Output Regulation of Distributed Parameter Systems With Time-Periodic Exosystems

Lassi Paunonen and Seppo Pohjolainen

***Abstract*— In this paper the output regulation of a linear distributed parameter system with a nonautonomous periodic exosystem is considered. It is shown that the solvability of the output regulation problem can be characterized by an infinite-dimensional Sylvester differential equation. Conditions are given for the existence of a controller solving the regulation problem along with a method for its construction.**

I. Introduction

Output regulation of distributed parameter systems has been studied extensively during the last thirty years [15], [17], [1], [3], [16] and many of the most essential results of finite-dimensional control theory have been generalized to infinite-dimensional systems. Even though the theory has been extended to infinite-dimensional systems, it has been customary to assume that the reference and disturbance signals are generated by a finite-dimensional exosystem

$$\dot{v} = Sv, \qquad v(0) = v_0 \in W. \tag{1}$$

Recently there has been interest in generalizing the theory to include more general classes of reference and disturbance signals [7] and in studying how general signals can a distributed parameter system regulate [8]. The direction of this research on more general signal generators has been to allow the signal generator (1) to be an infinite-dimensional differential equation and the operator S to be a diagonal [7], [5] or a block-diagonal [13], [12] generator of a C_0-group on a Hilbert space W.

In this paper we take a different direction in generalizing the classes of signals to be regulated by considering output regulation of a linear distributed parameter system together with a periodic nonautonomous exosystem. By this we mean that the signals to be regulated and rejected are generated by an exosystem of form

$$\dot{v}(t) = S(t)v(t), \qquad v(0) = v_0,$$

where $S(t)$ is a periodic function, i.e. there exists $T > 0$ such that $S(t + T) = S(t)$ for all $t \in \mathbb{R}$. Using this type of exosystems it is possible, for example, to regulate and reject all the usual signals generated by a finite-dimensional time-invariant exosystem but in addition it is possible to consider signals containing periodically modulated frequencies and signals related to other oscillatory phenomena, such as parametric resonance.

L. Paunonen and S. Pohjolainen are with Department of Mathematics, Tampere University of Technology, PO. Box 553, 33101 Tampere, Finland. Email: `lassi.paunonen@tut.fi`, `seppo.pohjolainen@tut.fi`

The theory presented in this paper generalizes the theory of output regulation of linear finite-dimensional time-invariant systems together with time-periodic exosystems presented in [18] to distributed parameter systems. On the other hand, the results also generalize the theory of output regulation of time-invariant distributed parameter systems developed in [4], [12] to the case of a time-dependent exosystem.

We consider a time-invariant infinite-dimensional system whose system operator generates an analytic semigroup and a finite-dimensional periodic signal generator. Our approach leads to an infinite-dimensional nonautonomous closed-loop system. To deal with these types of systems we use the theory of *evolution families* [2], [14] associated to nonautonomous Cauchy problems on Banach spaces.

Our main result is that the solvability of the periodic output regulation problem can be characterized by the properties of the periodic solution of a certain infinite-dimensional *Sylvester differential equation*. This result is a generalization of the results in the time-invariant case where the solvability of the output regulation problem can be characterized by the solvability of a constrained Sylvester equation [5].

One of the most essential tools in the theory are the infinite-dimensional Sylvester differential equations of form

$$\dot{\Sigma}(t) + \Sigma(t)S = A_e(t)\Sigma(t) + B_e(t) \tag{2}$$

where $A_e(t)$ and $B_e(t)$ are T-periodic operator-valued functions of the closed-loop system and S is a finite-dimensional operator associated to the signal generator. In order to use these equations we need results on their solvability. To this end we show that under suitable assumptions the Syvester differential equation (2) has a unique periodic solution.

In addition to characterizing the solvability of the periodic output regulation problem, we also present conditions for the existence of an observer-based controller solving the regulation problem and a method for its construction. This is done by generalizing a controller for time-invariant finite-dimensional linear systems given in [6] to our case.

The paper is organized as follows. In Section II we introduce notation, recall the definition of a strongly continuous evolution family and state the basic assumptions on the system, the exosystem and the controller. In Section III we formulate the periodic output regulation problem for infinite-dimensional systems and characterize its solvability using the properties of certain infinite-dimensional Sylvester differential equations. In Section IV we construct an observer-based controller solving the periodic output regulation problem. Section V contains concluding remarks.

II. Notation and Definitions

If X and Y are Banach spaces and $A : X \to Y$ is a linear operator, we denote by $\mathcal{D}(A)$ and $\mathcal{R}(A)$ the domain and the range of A, respectively. The space of bounded linear operators from X to Y is denoted by $\mathcal{L}(X,Y)$. If $A : X \to X$, then $\sigma(A)$ and $\rho(A)$ denote the spectrum and the resolvent set of A, respectively. For $\lambda \in \rho(A)$ the resolvent operator is given by $R(\lambda, A) = (\lambda I - A)^{-1}$. The space of continuous functions $f : \mathbb{R} \to X$ is denoted by $C(\mathbb{R}, X)$. The space of T-periodic continuous functions is defined as

$$C_T(\mathbb{R}, X) = \{ f : \mathbb{R} \to X \mid f \text{ is continuous}, \\ f(t+T) = f(t) \quad \forall t \in \mathbb{R} \} .$$

Similarly we denote by $C_T^1(\mathbb{R}, X)$ the space of T-periodic continuously differentiable functions. We denote by $C_T(\mathbb{R}, \mathcal{L}(X,Y))$ and $C_T(\mathbb{R}, \mathcal{L}_s(X,Y))$ the spaces of T-periodic functions with values in $\mathcal{L}(X,Y)$ which are uniformly or strongly continuous, respectively. Analogously we denote by $C_T^1(\mathbb{R}, \mathcal{L}(X,Y))$ and $C_T^1(\mathbb{R}, \mathcal{L}_s(X,Y))$ the spaces of functions which are continuously differentiable with respect to the uniform and strong operator topologies of $\mathcal{L}(X,Y)$, respectively. For an operator-valued function $A(\cdot) \in C_T(\mathbb{R}, \mathcal{L}_s(X,Y))$ we denote

$$\|A\|_\infty = \sup_{t \in [0,T]} \|A(t)\|.$$

In dealing with infinite-dimensional nonautonomous systems we need the concept of a strongly continuous evolution family [14, Ch. 5], [2, Sec. VI.9].

Definition 1 (A Strongly Continuous Evolution Family): A family of bounded operators $(U(t,s))_{t \geq s} \subset \mathcal{L}(X)$ is called a *strongly continuous evolution family* if

(a) $U(s,s) = I$ for $s \in \mathbb{R}$;
(b) $U(t,s) = U(t,r)U(r,s)$ for $t \geq r \geq s$;
(c) $\{ (t,s) \in \mathbb{R}^2 \mid t \geq s \} \ni (t,s) \mapsto U(t,s)$ is a strongly continuous mapping.

A strongly continuous evolution family is called *exponentially bounded* if there exists constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$\|U(t,s)\| \leq M e^{\omega(t-s)}$$

for all $t \geq s$. ■

Definition 2 (Parabolic Conditions): We say that a T-periodic family $(A(t))_{t \in \mathbb{R}}$ of linear operators on X satisfies the *parabolic conditions* if the following are satisfied for some $\mu \in \mathbb{R}$.

(P_1) The domain $\mathcal{D}(A(t)) =: \mathcal{D}(A)$ is independent of $t \in \mathbb{R}$ and dense in X.

(P_2) We have $\{ \lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \geq \mu \} \subset \rho(A(t))$ for every $t \in [0,T]$ and there exists a constant $M \geq 1$ such that

$$\|R(\lambda, A(t))\| \leq \frac{M}{|\lambda - \mu| + 1}, \qquad \operatorname{Re}\lambda \geq \mu, \ t \in [0,T].$$

(P_3) There exists a constant $L \geq 0$ such that for $t, s, r \in \mathbb{R}$

$$\|(A(t) - A(s))R(\mu, A(r))\| \leq L|t-s|. \qquad ■$$

The following lemma states a few useful properties of evolution families satisfying the parabolic conditions. [14, Ch. 5.6-8]

Lemma 3: If an operator family $(A(t))$ satisfies the parabolic conditions for some $\mu \in \mathbb{R}$, then there exists a unique exponentially bounded evolution family $U(t,s)$ such that $\mathcal{R}(U(t,s)) \subset \mathcal{D}(A)$, $\frac{\partial}{\partial t}U(t,s)x = A(t)U(t,s)x$ and $\frac{\partial}{\partial s}U(t,s)y = -U(t,s)A(s)y$ for all $t > s$, $x \in X$ and $y \in \mathcal{D}(A)$. Furthermore, if $f \in L^1(\mathbb{R}, X)$, the abstract Cauchy problem

$$\dot{x}(t) = A(t)x(t) + f(t), \qquad x(0) = x_0$$

has a unique mild solution

$$x(t) = U(t,0)x_0 + \int_0^t U(t-s)f(s)ds$$

for every $x_0 \in X$ and $t \geq 0$. If f is Hölder continuous, then $x(t)$ is the unique classical solution of the abstract Cauchy problem.

Throughout this paper we consider a linear distributed parameter system

$$\dot{x} = Ax + Bu + w_s(t), \qquad x(0) = x_0 \tag{3a}$$
$$y = Cx + Du + w_m(t) \tag{3b}$$

on a Banach space X. We assume A generates an analytic semigroup $T(t)$ on X and the rest of the operators are bounded, $B \in \mathcal{L}(U,X)$, $C \in \mathcal{L}(X,Y)$, $D \in \mathcal{L}(U,Y)$ where U and Y are finite-dimensional spaces. The disturbance signals $w_s(t)$ and $w_m(t)$ and the reference signal $y_{ref}(t)$ are generated by a time-periodic exosystem

$$\dot{w} = S(t)w, \qquad w(0) = w_0 \tag{4}$$

on $W = \mathbb{C}^q$ where $S(\cdot) \in C_T^1(\mathbb{R}, \mathcal{L}(W))$. The reference and disturbance signals are given by

$$w_s(t) = E_s(t)w(t), \qquad w_m(t) = F_m(t)w(t)$$
$$y_{ref}(t) = F_{ref}(t)w(t),$$

where $E_s(\cdot) \in C_T^1(\mathbb{R}, \mathcal{L}(W,X))$, $F_m(\cdot) \in C_T^1(\mathbb{R}, \mathcal{L}(W,Y))$ and $F_{ref}(\cdot) \in C_T^1(\mathbb{R}, \mathcal{L}(W,Y))$.

The Floquet-Lyapunov theory [9] states that there exists a constant matrix $S \in \mathbb{C}^{q \times q}$ and a T-periodic function $L_{FL}(\cdot) \in C_T^1(\mathbb{R}, \mathcal{L}(W))$ such that $L_{FL}(0) = I$, $L_{FL}(t)$ is invertible for every $t \in \mathbb{R}$ and if $U_S(t,0)$ is the fundamental matrix of (4), then

$$U_S(t,0) = L_{FL}(t)e^{St}.$$

The Lyapunov Reducibility Theorem [10] states that the change of variables $v(t) = L_{FL}(t)^{-1}w(t)$ transforms the time-periodic exosystem (4) into

$$\dot{v} = Sv, \qquad v(0) = v_0 \tag{5}$$

where $v_0 = L_{FL}(0)^{-1}w_0 = w_0$. We can also assume that S is in Jordan canonical form, because if this is not the case, we can write $S = TJT^{-1}$ and apply a the time-independent change of variables $\tilde{v} = T^{-1}v$ to obtain a system $\dot{\tilde{v}} = J\tilde{v}$.

We consider asymptotic regulation and disturbance rejection and because of this we are not concerned with reference and disturbance signals which decay asymptotically. We can therefore assume without loss of generality that the eigenvalues of the monodromy matrix $U_S(T,0)$ of $S(\cdot)$ have magnitude greater or equal to one. Since $U_S(T,0) = e^{ST}$, this is equivalent to the following assumption.

Assumption 4: We have $\operatorname{Re}\lambda \geq 0$ *for all* $\lambda \in \sigma(S)$.

Defining $E(t) = E_m(t)L_{FL}(t)$ and

$$F(t) = F_m(t)L_{FL}(t) - F_{ref}(t)L_{FL}(t)$$

for all $t \in [0,T]$ we have that $E(\cdot) \in C^1_T(\mathbb{R}, \mathcal{L}(W,X))$ and $F(\cdot) \in C^1_T(\mathbb{R}, \mathcal{L}(W,Y))$. The system (3) can now be written in a standard form

$$\dot{x} = Ax + Bu + E(t)v, \qquad x(0) = x_0 \tag{6a}$$

$$e = Cx + Du + F(t)v \tag{6b}$$

with the exosystem (5). We consider a T-periodic feedback controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ given by the equations

$$\dot{z} = \mathcal{G}_1(t)z + \mathcal{G}_2(t)e, \qquad z(0) = z_0 \tag{7a}$$

$$u = K(t)z \tag{7b}$$

on a Banach space Z where the operators $(\mathcal{G}_1(t))$ satisfy the parabolic conditions, and we have $\mathcal{G}_2(\cdot) \in C^1_T(\mathbb{R}, \mathcal{L}(Y,Z))$ and $K(\cdot) \in C^1_T(\mathbb{R}, \mathcal{L}_s(Z,U))$.

The closed-loop system on the Banach space $X_e = X \times Z$ with state $x_e(t) = (x(t), z(t))^T \in X_e$ can be written as

$$\dot{x}_e = A_e(t)x_e + B_e(t)v, \qquad x_e(0) = x_{e0} \tag{8a}$$

$$y = C_e(t)x_e + D_e(t)v \tag{8b}$$

where

$$A_e(t) = \begin{bmatrix} A & BK(t) \\ \mathcal{G}_2(t)C & \mathcal{G}_1(t) + \mathcal{G}_2(t)DK(t) \end{bmatrix},$$

$$B_e(t) = \begin{bmatrix} E(t) \\ \mathcal{G}_2(t)F(t) \end{bmatrix},$$

$C_e(t) = \begin{bmatrix} C & DK(t) \end{bmatrix}$ and $D_e(t) = F(t)$. Since the operators $A_e(t)$ are of form $A_e(t) = A^0_e(t) + A^1_e(t)$ where $(A^0_e(t))$ satisfy the parabolic conditions and $A^1_e(\cdot) \in C^1_T(\mathbb{R}, \mathcal{L}_s(X_e))$, it is straight-forward to verify that also the family $(A_e(t))$ of operators satisfies the parabolic conditions and thus there exists a strongly continuous evolution family $U_e(t,s)$ associated to the closed-loop system. For all initial states $x_{e0} \in X_e$ and $v_0 \in W$ the state of the closed-loop system is given by

$$x_e(t) = U_e(t,0)x_{e0} + \int_0^t U_e(t,s)B_e(s)v(s)ds.$$

III. Main Problem

The *Periodic Output Regulation Problem (PORP)* is stated as follows.

Problem 1 (Periodic Output Regulation Problem): Choose the parameters of a T-periodic controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ in such a way that the following are satisfied:

1) The evolution family $U_e(t,s)$ is exponentially stable, i.e. there exist constants $M_e \geq 1$ and $\omega_e > 0$ such that $\|U_e(t,s)\| \leq M_e e^{-\omega_e(t-s)}$;
2) For all initial values $x_{e0} \in X_e$ and $v_0 \in W$ the closed-loop system (8) and the exosystem (5) the regulation error satisfies

$$\lim_{t\to\infty} e(t) = 0. \qquad \blacksquare$$

To solve the Periodic Output Regulation Problem we need to consider the periodic Sylvester differential equation

$$\dot{\Sigma}(t) + \Sigma(t)S = A_e(t)\Sigma(t) + B_e(t)$$

on the interval $[0,T]$. By a *periodic solution* of this equation we mean a function $\Sigma(\cdot) \in C^1_T(\mathbb{R}, \mathcal{L}(W, X_e))$ such that $\mathcal{R}(\Sigma(t)) \subset \mathcal{D}(A_e)$ for all $t \in [0,T]$.

The following theorem is the main result of this paper. It characterizes the controllers solving the Periodic Output Regulation Problem in terms of the behaviour of the periodic solution of an infinite-dimensional Sylvester differential equation.

Theorem 5: Assume the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ *stabilizes the closed-loop system. The periodic Sylvester differential equation*

$$\dot{\Sigma}(t) + \Sigma(t)S = A_e(t)\Sigma(t) + B_e(t) \tag{9}$$

has a unique periodic solution $\Sigma_\infty(\cdot) \in C^1_T(\mathbb{R}, \mathcal{L}(W, X_e))$ *such that* $\mathcal{R}(\Sigma_\infty(t)) \subset \mathcal{D}(A_e)$ *for every* $t \in [0,T]$. *The controller* $(\mathcal{G}_1, \mathcal{G}_2, K)$ *solves the PORP if and only if this solution satisfies*

$$C_e(t)\Sigma_\infty(t) + D_e(t) = 0$$

for all $t \in [0,T]$.

The proof of the theorem is divided into parts. We will first show that under our assumptions the periodic Sylvester differential equation (9) has a unique periodic solution if the closed-loop system is stable. This is done in the following lemma. Subsequently in Lemma 7, we will show that the behaviour of the regulation error can be described using the solution of the Sylvester differential equation. After proving these two lemmas we present the proof of Theorem 5.

Lemma 6: Under the assumptions of Theorem 5 the periodic Sylvester differential equation (9) *has a unique periodic solution*

$$\Sigma_\infty(t) = \int_{-\infty}^t U_e(t,s)B_e(s)e^{S(s-t)}ds.$$

Proof: We will first show that the unique solution $\Sigma(\cdot) \in C^1([0,2T], \mathcal{L}(W, X_e))$ of (9) corresponding to the initial condition $\Sigma(0) = \Sigma_0 \in \mathcal{L}(W, X_e)$ is given by

$$\Sigma_e(t) = U_e(t,0)\Sigma_0 e^{-St} + \int_0^t U_e(t,s)B_e(s)e^{S(s-t)}ds \tag{10}$$

on $[0,2T]$ and that $\mathcal{R}(\Sigma_e(t)) \subset \mathcal{D}(A_e)$ for every $t \in (0,T]$.

Let $\lambda \in \sigma(S)$ and let $\{\phi_k\}_{k=1}^m$ be an orthonormal Jordan chain associated to this eigenvalue. We will show using induction that for every $k \in \{1,\dots,m\}$ the initial value problem

$$\frac{d}{dt}\Sigma_e(t)\phi_k + \Sigma_e(t)S\phi_k = A_e(t)\Sigma_e(t)\phi_k + B_e(t)\phi_k \tag{11a}$$

$$\Sigma_e(0)\phi_k = \Sigma_0\phi_k \tag{11b}$$

has a unique classical solution $\Sigma_e(t)\phi_k \in C^1([0,2T],X_e)$

$$\Sigma_e(t)\phi_k = U_e(t,0)\Sigma_0 e^{-St}\phi_k + \int_0^t U_e(t,s)B_e(s)e^{S(s-t)}\phi_k ds$$

on $[0,2T]$ and that $\mathcal{R}(\Sigma_e(t)) \subset \mathcal{D}(A_e)$ for all $t \in (0,T]$. Since the $\lambda \in \sigma(S)$ and the associated Jordan chain are arbitrary and since the Jordan chains of S form a basis of the space W, the result on W then follows from linearity. Since we assumed that S is in its Jordan canonical form we have that for $k \in \{1,\dots,m\}$ and $t \in \mathbb{R}$

$$e^{St}\phi_k = e^{\lambda t}\sum_{l=1}^{k}\frac{t^{k-l}}{(k-l)!}\phi_l$$

For $k=1$ the initial value problem (11) becomes

$$\frac{d}{dt}\Sigma_e(t)\phi_1 = (A_e(t)-\lambda I)\Sigma_e(t)\phi_1 + B_e(t)\phi_1,$$
$$\Sigma_e(0)\phi_1 = \Sigma_0\phi_1.$$

The evolution family associated to the operators $A_e(t)-\lambda I$ is given by $e^{-\lambda(t-s)}U_e(t,s)$ and we thus have from Theorem 5.7.1 in [14] that since $B_e(\cdot)\phi_1 \in C_T^1(\mathbb{R},X_e)$ this equation has a unique classical solution $\Sigma_e(t)\phi_1$ given by

$$\begin{aligned} & e^{-\lambda t}U_e(t,0)\Sigma_0\phi_1 + \int_0^t e^{-\lambda(t-s)}U(t,s)B_e(s)\phi_1 ds \\ = \ & U_e(t,0)\Sigma_0 e^{-St}\phi_1 + \int_0^t U(t,s)B_e(s)e^{S(s-t)}\phi_1 ds \end{aligned}$$

on $[0,2T]$ and that $\Sigma_e(t)\phi_1 \in \mathcal{D}(A_e)$ for all $t \in (0,T]$. Thus the claim holds for $k=1$. Assume now that for some $k \in \{1,\dots,m-1\}$ the initial value problem (11) has a unique classical solution $\Sigma_e(t)\phi_k$ given by

$$\begin{aligned} & U_e(t,0)\Sigma_0 e^{-St}\phi_k + \int_0^t U_e(t,s)B_e(s)e^{S(s-t)}\phi_k ds \\ = \ & e^{-\lambda t}U_e(t,0)\Sigma_0\sum_{l=1}^{k}\frac{(-t)^{k-l}}{(k-l)!}\phi_l \\ & + \int_0^t e^{-\lambda(t-s)}U_e(t,s)B_e(s)\sum_{l=1}^{k}\frac{(s-t)^{k-l}}{(k-l)!}\phi_l ds. \end{aligned}$$

on $[0,2T]$. Since $(S-\lambda I)\phi_{k+1} = \phi_k$, we have that for $k+1$ the initial value problem (11) becomes

$$\begin{aligned} \frac{d}{dt}\Sigma_e(t)\phi_{k+1} &= (A_e(t)-\lambda I)\Sigma_e(t)\phi_{k+1} - \Sigma_e(t)\phi_k \\ & \quad + B_e(t)\phi_{k+1}, \\ \Sigma_e(0)\phi_{k+1} &= \Sigma_0\phi_{k+1}. \end{aligned}$$

Since $\Sigma_e(\cdot)\phi_k, B_e(\cdot)\phi_{k+1} \in C^1([0,2T],X_e)$ we have from Theorem 5.7.1 in [14] that this equation has a unique classical solution given by

$$\begin{aligned} & \Sigma_e(t)\phi_{k+1} = e^{-\lambda t}U_e(t,0)\Sigma_0\phi_{k+1} \\ & \quad + \int_0^t e^{-\lambda(t-s)}U_e(t,s)\left(-\Sigma_e(s)\phi_k + B_e(s)\phi_{k+1}\right)ds \\ = \ & e^{-\lambda t}U_e(t,0)\Sigma_0\phi_{k+1} \\ & - \sum_{l=1}^{k}\int_0^t \frac{(-s)^{k-l}}{(k-l)!}e^{-\lambda(t-s)}U_e(t,s)e^{-\lambda s}U_e(s,0)\Sigma_0\phi_l ds \\ & - \sum_{l=1}^{k}\int_0^t e^{-\lambda(t-s)}U_e(t,s) \\ & \qquad \times \int_0^s \frac{(r-s)^{k-l}}{(k-l)!}e^{-\lambda(s-r)}U_e(s,r)B_e(r)\phi_l dr ds \\ & + \int_0^t e^{-\lambda(t-s)}U_e(t,s)B_e(s)\phi_{k+1}ds \\ = \ & e^{-\lambda t}U_e(t,0)\Sigma_0\phi_{k+1} \\ & - \sum_{l=1}^{k}e^{-\lambda t}U_e(t,0)\Sigma_0\phi_l\int_0^t \frac{(-s)^{k-l}}{(k-l)!}ds \\ & - \sum_{l=1}^{k}\int_0^t e^{-\lambda(t-r)}U_e(t,r)B_e(r)\phi_l\int_r^t \frac{(r-s)^{k-l}}{(k-l)!}ds dr \\ & + \int_0^t e^{-\lambda(t-s)}U_e(t,s)B_e(s)\phi_{k+1}ds \\ = \ & U_e(t)\Sigma_0 e^{-\lambda t}\sum_{l=1}^{k+1}\frac{(-t)^{k+1-l}}{(k+1-l)!}\phi_l \\ & + \int_0^t U_e(t,r)B_e(r)e^{-\lambda(r-t)}\sum_{l=1}^{k}\frac{(r-t)^{k+1-l}}{(k+1-l)!}\phi_l dr \\ = \ & U_e(t,0)\Sigma_0 e^{-St}\phi_{k+1} + \int_0^t U_e(t,r)B_e(r)e^{S(r-t)}\phi_{k+1}dr \end{aligned}$$

on $[0,2T]$ and that $\Sigma_e(t)\phi_{k+1} \in \mathcal{D}(A_e)$ for all $t \in (0,T]$. This shows that the claim holds for $k+1$ and concludes the proof that the unique solution of the Sylvester differential equation corresponding to the initial condition $\Sigma_e(0) = \Sigma_0$ is given by (10) on $[0,2T]$.

We will now show that $\Sigma_\infty(t)$ is a solution of the Sylvester differential equation on $[0,2T]$. Since

$$\begin{aligned} \Sigma_\infty(t) = U_e(t,0)\int_{-\infty}^0 U_e(0,s)B_e(s)e^{Ss}e^{-St}ds \\ + \int_0^t U_e(t,s)B_e(s)e^{S(s-t)}ds, \end{aligned}$$

it suffices to show that $\Sigma_\infty(0) = \int_{-\infty}^0 U_e(0,s)B_e(s)e^{Ss}ds$ is in $\mathcal{L}(W,X_e)$. Since the closed-loop system is stable, there exist constants $M_e \geq 1$ and $\omega_e < 0$ such that for all $t \geq s$ we have $\|U_e(t,s)\| \leq M_e e^{\omega_e(t-s)}$. Assumption 4 on the other hand implies that there exists a constant $M_S \geq 0$ such that

$\|e^{Ss}\| \leq M_S$ for all $s \leq 0$. Now

$$\|\int_{-\infty}^{0} U_e(0,s)B_e(s)e^{Ss}ds\| \leq \int_{-\infty}^{0} \|U_e(0,s)B_e(s)e^{Ss}\|ds$$
$$\leq M_e M_S \cdot \sup_{t\leq 0}\|B_e(t)\| \int_{-\infty}^{0} e^{-\omega_e s}ds = \frac{M_e M_S \cdot \|B_e\|_\infty}{|\omega_e|}.$$

and thus $\Sigma_\infty(0) \in \mathcal{L}(W, X_e)$.

To prove the periodicity of $\Sigma_\infty(t)$, let $t \in \mathbb{R}$. Then

$$\Sigma_\infty(t+T) = \int_{-\infty}^{t+T} U_e(t+T,s)B_e(s)e^{S(s-(t+T))}ds$$
$$= \int_{-\infty}^{t} U_e(t+T,s+T)B_e(s+T)e^{S(s+T-(t+T))}ds$$
$$= \int_{-\infty}^{t} U_e(t,s)B_e(s)e^{S(s-t)}ds = \Sigma_\infty(t).$$

This shows that $\Sigma_\infty(t)$ is periodic.

We have now proved that $\mathcal{R}(\Sigma_\infty(t)) \subset \mathcal{D}(A_e)$ for all $t \in (0,T]$ and that $\Sigma_\infty(t)$ is the unique solution of the Sylvester differential equation (9) on $[0,2T]$ corresponding to the initial condition $\Sigma_\infty(0)$ at $t=0$. Since $\Sigma_\infty(t)$ is periodic we have $\Sigma_\infty(\cdot) \in C_T^1(\mathbb{R}, \mathcal{L}(W,X_e))$ and $\mathcal{R}(\Sigma_\infty(t)) \subset \mathcal{D}(A_e)$ for all $t \in \mathbb{R}$ and thus $\Sigma_\infty(t)$ is the unique solution of the periodic Sylvester differential equation on $\mathbb{R}$ satisfying the initial condition $\Sigma_\infty(0)$ at $t=0$.

It remains to prove that the periodic Sylvester differential equation (9) has no other periodic solutions. To this end, let $\Sigma(t)$ be any periodic solution of the equation corresponding to an arbitrary initial condition $\Sigma(0) = \Sigma_0 \in \mathcal{L}(W,X_e)$, i.e.

$$\Sigma(t) = U_e(t,0)\Sigma_0 e^{-St} + \int_0^t U_e(t,s)B_e(s)e^{S(s-t)}ds.$$

The difference $\Delta(t) = \Sigma_\infty(t) - \Sigma(t)$ satisfies

$$\Delta(t) = \int_{-\infty}^{t} U_e(t,s)B_e(s)e^{S(s-t)}ds - U_e(t,0)\Sigma_0 e^{-St}$$
$$- \int_0^t U_e(t,s)B_e(s)e^{S(s-t)}ds$$
$$= \int_{-\infty}^{0} U_e(t,s)B_e(s)e^{S(s-t)}ds - U_e(t,0)\Sigma_0 e^{-St}$$
$$= U_e(t,0)\int_{-\infty}^{0} U_e(0,s)B_e(s)e^{Ss}e^{-St}ds$$
$$- U_e(t,0)\Sigma_0 e^{-St}$$
$$= U_e(t,0)\Sigma_\infty(0)e^{-St} - U_e(t,0)\Sigma_0 e^{-St}$$
$$= U_e(t,0)\Delta(0)e^{-St}.$$

Thus

$$\|\Delta(t)\| \leq M_e M_S e^{\omega_e t}\|\Delta(0)\|$$

and the assumption $\omega_e < 0$ implies $\lim_{t\to\infty}\Delta(t) = 0$. Since $\Sigma(t)$ is periodic and since $\lim_{t\to\infty}\|\Sigma(t) - \Sigma_\infty(t)\| = 0$, we must have $\Sigma(t) \equiv \Sigma_\infty(t)$. This concludes that no other periodic solution than $\Sigma_\infty(t)$ may exist. ■

The next lemma shows that the solution $\Sigma_\infty(t)$ of the periodic Sylvester differential equation describes the behaviour of the state of the closed-loop system and the regulation error. Moreover, the formula for the state $x_e(t)$ of the closed-loop system shows that if the closed-loop system is stable, the state of the closed-loop system approaches the function $\Sigma_\infty(t)v(t)$, where $v(t)$ is the state of the exosystem. Because of this, the asymptotic behaviour of $\Sigma_\infty(t)v(t)$ can be seen as a dynamic steady state of the closed-loop system. Lemma 7 is a generalization of the corresponding result for time-invariant systems [5], [12].

Lemma 7: Assume the Sylvester differential equation (9) *has a unique periodic solution* $\Sigma_\infty(\cdot) \in C_T^1(\mathbb{R}, \mathcal{L}(W,X_e))$ *with* $\mathcal{R}(\Sigma_\infty(t)) \subset \mathcal{D}(A_e)$ *for all* $t \in [0,T]$. *For all* $t \geq 0$ *and for all initial values* $x_{e0} \in X_e$ *and* $v_0 \in W$ *the regulation error* $e(t)$ *satisfies*

$$e(t) = C_e(t)U_e(t,0)(x_{e0} - \Sigma_\infty(0)v_0)$$
$$+ (C_e(t)\Sigma_\infty(t) + D_e(t))\,v(t),$$

where $v(t) = e^{St}v_0$ *is the state of the exosystem. If the closed-loop system is stable, then the state* $x_e(t)$ *of the closed-loop system and the regulation error satisfy*

$$\lim_{t\to\infty}\|x_e(t) - \Sigma_\infty(t)v(t)\| = 0,$$
$$\lim_{t\to\infty}\|e(t) - (C_e(t)\Sigma_\infty(t) + D_e(t))v(t)\| = 0.$$

Proof: For any initial conditions $x_{e0} \in X_e$ and $v_0 \in W$ and for any $t \geq 0$ the state of the closed-loop system (8) is given by

$$x_e(t) = U_e(t,0)x_{e0} + \int_0^t U_e(t,s)B_e(s)e^{Ss}v_0 ds.$$

Using the Sylvester differential equation (9) we see that

$$U_e(t,s)B_e(s)e^{Ss}v_0$$
$$= U_e(t,s)(\dot{\Sigma}_\infty(s) + \Sigma_\infty(s)S - A_e(s)\Sigma_\infty(s))e^{Ss}v_0$$
$$= U_e(t,s)\dot{\Sigma}_\infty(s)e^{Ss}v_0 + U_e(t,s)\Sigma_\infty(s)Se^{Ss}v_0$$
$$- U_e(t,s)A_e(s)\Sigma_\infty(s)e^{Ss}v_0$$
$$= \frac{d}{ds}U_e(t,s)\Sigma_\infty(s)e^{Ss}v_0.$$

Using this we see that the state of the closed-loop system (8) is given by

$$x_e(t) = U_e(t,0)x_{e0} + \int_0^t U_e(t,s)B_e(s)e^{Ss}v_0 ds$$
$$= U_e(t,0)x_{e0} + \Sigma_\infty(t)e^{St}v_0 - U_e(t,0)\Sigma_\infty(0)v_0$$
$$= U_e(t,0)(x_{e0} - \Sigma_\infty(0)v_0) + \Sigma_\infty(t)v(t).$$

This further implies that the regulation error is given by

$$e(t) = C_e(t)x_e(t) + D_e v(t)$$
$$= C_e(t)U_e(t,0)(x_{e0} - \Sigma_\infty(0)v_0)$$
$$+ (C_e(t)\Sigma_\infty(t) + D_e(t))\,v(t).$$

If the closed-loop system is stable, there exist constants $M_e \geq 1$ and $\omega_e < 0$ such that for all $t \geq s$ we have

$\|U_e(t,s)\| \leq M_e e^{\omega_e(t-s)}$. Using the previous formulas for $x_e(t)$ and $e(t)$ we see that

$$\begin{aligned}\|x_e(t) - \Sigma_\infty(t)v(t)\| &= \|U_e(t,0)(x_{e0} - \Sigma_\infty(0)v_0)\| \\ &\leq M_e e^{\omega_e t}\|x_{e0} - \Sigma_\infty(0)v_0\| \longrightarrow 0\end{aligned}$$

and

$$\begin{aligned}&\|e(t) - (C_e(t)\Sigma_\infty(t) + D_e(t))v(t)\| \\ =\ &\|C_e(t)U_e(t,0)(x_{e0} - \Sigma_\infty(0)v_0)\| \\ \leq\ &M_e e^{\omega_e t}\|C_e\|_\infty \cdot \|x_{e0} - \Sigma_\infty(0)v_0\| \longrightarrow 0\end{aligned}$$

as $t \to \infty$ since $\omega_e < 0$. This concludes the proof. ■

We can now present the proof of Theorem 5.

Proof of Theorem 5: The unique solvability of the periodic Sylvester differential equation (9) follows directly from Lemma 6.

Assume the unique periodic solution $\Sigma_\infty(t)$ of (9) satisfies $C_e(t)\Sigma_\infty(t) + D_e(t) = 0$ for all $t \in [0,T]$. Since the functions are T-periodic, this is satisfied for all $t \in \mathbb{R}$. Using this and Lemma 7 we have that for all initial values $x_{e0} \in X_e$ and $v_0 \in W$ that

$$\|e(t)\| = \|e(t) - (C_e(t)\Sigma_\infty(t) + D_e(t))v(t)\| \longrightarrow 0$$

as $t \to \infty$ and thus the controller solves the Periodic Output Regulation Problem.

On the other hand, assume that the controller solves the Periodic Output Regulation Problem and let $v_0 \in W$, $t_0 \in [0,T)$ and $n \in \mathbb{N}$. Let $e(t)$ be the regulation error originating from the initial conditions $0 \in X_e$ and $v_0 \in W$ of the closed-loop system and the exosystem, respectively, and let $t = t_0 + nT$. We then have using Lemma 7 that

$$\begin{aligned}&\|(C_e(t_0)\Sigma(t_0) + D_e(t_0))\, e^{St}v_0\| \\ =\ &\|(C_e(t)\Sigma(t) + D_e(t))e^{St}v_0\| \\ \leq\ &\|(C_e(t)\Sigma(t) + D_e(t))\, e^{St}v_0 - e(t)\| + \|e(t)\| \longrightarrow 0\end{aligned}$$

as $n \to \infty$. Let $\lambda \in \sigma(S)$ and let $\{\phi_k\}_{k=1}^m$ be an orthonormal Jordan chain associated to this eigenvalue. We will show that $(C_e(t_0)\Sigma(t_0)+D_e(t_0))\phi_k = 0$ for all $k \in \{1,\ldots,m\}$. Since the eigenvalue and the associated Jordan chain were arbitrary and since the generalized eigenvectors of S form a basis of the space W, this implies that $C_e(t_0)\Sigma(t_0) + D_e(t_0) = 0$. For $k \in \{1,\ldots,m\}$ and $t \in \mathbb{R}$ we have

$$e^{St}\phi_k = e^{\lambda t}\sum_{l=1}^{k}\frac{t^{k-l}}{(k-l)!}\phi_l.$$

Now

$$\begin{aligned}0 &= \lim_{n\to\infty}\ \|(C_e(t_0)\Sigma(t_0) + D_e(t_0))\, e^{St}\phi_1\| \\ &= \|(C_e(t_0)\Sigma(t_0) + D_e(t_0))\, \phi_1\| \cdot \lim_{n\to\infty} e^{\mathrm{Re}\,\lambda(t_0+nT)}\end{aligned}$$

and $(C_e(t_0)\Sigma(t_0) + D_e(t_0))\,\phi_1 = 0$ since we have $\mathrm{Re}\,\lambda \geq 0$ by Assumption 4. Using this we get

$$\begin{aligned}0 &= \lim_{n\to\infty}\ \|(C_e(t_0)\Sigma(t_0) + D_e(t_0))\, e^{St}\phi_2\| \\ &= \|(C_e(t_0)\Sigma(t_0) + D_e(t_0))\, \phi_2\| \cdot \lim_{n\to\infty} e^{\mathrm{Re}\,\lambda(t_0+nT)}\end{aligned}$$

and thus $(C_e(t_0)\Sigma(t_0) + D_e(t_0))\,\phi_2 = 0$ by Assumption 4. Continuing this we finally get

$$\begin{aligned}0 &= \lim_{n\to\infty}\ \|(C_e(t_0)\Sigma(t_0) + D_e(t_0))\, e^{St}\phi_m\| \\ &= \|(C_e(t_0)\Sigma(t_0) + D_e(t_0))\, \phi_m\| \cdot \lim_{n\to\infty} e^{\mathrm{Re}\,\lambda(t_0+nT)}\end{aligned}$$

which implies $(C_e(t_0)\Sigma(t_0) + D_e(t_0))\,\phi_m = 0$ by Assumption 4. Since the $\lambda \in \sigma(S)$ and the associated Jordan chain were arbitrary, we have that $C_e(t_0)\Sigma(t_0) + D_e(t_0) = 0$. Since $t_0 \in [0,T)$ was arbitrary, this finally shows that $C_e(t)\Sigma(t) + D_e(t) = 0$ for every $t \in [0,T]$. ■

We conclude this Section by considering the formula

$$x_e(t) = U_e(t,0)(x_{e0} - \Sigma_\infty(0)v_0) + \Sigma_\infty(t)v(t).$$

for the state of the closed-loop system appearing in the proof of Lemma 7 in greater detail. The next theorem shows the converse of Lemma 7, i.e. that if the state of the closed-loop system can be given in this form, then the operator-valued function $\Sigma_\infty(\cdot)$ appearing in the formula satisfies the periodic Sylvester differential equation.

Theorem 8: If there exists an operator-valued function $\Sigma(\cdot) \in C_T^1(\mathbb{R}, \mathcal{L}(W,Z))$ such that $\mathcal{R}(\Sigma(t)) \subset \mathcal{D}(A_e)$ for all $t \in [0,T]$ and the state of the closed-loop system satisfies

$$x_e(t) = U_e(t,0)(x_{e0} + \Sigma(0)v_0) + \Sigma(t)e^{St}v_0$$

for any $x_{e0} \in X_e$, $v_0 \in W$ and $t \geq 0$, then for all $t \in [0,T]$

$$\dot{\Sigma}(t) + \Sigma(t)S = A_e(t)\Sigma(t) + B_e(t).$$

Proof: Let $x_{e0} \in X_e$, $v_0 \in W$ and $t \in (0,2T)$. Differentiating the formula for the state of the closed-loop system we obtain

$$\begin{aligned}\dot{x}_e(t) &= \frac{d}{dt}\left(U_e(t,0)(x_{e0} + \Sigma(0)v_0) + \Sigma(t)e^{St}v_0\right) \\ &= A_e(t)U_e(t,0)(x_{e0} + \Sigma(0)v_0) + \dot{\Sigma}(t)e^{St}v_0 + \Sigma(t)Se^{St}v_0.\end{aligned}$$

On the other hand, since $x_e(t)$ is the state of the closed-loop system and since $B_e(\cdot)v(\cdot)$ is Hölder continuous on $[0,2T]$, we have that

$$\begin{aligned}\dot{x}_e(t) &= A_e(t)x_e(t) + B_e(t)v(t) \\ &= A_e(t)U_e(t,0)(x_{e0} + \Sigma(0)v_0) + A_e(t)\Sigma(t)e^{St}v_0 \\ &\quad + B_e(t)e^{St}v_0\end{aligned}$$

Combining these formulas for $\dot{x}_e(t)$ we obtain

$$\left(\dot{\Sigma}(t) + \Sigma(t)S - A_e(t)\Sigma(t) - B_e(t)\right)e^{St}v_0 = 0.$$

Since $v_0 \in W$ was arbitrary and since e^{St} is invertible for every $t \in \mathbb{R}$, this and the periodicity of the functions imply that for all $t \in \mathbb{R}$ we have

$$\dot{\Sigma}(t) + \Sigma(t)S = A_e(t)\Sigma(t) + B_e(t).$$ ■

IV. Controller Design

In this section we show how to construct an observer-based controller solving the Periodic Output Regulation Problem. This construction generalizes the one for finite-dimensional time-invariant systems presented in [6]. The next theorem is the main result of the section.

Theorem 9: Assume that the pair (A, B) is exponentially stabilizable, that there exists a periodic output injection $L(\cdot) \in C_T^1(\mathbb{R}, \mathcal{L}(Y, X_e))$ such that the system

$$\dot{y} = \left(\begin{bmatrix} A & E(t) \\ & S \end{bmatrix} - L(t) \begin{bmatrix} C & F(t) \end{bmatrix} \right) y \tag{12}$$

on $X \times W$ is exponentially stable and assume the constrained Sylvester differential equation

$$\dot{X}(t) + X(t)S = AX(t) + BU(t) + E(t) \tag{13a}$$
$$0 = CX(t) + DU(t) + F(t) \tag{13b}$$

has a periodic solution $X(\cdot) \in C_T^1(\mathbb{R}, \mathcal{L}(W, X))$ and $U(\cdot) \in C_T^1(\mathbb{R}, \mathcal{L}(W, U))$. Under these assumptions the Periodic Output Regulation Problem is solved by a controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ with parameters

$$\mathcal{G}_1(t) = \begin{bmatrix} A & E(t) \\ & S \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \cdot \begin{bmatrix} K_1 & K_2(t) \end{bmatrix} - L(t) \left(\begin{bmatrix} C & F(t) \end{bmatrix} + DK(t) \right)$$

and $\mathcal{G}_2(t) = L(t)$ on $Z = X \times W$ where $K_1 \in \mathcal{L}(X, U)$ is such that $A + BK_1$ generates an exponentially stable semigroup, $K_2(t) = U(t) - K_1 X(t)$ and $K(t) = \begin{bmatrix} K_1 & K_2(t) \end{bmatrix}$.

Proof: We will first show that the closed-loop system is exponentially stable. The closed-loop system operator $A_e(t)$ is equal to

$$\begin{bmatrix} A & BK_1 & BK_2(t) \\ L_1(t)C & A + BK_1 - L_1(t)C & E(t) + BK_2(t) - L_1(t)F(t) \\ L_2(t)C & -L_2(t)C & S - L_2(t)F(t) \end{bmatrix}$$

where we have denoted

$$L(t) = \begin{bmatrix} L_1(t) \\ L_2(t) \end{bmatrix}.$$

Applying a time-invariant similarity transform

$$T = \begin{bmatrix} I & 0 & 0 \\ -I & I & 0 \\ 0 & 0 & I \end{bmatrix}, \qquad T^{-1} = \begin{bmatrix} I & 0 & 0 \\ I & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

we can define $\widetilde{A}_e(t) = TA_e(t)T^{-1}$. This operator is equal to

$$\begin{bmatrix} A + BK_1 & BK_1 & BK_2(t) \\ 0 & A - L_1(t)C & E(t) - L_1(t)F(t) \\ 0 & -L_2(t)C & S - L_2(t)F(t) \end{bmatrix}.$$

Clearly the closed-loop system is stable whenever the system

$$\dot{\tilde{x}}_e = \widetilde{A}_e(t)\tilde{x}_e, \qquad \tilde{x}_e(0) = \tilde{x}_{e0} \in X_e$$

is exponentially stable. Also it is straight-forward to show that this is the case since $K_2(\cdot) \in C_T^1(\mathbb{R}, \mathcal{L}(W, X))$, since $A + BK_1$ generates an exponentially stable semigroup and since the system (12) is exponentially stable.

By Theorem 5 it remains to show that the unique periodic solution of the Sylvester differential equation

$$\dot{\Sigma}(t) + \Sigma(t)S = A_e(t)\Sigma(t) + B_e(t) \tag{14}$$

satisfies $C_e(t)\Sigma(t) + D_e(t) = 0$ for all $t \in [0, T]$. Let $X(t)$ and $U(t)$ be the solution of the equation (13) and denote

$$\Gamma(t) = \begin{bmatrix} X(t) \\ I \end{bmatrix}.$$

Since

$$U(t) = K_1 X(t) + K_2(t) = K(t)\Gamma(t),$$

it is sufficient to show that

$$\dot{\Gamma}(t) + \Gamma(t)S = \mathcal{G}_1(t)\Gamma(t) + \mathcal{G}_2(t)(CX(t) + DK(t)\Gamma(t) + F(t)).$$

for all $t \in [0, T]$. If this is the case, then

$$\Sigma_\infty(t) = \begin{bmatrix} X(t) \\ \Gamma(t) \end{bmatrix}$$

is the unique periodic solution of the Sylvester differential equation (14) and

$$C_e(t)\Sigma_\infty(t) + D_e(t) = CX(t) + DK(t)\Gamma(t) + F(t) = 0$$

follows from (13b). A direct computation yields

$$\begin{aligned} \mathcal{G}_1(t)\Gamma(t) &= \begin{bmatrix} (A + BK_1)X(t) + E(t) + BK_2(t) \\ S \end{bmatrix} \\ &\quad - L(t)(CX(t) + F(t) + DK(t)\Gamma(t)) \\ &= \begin{bmatrix} AX(t) + BK(t)\Gamma(t) + E(t) \\ S \end{bmatrix} \\ &\quad - L(t)(CX(t) + DK(t)\Gamma(t) + F(t)) \\ &= \begin{bmatrix} AX(t) + BU(t) + E(t) \\ S \end{bmatrix} \\ &\quad - L(t)(CX(t) + DK(t)\Gamma(t) + F(t)) \\ &= \begin{bmatrix} \dot{X}(t) + X(t)S \\ 0 + S \end{bmatrix} - \mathcal{G}_2(t)(CX(t) + DK(t)\Gamma(t) + F(t)) \\ &= \dot{\Gamma}(t) + \Gamma(t)S - \mathcal{G}_2(t)(CX(t) + DK(t)\Gamma(t) + F(t)). \end{aligned}$$

This concludes the proof. ■

We conclude this section with an example illustrating the choice of the stabilizing output injection $L(t)$ in (12) in a special case where the original plant is exponentially stable and we do not have any disturbance signals to reject. In a more general case for example the results in [11] can be used to find the function $L(\cdot)$ to stabilize the system (12).

Example 10: Assume that A generates an exponentially stable analytic semigroup and $\dim Y = 1$. Consider a one-dimensional exosystem $\dot{w} = S(t)w$ such that $E_s(t) \equiv 0$, $F_m(t) \equiv 0$ and $F_{ref}(t) \equiv 1$. If $L_{FL}(\cdot)$ is the function from the Lyapunov Reducibility Theorem, the exosystem can be transformed into the form $\dot{v} = sv$, where $s \in \mathbb{C}$ and we have

$E(t) \equiv 0$ and $F(t) = -F_{ref}(t)L_{FL}(t) = -L_{FL}(t)$. If we now choose $\alpha > \operatorname{Re} s$ and

$$L(t) = \begin{bmatrix} 0 \\ -\alpha L_{FL}(t)^{-1} \end{bmatrix}$$

for all $t \in \mathbb{R}$, we have $L(\cdot) \in C_T^1(\mathbb{R}, \mathcal{L}(\mathbb{C}, X \times \mathbb{C}))$ and

$$\begin{aligned} &\begin{bmatrix} A & E(t) \\ & S \end{bmatrix} - L(t) \begin{bmatrix} C & F(t) \end{bmatrix} \\ = &\begin{bmatrix} A & \\ \alpha L_{FL}(t)^{-1}C & s - \alpha L_{FL}(t)^{-1}L_{FL}(t) \end{bmatrix} \\ = &\begin{bmatrix} A & \\ \alpha L_{FL}(t)^{-1}C & s - \alpha \end{bmatrix}. \end{aligned}$$

Since A generates an exponentially stable semigroup, since $\operatorname{Re}(s - \alpha) < 0$ and since $\alpha L_{FL}(t)^{-1}C \in C_T^1(\mathbb{R}, \mathcal{L}(X, \mathbb{C}))$, it is straight-forward to verify that the evolution family associated to the system (12) is exponentially stable. ■

V. CONCLUSIONS

In this paper we have considered the infinite-dimensional Periodic Output Regulation Problem consisting of regulation and disturbance rejection of a time-invariant distributed parameter system and a periodic nonautonomous finite-dimensional exosystem. We have characterized the solvability of this problem using the unique periodic solution of an infinite-dimensional Sylvester differential equation.

The most important topic for further research is to develope further the results on the solvability of the Sylvester differential equation. If we can weaken the conditions on the existence of the unique periodic solution of the Sylvester differential equation, the theory can be applied to a more general class of distributed parameter systems.

Another important topic is the stablization of the closed-loop system. The stabilization of a time-dependent system is more complicated than the stabilization problem in the time-invariant case, but this problem has been studied also in infinite-dimensional spaces [11].

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