On Infinite-Dimensional Sylvester Equation and The Internal Model Principle

L. Paunonen[*], *S. Pohjolainen and T. Hämäläinen*

Abstract

In this paper we study the decomposing of certain infinite-dimensional Sylvester equations. This property of the equations is closely related to robust output regulation of infinite-dimensional systems. When the signal generator has discrete spectrum and a complete set of orthonormal eigenvectors, some sufficient conditions for the decomposing of the Sylvester equations are already known. In this paper we show that these are also necessary conditions. We also study how these conditions are related to an infinite-dimensional version of the internal model of finite-dimensional control theory. We show that under certain assumptions on the spectra of the closed-loop system and the signal generator the conditions are equivalent to the internal model.

1 Introduction

Recently there has been much work on infinite-dimensional robust regulation theory [6, 3]. In [8, 3] the finite-dimensional robust controller theory by Huang [5] has been partly generalized to infinite-dimensional systems. The key idea is that the closed-loop state $x_e(t)$ approaches a dynamic steady state of the form $\Sigma v(t)$ as $t \to \infty$. Here Σ is the solution of the associated Sylvester equations and $v(t)$ is the state of the exosystem. The steady state operator Σ can be decomposed into two parts Π and Γ according to the decomposition of the extended state space to the state

[*]Tampere University of Technology, P.O. Box 553, 33101 Tampere, Finland. Email: lassi.paunonen@tut.fi

spaces of the system and the controller. The Sylvester equations can be decomposed accordingly into

$$\Pi S = A\Pi + BK\Gamma + E \tag{1a}$$

$$\Gamma S = \mathcal{G}_1\Gamma + \mathcal{G}_2(C\Pi + DK\Gamma + F) \tag{1b}$$

The regulation error $e(t)$ goes to zero as $t \to \infty$ if $C\Pi + DK\Gamma + F = 0$. To achieve this, the controller parameters $(\mathcal{G}_1, \mathcal{G}_2, K)$ are chosen such that the above equations decompose into

$$\Pi S = A\Pi + BK\Gamma + E \tag{2a}$$

$$\Gamma S = \mathcal{G}_1\Gamma \tag{2b}$$

$$0 = C\Pi + DK\Gamma + F \tag{2c}$$

This also leads naturally to robust regulation if we choose the controller parameters such that the equations (1) and (2) are equivalent for all suitable perturbations of operators A, B, C, D, E and F. Then the equation (2c) implies that the regulation error goes to zero as $t \to \infty$ for all these perturbations and thus the regulation property is robust.

In this paper we study this equivalence of the Sylvester equations. It is already shown in [3] that the equations (1) and (2) are equivalent for all suitable operators if

$$\mathcal{R}(sI - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) = \{0\} \qquad \forall s \in \sigma(S) \tag{3a}$$

$$\mathcal{N}(\mathcal{G}_2) = \{0\} \tag{3b}$$

In this paper we show that the conditions (3) are also necessary for the equations (1) and (2) to be equivalent.

This result is related to the Internal Model Principle of control theory, which states that any feedback controller which stabilizes the closed-loop system also solves the robust output regulation problem if and only if it contains a a suitably reduplicated copy of the dynamics of the signal generator [2]. This is a well-known result for finite-dimensional systems and has also been studied in the case of distributed parameter systems with finite- and infinite-dimensional exosystems [1, 6].

In this paper we will also study the conditions (3) in more detail and show that under certain assumptions they are equivalent to the classical definition of the internal model in finite-dimensional control theory. This definition uses the Jordan canonical forms of the system operators of the signal generator and the controller. However, if the exosystem has discrete spectrum and a complete set of orthonormal eigenvectors, the definition can be reformulated in such a way that it also makes sense in the infinite-dimensional case.

2 Notation and Definitions

If X and Y are Banach spaces and $A : X \to Y$ is a linear operator, we denote by $\mathcal{D}(A)$, $\mathcal{N}(A)$ and $\mathcal{R}(A)$ the domain, kernel and range of A, respectively. The space of bounded linear operators from X to Y is denoted by $\mathcal{L}(X, Y)$. If $A : X \to X$, then $\sigma(A)$, $\sigma_p(A)$ and $\rho(A)$ denote the spectrum, the point spectrum and the resolvent set of A, respectively. For $\lambda \in \rho(A)$ the resolvent operator is given by $R(\lambda, A) = (\lambda I - A)^{-1}$.

Let X, Y, U be Banach-spaces and let W be a Hilbert space. We consider a linear system

$$\begin{aligned} \dot{x} &= Ax + Bu + Ev, \qquad x(0) = x_0 \in \mathcal{D}(A) \\ e &= Cx + Du + Fv \end{aligned}$$

where we have the state of the system $x(t) \in X$, the regulation error $e(t) \in Y$ and the input $u(t) \in U$ for all $t \geq 0$. We assume that $A : \mathcal{D}(A) \subset X \to X$ generates a C_0-semigroup on X and the other operators are bounded, $B \in \mathcal{L}(U, X)$, $C \in \mathcal{L}(X, Y)$, $D \in \mathcal{L}(U, Y)$, $E \in \mathcal{L}(W, X)$ and $F \in \mathcal{L}(W, Y)$. In the above system $v(t) \in W$ is the state of the exosystem

$$\dot{v} = Sv, \qquad v(0) = v_0 \in \mathcal{D}(S),$$

The system operator S of the exosystem is defined as

$$Sv = \sum_{k=-\infty}^{\infty} i\omega_k \langle v, \phi_k \rangle \phi_k, \qquad \mathcal{D}(S) = \left\{ v \in W \;\middle|\; \sum_{k=-\infty}^{\infty} \omega_k^2 |\langle v, \phi_k \rangle|^2 < \infty \right\},$$

where the sequence $(\omega_k)_{k \in \mathbb{Z}} \subset \mathbb{R}$ has no finite accumulation points, $\omega_k \neq \omega_n$ for all $k \neq n$ and $\{\phi_k\}_{k \in \mathbb{Z}}$ is an orthonormal basis of W. We also assume $\sigma(S) \cap \sigma(A) = \varnothing$. The dynamic feedback controller on a Banach-space Z is of form

$$\begin{aligned} \dot{z} &= \mathcal{G}_1 z + \mathcal{G}_2 e, \qquad z(0) = z_0 \in \mathcal{D}(\mathcal{G}_1) \\ u &= Kz \end{aligned}$$

where $\mathcal{G}_1 : \mathcal{D}(\mathcal{G}_1) \subset Z \to Z$ generates a C_0-semigroup on Z, $\mathcal{G}_2 \in \mathcal{L}(Y, Z)$ and $K \in \mathcal{L}(Z, U)$. The closed-loop system on $X \times Z$ with state $x_e(t) = (x(t), z(t))^T$ is given by

$$\begin{aligned} \dot{x}_e &= A_e x_e + B_e v, \qquad x_e(0) = \begin{bmatrix} x_0 \\ z_0 \end{bmatrix} \\ e &= C_e x_e + D_e v, \end{aligned}$$

where $C_e = \begin{bmatrix} C & DK \end{bmatrix}$, $D_e = F$,

$$A_e = \begin{bmatrix} A & BK \\ \mathcal{G}_2 C & \mathcal{G}_1 + \mathcal{G}_2 DK \end{bmatrix} \quad \text{and} \quad B_e = \begin{bmatrix} E \\ \mathcal{G}_2 F \end{bmatrix}.$$

For $\lambda \in \rho(A)$ the transfer function of the plant is $P(\lambda) = CR(\lambda, A)B + D \in \mathcal{L}(U, Y)$. Since we assumed $\sigma(A) \cap \sigma(S) = \varnothing$, we have that $P(s)$ is well-defined for all $s \in \sigma(S)$.

Throughout this paper we denote by $\mathcal{O}$ a list (A, B, C, D, E, F) of operators such that $A : \mathcal{D}(A) \subset X \to X$ generates a C_0-semigroup with $\sigma(A) \cap \sigma(S) = \varnothing$, $B \in \mathcal{L}(U, X)$, $C \in \mathcal{L}(X, Y)$, $D \in \mathcal{L}(U, Y)$, $E \in \mathcal{L}(W, X)$ and $F \in \mathcal{L}(W, Y)$.

3 Decomposing of the Sylvester Equations

In this section we study the infinite-dimensional Sylvester equations

$$\Pi S = A\Pi + BK\Gamma + E \tag{4a}$$
$$\Gamma S = \mathcal{G}_1\Gamma + \mathcal{G}_2(C\Pi + DK\Gamma + F) \tag{4b}$$

and

$$\Pi S = A\Pi + BK\Gamma + E \tag{5a}$$
$$\Gamma S = \mathcal{G}_1\Gamma \tag{5b}$$
$$0 = C\Pi + DK\Gamma + F, \tag{5c}$$

for operators $\mathcal{O}$. The operator equations are considered in $\mathcal{D}(S)$ and the operators $\Pi \in \mathcal{L}(W,X)$ and $\Gamma \in \mathcal{L}(W,Z)$ are assumed to satisfy $\Pi(\mathcal{D}(S)) \subset \mathcal{D}(A)$ and $\Gamma(\mathcal{D}(S)) \subset \mathcal{D}(\mathcal{G}_1)$.

We are interested in finding necessary and sufficient conditions for the operators $\mathcal{G}_1$ and $\mathcal{G}_2$ such that the equations (4) and (5) are equivalent for all operators $\mathcal{O}$. By this equivalence we mean that if one of the equations (4) and (5) has a solution (Π, Γ) such that $\Pi \in \mathcal{L}(W,X)$, $\Gamma \in \mathcal{L}(W,Z)$, $\Pi(\mathcal{D}(S)) \subset \mathcal{D}(A)$ and $\Gamma(\mathcal{D}(S)) \subset \mathcal{D}(\mathcal{G}_1)$, then it is also a solution to the other equation.

To make this consideration meaningful, we present assumptions under which the equations (4) have a solution. These assumptions are in no way minimal and they actually guarantee the existence of a unique bounded solution (Π, Γ).

Lemma 1 ([3, Lem 3]). *If $\mathcal{O}$ are operators such that A_e generates a strongly stable C_0-semigroup, $B_e\phi_k \in \mathcal{R}(i\omega_k I - A_e)$ for all $k \in \mathbb{Z}$ and*

$$\sum_{k\in\mathbb{Z}} \|R(i\omega_k, A_e)B_e\phi_k\|^2 < \infty,$$

then there exist operators $\Pi \in \mathcal{L}(W,X)$ and $\Gamma \in \mathcal{L}(W,Z)$ with $\Pi(\mathcal{D}(S)) \subset \mathcal{D}(A)$, $\Gamma(\mathcal{D}(S)) \subset \mathcal{D}(\mathcal{G}_1)$ such that the equations (4) are satisfied.

It is shown in [3] that (4) and (5) are equivalent for all operators $\mathcal{O}$ if the operators $\mathcal{G}_1$ and $\mathcal{G}_2$ satisfy

$$\mathcal{R}(sI - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) = \{0\} \qquad \forall s \in \sigma(S) \tag{6a}$$
$$\mathcal{N}(\mathcal{G}_2) = \{0\} \tag{6b}$$

We will now show that the conditions (6) are also a necessary condition for this equivalence. The following is the main result of this section.

Theorem 2. *The equations (4) and (5) are equivalent for all operators $\mathcal{O}$ if and only if (6) hold.*

We prove this theorem in parts. The following two lemmas prove the "only if" -part of the theorem.

Lemma 3. *If the equations* (4) *and* (5) *are equivalent for all operators* $\mathcal{O}$, *then* $\mathcal{R}(sI-\mathcal{G}_1)\cap\mathcal{R}(\mathcal{G}_2)=\{0\}$ *for all* $s\in\sigma(S)$.

Proof. Let $s\in\sigma(S)$ and $v\in\mathcal{R}(sI-\mathcal{G}_1)\cap\mathcal{R}(\mathcal{G}_2)$. Then there exist $z\in\mathcal{D}(\mathcal{G}_1)$ and $y\in Y$ such that

$$v=(sI-\mathcal{G}_1)z=\mathcal{G}_2y.$$

Let $A:\mathcal{D}(A)\subset X\to X$, $B\in\mathcal{L}(U,X)$, $C\in\mathcal{L}(X,Y)$, $D\in\mathcal{L}(U,Y)$ be any operators such that A generates a C_0-semigroup and $\sigma(A)\cap\sigma(S)=\varnothing$. In the following we will choose Π,Γ,E,F such that

$$\Pi S=A\Pi+BK\Gamma+E \tag{7a}$$
$$\Gamma S=\mathcal{G}_1\Gamma+\mathcal{G}_2(C\Pi+DK\Gamma+F) \tag{7b}$$

are satisfied. We can then use the equivalence of (4) and (5) to show that $v=0$.

We have $s=i\omega_k$ for some $k\in\mathbb{Z}$ and thus $\phi_k\in\mathcal{N}(sI-S)$. Define $\Gamma\in\mathcal{L}(W,Z)$ such that

$$\Gamma w=\langle w,\phi_k\rangle z\qquad\forall w\in W.$$

Since we assumed that $z\in\mathcal{D}(\mathcal{G}_1)$, we have $\mathcal{R}(\Gamma)\subset\mathcal{D}(\mathcal{G}_1)$.

Choose $\Pi=0\in\mathcal{L}(W,X)$ and $E=-BK\Gamma\in\mathcal{L}(W,X)$. Then we have $\mathcal{R}(\Pi)=\{0\}\subset\mathcal{D}(A)$ and equation (7a) is satisfied with these choices of operators.

Choose $F\in\mathcal{L}(W,Y)$ such that for all $w\in W$ we have

$$Fw=\langle w,\phi_k\rangle y-C\Pi w-DK\Gamma w\qquad\Leftrightarrow\qquad\langle w,\phi_k\rangle y=C\Pi w+DK\Gamma w+Fw$$

Now we have for all $w\in\mathcal{D}(S)$

$$\begin{aligned}(\Gamma S-\mathcal{G}_1\Gamma)w&=\langle Sw,\phi_k\rangle z-\mathcal{G}_1\langle w,\phi_k\rangle z=s\langle w,\phi_k\rangle z-\langle w,\phi_k\rangle\mathcal{G}_1 z\\&=\langle w,\phi_k\rangle(sI-\mathcal{G}_1)z=\langle w,\phi_k\rangle\mathcal{G}_2y=\mathcal{G}_2(\langle w,\phi_k\rangle y)\\&=\mathcal{G}_2(C\Pi w+DK\Gamma w+Fw)=\mathcal{G}_2(C\Pi+DK\Gamma+F)w\end{aligned}$$

and thus also equation (7b) is satisfied.

By our assumption the equations (7) now that imply $\Gamma S-\mathcal{G}_1\Gamma=0$ and thus

$$0=(\Gamma S-\mathcal{G}_1\Gamma)\phi_k=(sI-\mathcal{G}_1)\Gamma\phi_k=(sI-\mathcal{G}_1)\langle\phi_k,\phi_k\rangle z=(sI-\mathcal{G}_1)z=v.$$

Therefore $\mathcal{R}(sI-\mathcal{G}_1)\cap\mathcal{R}(\mathcal{G}_2)=\{0\}$. Since $s\in\sigma(S)$ was arbitrary, this completes the proof. $\square$

Lemma 4. *If the equations* (4) *and* (5) *are equivalent for all operators* $\mathcal{O}$*, then* $\mathcal{N}(\mathcal{G}_2) = \{0\}$.

Proof. Let $y \in \mathcal{N}(\mathcal{G}_2)$. Let $A : \mathcal{D}(A) \subset X \to X$, $B \in \mathcal{L}(U, X)$, $C \in \mathcal{L}(X, Y)$, $D \in \mathcal{L}(U, Y)$ be any operators such that A generates a C_0-semigroup and $\sigma(A) \cap \sigma(S) = \varnothing$ and choose

$$\begin{aligned} E &= 0 \in \mathcal{L}(W, X) & \Pi &= 0 \in \mathcal{L}(W, X) \\ \Gamma &= 0 \in \mathcal{L}(W, Z) & F &= \langle \cdot, \phi \rangle y \in \mathcal{L}(W, Y), \end{aligned}$$

where $\phi \in W$ such that $\|\phi\| = 1$. We now have $\Pi(\mathcal{D}(S)) = \{0\} \subset \mathcal{D}(A)$ and $\Gamma(\mathcal{D}(S)) = \{0\} \subset \mathcal{D}(\mathcal{G}_1)$ and thus the equations

$$\Pi S = A\Pi + BK\Gamma + E \tag{8a}$$

$$\Gamma S = \mathcal{G}_1 \Gamma + \mathcal{G}_2(C\Pi + DK\Gamma + F) \tag{8b}$$

are well-defined. It is easy to see that (8a) is satisfied since $\Pi = 0$, $\Gamma = 0$ and $E = 0$. Using $\mathcal{G}_2 y = 0$ we see that for all $w \in \mathcal{D}(S)$ we have $\Gamma S w = 0$ and

$$(\mathcal{G}_1 \Gamma + \mathcal{G}_2(C\Pi + DK\Gamma + F))w = \mathcal{G}_2(\langle w, \phi \rangle y) = \langle w, \phi \rangle \mathcal{G}_2 y = 0$$

This implies that also (8b) is satisfied.

By our assumption the equations (8) now imply $C\Pi + DK\Gamma + F = 0$ and thus also $F = 0$ since $\Pi = 0$ and $\Gamma = 0$. This implies

$$y = \langle \phi, \phi \rangle y = F\phi = 0.$$

Since y was arbitrary, this concludes that $\mathcal{N}(\mathcal{G}_2) = \{0\}$. $\square$

The "if"-part of theorem 2 is shown in [3]. We include the proof for completeness.

Theorem 5 ([3, Thm 7]). *If* (6) *are satisfied, then the equations* (4) *and* (5) *are equivalent for all operators* $\mathcal{O}$.

Proof. It is sufficient to show that for all suitable operators (4b) implies (5b) and (5c). Assume (4b) holds. If we apply both sides of this equation to ϕ_k we have

$$(i\omega_k I - \mathcal{G}_1)\Gamma \phi_k = \mathcal{G}_2(C\Pi + DK\Gamma + F)\phi_k$$

for all $k \in \mathbb{Z}$. Now (6a) implies that

$$\mathcal{G}_2(C\Pi + DK\Gamma + F)\phi_k = 0$$

and using (6b) we get $(C\Pi + DK\Gamma + F)\phi_k = 0$ for all $k \in \mathbb{Z}$. Since $\{\phi_k\}$ is a basis of W, this implies $C\Pi + DK\Gamma + F = 0$. Substituting this into (4b) also concludes that $\Gamma S = \mathcal{G}_1 \Gamma$. $\square$

4 Connection to the Internal Model Principle

In this section we study the conditions

$$\mathcal{R}(sI-\mathcal{G}_1)\cap\mathcal{R}(\mathcal{G}_2)=\{0\}\qquad \forall s\in\sigma(S) \tag{9a}$$

$$\mathcal{N}(\mathcal{G}_2)=\{0\} \tag{9b}$$

We will show that if $\dim Y<\infty$ and spectra of the closed-loop system and the signal generator are disjoint, then these conditions are equivalent to

$$\dim\mathcal{N}(sI-\mathcal{G}_1)=\dim Y\qquad \forall s\in\sigma(S). \tag{10}$$

Equation (10) is closely related to the concept of internal model in finite-dimensional control theory. If $\dim Y=p$, then a controller incorporates an internal model of the signal generator if the following is satisfied [2]: *If $s\in\sigma(S)$ is an eigenvalue of S such that $d(s)$ is the dimension of the largest Jordan block associated to s, then $s\in\sigma(\mathcal{G}_1)$ and $\mathcal{G}_1$ has at least p Jordan blocks of dimension $\geq d(s)$ associated to s.*

If the signal generator has discrete spectrum and a complete set of orthonormal eigenvectors, then this reduces to the following condition.

$$\dim\mathcal{N}(sI-\mathcal{G}_1)\geq\dim Y\qquad \forall s\in\sigma(S)$$

We can see that even though the definition of the internal model is given using the Jordan normal form, the previous condition also makes sense if the the controller is infinite-dimensional.

The following theorem is the main result of this section

Theorem 6. *Let $\sigma(S)\cap\sigma(A_e)=\varnothing$ and $\dim Y<\infty$. The conditions* (9) *hold if and only if* (10) *holds.*

We will prove this theorem by proving a series of lemmas. Since these are also useful results considered separately, we will prove them using weaker assumptions whenever possible. We will start by proving the following useful lemma.

Lemma 7. *If $\sigma(S)\cap\sigma_p(A_e)=\varnothing$, then the operator $(P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)}$ is injective for all $s\in\sigma(S)$.*

Proof. Let $s\in\sigma(S)$ and let $z\in\mathcal{N}(sI-\mathcal{G}_1)$ be such that $P(s)Kz=0$. Choose $x=R(s,A)BKz\in\mathcal{D}(A)$. Now

$$\begin{aligned}(sI-A_e)\begin{bmatrix}x\\ z\end{bmatrix}&=\begin{bmatrix}(sI-A)x-BKz\\ -\mathcal{G}_2Cx+(sI-\mathcal{G}_1)z-\mathcal{G}_2DKz\end{bmatrix}\\ &=\begin{bmatrix}BKz-BKz\\ -\mathcal{G}_2(CR(s,A)B+D)Kz+(sI-\mathcal{G}_1)z\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}\end{aligned}$$

Since $s\in\sigma(S)$, we know that $s\notin\sigma_p(A_e)$ and thus $sI-A_e$ is injective. This implies that $z=0$. From this we can conclude that the restriction of $P(s)K$ to $\mathcal{N}(sI-\mathcal{G}_1)$ is an injection. $\square$

The following lemma states that if (9) are satisfied, then the spaces $\mathcal{N}(sI-\mathcal{G}_1)$ are isomorphic to Y for all $s \in \sigma(S)$. This concludes that under assumptions of theorem 6 the conditions (9) imply that (10) holds, but the result is more general in the sense that it doesn't require Y to be finite-dimensional.

Lemma 8. *If $\sigma(S) \cap \sigma(A_e) = \varnothing$ and the conditions* (9) *are satisfied, then the operator $(P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)}$ is an isomorphism between $\mathcal{N}(sI-\mathcal{G}_1)$ and Y for all $s \in \sigma(S)$.*

Proof. Let $s \in \sigma(S)$. From lemma 7 we see that $(P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)}$ is injective and thus it is sufficient to prove that it is also surjective.

Since $\sigma(S) \cap \sigma(A_e) = \varnothing$, we have $s \in \rho(A_e)$ and $sI - A_e$ is surjective. Thus for all $z \in Z$ there exist $x_1 \in \mathcal{D}(A), z_1 \in \mathcal{D}(\mathcal{G}_1)$ such that

$$\begin{bmatrix} 0 \\ z \end{bmatrix} = (sI - A_e) \begin{bmatrix} x_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} (sI-A)x_1 - BKz_1 \\ -\mathcal{G}_2 C x_1 + (sI - \mathcal{G}_1) z_1 - \mathcal{G}_2 DKz_1 \end{bmatrix}$$

Since $\sigma(S) \cap \sigma(A) = \varnothing$, we have $s \in \rho(A)$ and we get from the first equation that $x_1 = R(s,A)BKz_1$. Thus

$$z = -\mathcal{G}_2 CR(s,A)BKz_1 + (sI-\mathcal{G}_1)z_1 - \mathcal{G}_2 DKz_1 = (sI-\mathcal{G}_1)z_1 - \mathcal{G}_2 P(s)Kz_1. \quad (11)$$

Let $y \in Y$. Then $z = -\mathcal{G}_2 y \in \mathcal{R}(\mathcal{G}_2) \subset Z$ and we can choose $z_1 \in \mathcal{D}(\mathcal{G}_1)$ such that (11) holds. Now

$$\begin{aligned} & -\mathcal{G}_2 y = (sI-\mathcal{G}_1)z_1 - \mathcal{G}_2 P(s)Kz_1 \\ \Leftrightarrow \quad & \underbrace{-\mathcal{G}_2 y + \mathcal{G}_2 P(s)Kz_1}_{\in \mathcal{R}(\mathcal{G}_2)} = \underbrace{(sI-\mathcal{G}_1)z_1}_{\in \mathcal{R}(sI-\mathcal{G}_1)} \\ \Leftrightarrow \quad & \begin{cases} \mathcal{G}_2 y = \mathcal{G}_2 P(s)Kz_1 \\ 0 = (sI-\mathcal{G}_1)z_1 \end{cases} \\ \Leftrightarrow \quad & \begin{cases} y = P(s)Kz_1 \\ 0 = (sI-\mathcal{G}_1)z_1 \end{cases} \end{aligned}$$

because $\mathcal{R}(sI-\mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) = \{0\}$ and $\mathcal{N}(\mathcal{G}_2) = \{0\}$. This means that for every $y \in Y$ there exists $z_1 \in \mathcal{N}(sI-\mathcal{G}_1)$ such that $y = P(s)Kz_1$ and thus the operator $(P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)}$ is surjective. □

We will now show that if $\dim Y < \infty$, then (10) also imply that the conditions (9) hold. For this it is sufficient to assume that $\sigma(S) \cap \sigma_p(A_e) = \varnothing$. This is satisfied, for example, whenever the closed-loop system is strongly stable, because then $\sigma_p(A_e) \subset \mathbb{C}^-$ [4]. The proof is divided into the following two lemmas.

Lemma 9. *If $\sigma(S) \cap \sigma_p(A_e) = \varnothing$, $\dim Y < \infty$ and* (10) *holds, then we have $\mathcal{R}(sI-\mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) = \{0\}$ for all $s \in \sigma(S)$.*

Proof. Let $s \in \sigma(S)$ and $v \in \mathcal{R}(sI-\mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2)$. Then there exist $y \in Y$ and $z \in \mathcal{D}(\mathcal{G}_1)$ such that

$$v = \mathcal{G}_2 y = (sI-\mathcal{G}_1)z.$$

We will first show that there exists $z_1 \in \mathcal{D}(\mathcal{G}_1)$ such that

$$v = \mathcal{G}_2 P(s) K z_1 = (sI - \mathcal{G}_1) z_1.$$

From lemma 7 we get that $(P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)}$ is injective and since $\dim \mathcal{N}(sI-\mathcal{G}_1) = \dim Y$ we have that it is invertible. Because of this we can choose $z_0 \in \mathcal{N}(sI - \mathcal{G}_1)$ such that

$$P(s)Kz_0 = y - P(s)Kz \in Y \qquad \Leftrightarrow \qquad y = P(s)K(z + z_0).$$

We then have

$$\mathcal{G}_2 P(s) K(z + z_0) = \mathcal{G}_2 y = v = (sI - \mathcal{G}_1)z = (sI - \mathcal{G}_1)(z + z_0)$$

and we can choose $z_1 = z + z_0$.

Choose $x_1 = R(s, A)BKz_1 \in \mathcal{D}(A)$. As in the proof of lemma 7, we see that

$$(sI - A_e) \begin{bmatrix} x_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ -\mathcal{G}_2 P(s)Kz_1 + (sI - \mathcal{G}_1)z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since $sI - A_e$ is injective, we have $z_1 = 0$ and thus

$$v = (sI - \mathcal{G}_1)z_1 = 0.$$

This concludes that $\mathcal{R}(sI - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) = \{0\}$. $\square$

Lemma 10. *If $\sigma(S) \cap \sigma_p(A_e) = \varnothing$, $\dim Y < \infty$ and (10) holds, then $\mathcal{N}(\mathcal{G}_2) = \{0\}$.*

Proof. Let $y \in \mathcal{N}(\mathcal{G}_2)$ and $s \in \sigma(S)$. From lemma 7 we get that $(P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)}$ is injective and since $\dim \mathcal{N}(sI - \mathcal{G}_1) = \dim Y$, it is invertible. This implies that there exists $z_1 \in \mathcal{N}(sI - \mathcal{G}_1)$ such that

$$y = P(s)Kz_1$$

and thus $\mathcal{G}_2 P(s)Kz_1 = 0$. Choose $x_1 = R(s, A)BKz_1 \in \mathcal{D}(A)$. As in the proof of lemma 7, we see that

$$(sI - A_e) \begin{bmatrix} x_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ -\mathcal{G}_2 P(s)Kz_1 + (sI - \mathcal{G}_1)z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since $s \in \sigma(S)$ and $\sigma(S) \cap \sigma_p(A_e) = \varnothing$, we have that $sI - A_e$ is injective and $z_1 = 0$. This also implies

$$y = P(s)Kz_1 = 0,$$

and thus $\mathcal{N}(\mathcal{G}_2) = \{0\}$. $\square$

Finally we will note that if $\sigma(S) \cap \sigma_p(A_e) = \varnothing$, then $\dim Y$ is necessarily an upper bound for the dimensions of $\mathcal{N}(sI - \mathcal{G}_1)$ for all $s \in \sigma(S)$. This is stated in the next lemma.

Lemma 11. *If $\sigma(S) \cap \sigma_p(A_e) = \varnothing$, then $\mathcal{N}(sI - \mathcal{G}_1) \leq \dim Y$ for all $s \in \sigma(S)$.*

Proof. Let $s \in \sigma(S)$. We have from lemma 7 that

$$(P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)} \in \mathcal{L}(\mathcal{N}(sI - \mathcal{G}_1), Y)$$

is injective. Using the Rank-Nullity Theorem [7, Thm 4.7.7] we can conclude that

$$\begin{aligned}\dim \mathcal{N}(sI - \mathcal{G}_1) &= \dim \mathcal{R}\left((P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)}\right) + \dim \mathcal{N}\left((P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)}\right) \\ &= \dim \mathcal{R}\left((P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)}\right) \leq \dim Y.\end{aligned}$$

□

5 Conclusions

Necessary and sufficient conditions for decomposing of infinite-dimensional Sylvester equations have been presented. This property is closely related to robust output regulation of infinite-dimensional systems. It was also proved that under certain assumptions these conditions are equivalent to an infinite-dimensional version of the Internal Model Principle of finite-dimensional control theory.

In this paper we have only considered bounded solutions of the Sylvester equations. A more general theory of robust regulation may require the dynamic steady state operator Σ to be unbounded. Because of this, the results presented in this paper should be extended to allow unbounded operators Π and Γ.

It was also assumed that the signal generator has a simple structure in the sense that it has pure point spectrum and a complete set of orthonormal eigenvectors. One area of further research is considering more general signal generators.

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