if" part of the lemma. Let $v_0 \in W$. If $(C_e\Sigma + D_e)T_S(t)v_0 \to 0$ as $t \to \infty$, then the nondecay condition implies that in particular for $t_0 = 0$ in (7) we have

$$0 = (C_e\Sigma + D_e)T_S(0)v_0 = (C_e\Sigma + D_e)v_0.$$

Since $v_0 \in W$ was arbitrary, we must have $C_e\Sigma + D_e = 0$. This concludes the proof. □

We can now collect the above results to prove Theorem 1.

Proof of Theorem 1. We will first show that (b) implies (a). Assume the regulation constraint (6) is satisfied. Since $T_e(t)$ is strongly stable, we have from Corollary 4 that for all initial states $x_{e0} \in X_e$ and $v_0 \in W$

$$\lim_{t\to\infty} \|e(t)\| = \lim_{t\to\infty} \|e(t) - (C_e\Sigma + D_e)v(t)\| = 0,$$

since $C_e\Sigma + D_e = 0$. Thus the controller solves the output regulation problem.

It remains to prove that (a) implies (b). Assume that the controller solves the output regulation problem and that $\Sigma \in \mathcal{L}(W, X_e)$ is a solution of the Sylvester equation (5). Since the regulation error decays to zero asymptotically for all initial states of the closed-loop system and the exosystem, Corollary 4 implies that for all $x_{e0} \in X_e$ and $v_0 \in W$ we must have

$$\begin{aligned}&\|(C_e\Sigma + D_e)T_S(t)v_0\| \\ &\qquad \le \|(C_e\Sigma + D_e)T_S(t)v_0 - e(t)\| + \|e(t)\| \stackrel{t\to\infty}{\longrightarrow} 0,\end{aligned}$$

and thus $\lim_{t\to\infty}(C_e\Sigma + D_e)T_S(t)v_0 = 0$ for every $v_0 \in W$. This together with Lemma 5 concludes that the regulation constraint (6) is satisfied. □

We close the section by showing that even though the solution of the Sylvester equation (5) in Theorem 1 may not be unique, the different solutions still lead to the same dynamic steady state of the closed-loop system.

Theorem 6. Assume the closed-loop system is strongly stable and the Sylvester equation (5) has a (possibly nonunique) solution. Then asymptotic the asymptotic behaviour of the state of the closed-loop system is unique.

Proof. Let $\Sigma_1 v(t)$ and $\Sigma_2 v(t)$ be two dynamic steady states of the closed-loop system corresponding to two solutions $\Sigma_1, \Sigma_2 \in \mathcal{L}(W, X_e)$ of the Sylvester equation (5) and the initial state $v_0 \in W$ of the exosystem. For any $v \in \mathcal{D}(S)$ we have

$$\begin{cases} \Sigma_1 S v = A_e \Sigma_1 v + B_e v \\ \Sigma_2 S v = A_e \Sigma_2 v + B_e v \end{cases}$$

$$\Rightarrow \qquad (\Sigma_1 - \Sigma_2)Sv = A_e(\Sigma_1 - \Sigma_2)v.$$

Denote $\Delta = \Sigma_1 - \Sigma_2$. For any $t > 0$ and $w \in \mathcal{D}(S)$

$$\begin{aligned}\Delta T_S(t)w - T_e(t)\Delta w &= \Big[T_e(t-s)\Delta T_S(s)w\Big]_{s=0}^{t} \\ &= \int_0^t \frac{d}{ds}\Big(T_e(t-s)\Delta T_S(s)w\Big)ds \\ &= \int_0^t T_e(t-s)\left(-A_e\Delta + \Delta S\right)T_S(s)w ds = 0\end{aligned}$$

and thus

$$(\Sigma_1 - \Sigma_2)T_S(t)w = T_e(t)(\Sigma_1 - \Sigma_2)w$$

for all $t \ge 0$. Since for all $t \ge 0$ the operators on both sides of the equation are in $\mathcal{L}(W, X_e)$ and since $\mathcal{D}(S)$ is dense in W, the above identity holds for all $w \in W$. This shows that the difference $\Sigma_1 v(t) - \Sigma_2 v(t)$ between the dynamic steady states satisfies

$$\begin{aligned}\|\Sigma_1 v(t) - \Sigma_2 v(t)\| &= \|(\Sigma_1 - \Sigma_2)T_S(t)v_0\| \\ &= \|T_e(t)(\Sigma_1 - \Sigma_2)v_0\| \to 0\end{aligned}$$

as $t \to \infty$ since $T_e(t)$ is strongly stable. This shows that the two dynamic steady states are the same and concludes the proof. □

5. MINIMALITY OF THE NONDECAY CONDITION

In this section we show that Theorem 1 may fail to be true only if the exosystem does not satisfy the nondecay condition in Definition 2. Since we saw in the previous section that for any stable closed-loop system the regulation error $e(t)$ behaves asymptotically as

$$e(t) \sim (C_e\Sigma + D_e)v(t),$$

it is evident that part (b) always implies part (a) in Theorem 1. The following result shows that part (b) may fail to follow from part (a) only if the exosystem does not satisfy the nondecay condition.

Theorem 7. Assume that the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ is such that A_e generates a strongly stable C_0-semigroup on X_e and that the Sylvester equation

$$\Sigma S = A_e\Sigma + B_e$$

has a solution $\Sigma \in \mathcal{L}(W, X_e)$ satisfying $\Sigma(\mathcal{D}(S)) \subset \mathcal{D}(A_e)$. If the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ solves the output regulation problem and if $C_e\Sigma + D_e \neq 0$, then the exosystem does not satisfy the nondecay condition in Definition 2.

Proof. Choose $Q = C_e\Sigma + D_e \in \mathcal{L}(W, Y)$ and let $v_0 \in W$ be such that $(C_e\Sigma + D_e)v_0 \neq 0$. Then Corollary 4 and the fact that the controller solves the output regulation problem imply

$$\begin{aligned}&\|QT_S(t)v_0\| = \|(C_e\Sigma + D_e)T_S(t)v_0\| \\ &\qquad \le \|(C_e\Sigma + D_e)T_S(t)v_0 - e(t)\| + \|e(t)\| \stackrel{t\to\infty}{\longrightarrow} 0,\end{aligned}$$

but $QT_S(0)v_0 = (C_e\Sigma + D_e)v_0 \neq 0$ by assumption. This concludes that the exosystem does not satisfy the nondecay condition. □

6. THE IMPLICATIONS OF NONDECAY CONDITIONS

In this section we study the consequences of the nondecay condition imposed on the exosystem. In particular we study the uniqueness of the solution of the Sylvester equation (5). For the main result giving conditions for the uniqueness of the solution of the Sylvester equation (5) we need a slightly modified version of the nondecay condition. The only difference between the condition in the next theorem compared to nondecay condition in Definition 2 is that now the range space of the "output operator" Q of the exosystem is assumed to be X_e instead of Y.

Theorem 8. Assume that the closed-loop system is strongly stable and that the semigroup $T_S(t)$ generated by the operator S is such that for all $Q \in \mathcal{L}(W, X_e)$ and all $v_0 \in W$ we have

$$QT_S(t)v_0 \stackrel{t\to\infty}{\longrightarrow} 0 \quad \Rightarrow \quad QT_S(t_0)v_0 = 0 \;\; \forall t_0 \ge 0. \tag{13}$$

Then the Sylvester equation $\Sigma S = A_e\Sigma + B_e$ may have at most one solution.

Proof. Assume the Sylvester equation has a possibly nonunique solution. In the proof of Theorem 6 we saw that if $\Sigma_1, \Sigma_2 \in \mathcal{L}(W, X_e)$ are two solutions of the equation, then for any $v_0 \in W$ we have

$$\|(\Sigma_1 - \Sigma_2)T_S(t)v_0\| = \|T_e(t)(\Sigma_1 - \Sigma_2)v_0\| \to 0$$

as $t \to \infty$ due to the strong stability of $T_e(t)$. Since the exosystem satisfies the nondecay condition (13) and since

$\Sigma_1 - \Sigma_2 \in \mathcal{L}(W, X_e)$, this implies that in particular for $t_0 = 0$ we obtain

$$0 = (\Sigma_1 - \Sigma_2)T_S(t_0)v_0 = (\Sigma_1 - \Sigma_2)v_0.$$

Since $v_0 \in W$ was arbitrary, this concludes that $\Sigma_1 = \Sigma_2$ and thus the Sylvester equation may have at most one solution. □

7. CONDITIONS FOR NONDECAY

In this section we state some necessary and sufficient conditions for the nondecay condition in Definition 2. The following theorem shows that if the reference and disturbance signals are *almost periodic* functions (see Arendt et al. [2001] for the definition), then the exosystem satisfies the nondecay condition.

Theorem 9. If for all $Q \in \mathcal{L}(W,Y)$ and $v_0 \in W$ the functions $QT_S(\cdot)v_0$ are almost periodic functions, then the nondecay condition in Definition 2 is satisfied.

Proof. Let $Q \in \mathcal{L}(W,Y)$ and $v_0 \in W$ be such that $QT_S(t)v_0 \to 0$ as $t \to \infty$. Since $QT_S(\cdot)v_0$ is an almost periodic function, there exists a sequence $(t_n)_{\mathbb{N}}$ such that $t_n \to \infty$ and $\|QT_S(t_n+s)v_0 - QT_S(s)\| < \frac{1}{n}$ for all $s \in \mathbb{R}$ and $n \in \mathbb{N}$ [Arendt et al., 2001, Sec. 4.5]. Therefore for all $t_0 \geq 0$ we have

$$\begin{aligned}\|QT_S(t_0)v_0\| \leq\ & \|QT_S(t_n+t_0)v_0 - QT_S(t_0)v_0\| \\ & + \|QT_S(t_n+t_0)v_0\| \\ <\ & \frac{1}{n} + \|QT_S(t_n+t_0)v_0\| \to 0\end{aligned}$$

as $n \to \infty$. This implies $QT_S(t_0)v_0 = 0$. Since $t_0 \geq 0$ was arbitrary, the condition (7) is satisfied. This concludes the proof. □

The next theorem states that an exosystem consisting of an infinite number of finite-dimensional Jordan blocks satisfies the nondecay condition. Such exosystems have been studied in Paunonen and Pohjolainen [2010, Submitted]. An exosystem containing nontrivial Jordan blocks is capable of generating polynomially increasing reference and disturbance signals.

Theorem 10. The infinite-dimensional Jordan exosystem studied in Paunonen and Pohjolainen [2010] satisfies the nondecay condition in Definition 2.

Proof. Since the spaces $W_k = \operatorname{span}\{\phi_k^l\}_{l=1}^{n_k}$ are $T_S(t)$-invariant for all $k \in \mathbb{Z}$, we can consider $v_0 \in W_k$ separately for $k \in \mathbb{Z}$. For any $Q \in \mathcal{L}(W,Y)$ and for all $k \in \mathbb{Z}$ and $v_0 \in W_k$ we have

$$QT_S(t)v_0 = e^{i\omega_k t}\sum_{l=1}^{n_k}\langle v_0, \phi_k^l\rangle \sum_{j=1}^{l}\frac{t^{l-j}}{(l-j)!}Q\phi_k^j \qquad (14a)$$

$$= e^{i\omega_k t}\sum_{j=0}^{n_k-1} t^j \cdot \frac{1}{j!}\sum_{l=j+1}^{n_k}\langle v_0, \phi_k^l\rangle Q\phi_k^{l-j} \qquad (14b)$$

If $QT_S(t)v_0 \to 0$, it is easy to see that we must have

$$\sum_{l=j+1}^{n_k}\langle v_0, \phi_k^l\rangle Q\phi_k^{l-j} = 0 \qquad \forall j \in \{0, \ldots, n_k - 1\}.$$

However, by (14) this also implies $QT_S(t_0)v_0 = 0$ for all $t_0 \geq 0$. □

Remark. It should be noted that in the case where all of the Jordan blocks of the exosystem have dimensions $n_k = 1$, the exosystem in Paunonen and Pohjolainen [2010] generates almost periodic functions, and also Theorem 9 could be used to conclude that the exosystem satisfies the nondecay condition.

The following simple example shows that the nondecay condition may not be satisfied even if S generates a strongly continuous group of isometries on a Hilbert space. In this case the group $T_S(t)$ is completely unstable, i.e., $T_S(t)v_0 \to 0$ as $t \to \infty$ if and only if $v_0 = 0$, but for all $F \in \mathcal{L}(W, \mathbb{C})$ we have $FT_S(t)v_0 \to 0$ as $t \to \infty$.

Counterexample 11. Let $W = L^2(a,b)$ for some $a < b$ and let $S \in \mathcal{L}(W)$ be a multiplication operator defined by $(Sv)(\xi) = i\xi v(\xi)$ for all $v \in W$. It is easy to see that $\sigma(S) = \sigma_c(S) = [ia, ib] \subset i\mathbb{R}$ and that S is skew-adjoint. The operator S generates a multiplication group $T_S(t)$ defined by $(T_S(t)v)(\xi) = e^{i\xi t}v(\xi)$ on W, and this group is isometric since

$$\|T_S(t)v\|_{L^2}^2 = \int_a^b |e^{i\xi t}v(\xi)|^2 d\xi = \int_a^b |v(\xi)|^2 d\xi = \|v\|_{L^2}^2$$

for all $v \in W$ and $t \in \mathbb{R}$.

Let $Y = \mathbb{C}$ and $Q \in \mathcal{L}(W, \mathbb{C})$. By the Riesz Representation Theorem there exists $w \in W$ such that

$$Qv = \int_a^b v(\xi)w(\xi)d\xi, \qquad \forall v \in W.$$

For any initial state $v_0 \in W$ we then have

$$QT_S(t)v_0 = \int_a^b e^{i\xi t}v_0(\xi)w(\xi)d\xi \to 0$$

as $t \to \infty$ due to the Riemann-Lebesgue Lemma, since $v_0(\cdot)w(\cdot) \in L^1(a,b)$. This concludes that this exosystem does not satisfy the nondecay condition in Definition 2.

In the above example, the nondecay condition failed because the semigroup $T_S(t)$ was weakly stable. The following result shows that already a single weakly stable orbit $T_S(t)v_0$ with $v_0 \neq 0$ contradicts the nondecay condition.

Theorem 12. If $T_S(t)$ satisfies the nondecay condition, then it has no weakly stable orbits $T_S(t)v_0$ with $v_0 \neq 0$.

Proof. Assume $v_0 \neq 0$ is such that $T_S(t)v_0$ is a weakly stable orbit. Take $y \in Y$ such that $y \neq 0$ and $w \in W^*$ such that $\langle v_0, w\rangle \neq 0$, and choose $Q = \langle \cdot, w\rangle y$. We now have

$$QT_S(t)v_0 = \langle T_S(t)v_0, w\rangle y \to 0 \quad \text{as} \quad t \to \infty$$

due to the weak stability of the orbit $T_S(t)v_0$. However, for $t_0 = 0$ we have $QT_S(t_0)v_0 = \langle v_0, w\rangle y \neq 0$, and thus the semigroup $T_S(t)$ does not satisfy the nondecay condition (7). This concludes the proof. □

If W is a Hilbert space and if $Y = \mathbb{C}^p$, then the Riesz Representation Theorem implies that any output operator $Q \in \mathcal{L}(W,Y)$ is necessarily of the form

$$Qv = (\langle v, w_1\rangle,\ \langle v, w_2\rangle,\ \ldots,\ \langle v, w_p\rangle)^T, \qquad v \in W$$

for some $\{w_k\}_{k=1}^p \subset W$. In such a case the nondecay condition in Definition 2 is therefore equivalent to the requirement that the semigroup $T_S(t)$ satisfies

$$\langle T_S(t)v_0, w\rangle \xrightarrow{t\to\infty} 0 \quad \Rightarrow \quad \langle T_S(t_0)v_0, w\rangle = 0 \ \ \forall t_0 \geq 0.$$

for all $w, v_0 \in W$.

8. CONCLUSIONS

In this paper we have studied the theory of output regulation of distributed parameter systems in the case where the exosystem itself is a very general infinite-dimensional linear system. In particular we showed that in order to characterize the solvability of the output regulation problem it is sufficient to assume that the exosystem does not produce signals that decay to zero asymptotically.

The results concerning the regulator equations and the solvability of the Sylvester equations can be readily extended for unbounded solutions Σ. In Paunonen and Pohjolainen [2010, Submitted] the authors of the current paper demonstrated that allowing the solution of the regulator equations to be an operator in $\Sigma \in \mathcal{L}(\mathcal{D}(S^m), X_e)$ for some $m \in \mathbb{N}_0$ allows weakening of the conditions for the solvability of the output regulation problem. It was also shown that the solutions in this class require that the considered reference and disturbance signals correspond to initial states $v_0 \in \mathcal{D}(S^m)$ of the exosystem. For diagonal and block diagonal exosystems this in some cases directly corresponds to setting minimal requirements for the smoothness of the considered exogeneous signals, see Paunonen and Pohjolainen [Submitted].

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