

# Robustness Properties of Controllers with Reduced Order Internal Models

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**Abstract**—In this paper we study robust output regulation for distributed parameter systems. In particular we are interested in the internal model principle, which can be used in characterizing controllers that achieve robust output tracking and disturbance rejection for a linear system. We show that if we do not require robustness with respect to arbitrary perturbations, then there may exist robust controllers that do not contain a full-sized internal model of the exosystem’s dynamics. The existence of such controllers depends on the class of admissible perturbations. Our approach also establishes a convenient way of testing the robustness of a controller with respect to given perturbations. The theoretic results are applied to analyzing the robustness properties of controllers for a system of two shock absorber models, and for a one-dimensional heat equation.

## I. INTRODUCTION

In this paper we consider robust output regulation for a distributed parameter system of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \in X \quad (1a)$$

$$y(t) = Cx(t) + Du(t). \quad (1b)$$

This problem consists of robust output tracking and disturbance rejection for reference and disturbance signals generated by a signal generator (also called the exosystem) of the form

$$\dot{v}(t) = Sv(t), \quad v(0) = v_0 \in W \quad (2a)$$

$$y_{ref}(t) = F_r v(t). \quad (2b)$$

The robust output problem has been studied actively for over three decades, see [1], [2], [3], [4] and references therein. We are especially interested in the *internal model principle* introduced for finite-dimensional systems in the 1970’s by Francis and Wonham, and Davison. This well-known result gives a characterization for controllers achieving robust output tracking and disturbance rejection for linear finite-dimensional systems [5], [6]. Recently, the internal model principle was also generalized for infinite-dimensional linear systems [7].

One of the implications of the internal model principle is that a robust controller must contain at least  $p$  copies of the dynamics of the signal generators, where  $p$  is the number of outputs of the plant (1). In practice this means that for any Jordan block of  $S$  in the exosystem (2) associated to an eigenvalue  $i\omega$ , the system operator of the controller must have at least  $p$  Jordan blocks of greater or equal size associated  $i\omega$ . In this paper we study the

robustness properties of controllers that do not contain “full” internal models, i.e., the number of copies of the exosystem’s dynamics is smaller than  $p$ . The main motivation for our study is the possibility of reducing the size of the internal model in the controller. In particular, we will see that the number of copies of the exosystem’s dynamics required in the controller is dependent on the class of admissible perturbations. Therefore, if we do not require robustness with respect to all possible uncertainties, there may exist robust controllers that contain less than  $p$  copies of the exosystem’s dynamics. We will see an example of this in Section V, where the one-dimensional controller is in particular robust with respect to small perturbations in the damping coefficients of the shock absorber models.

As our main results we present a new characterization for the robustness of a control law, and a method for testing the robustness of a given controller with respect to particular perturbations. The testing of robustness can be accomplished by checking the solvability of certain linear operator equations. Our conditions for robustness also show that the perturbations in the parameters of the plant affect the behaviour of the closed-loop system only through the change of the transfer function  $P(\lambda) = CR(\lambda, A)B + D$  of the plant at the frequencies  $i\omega_k$  of the signal generator.

In this paper we consider systems and controllers that are linear systems on infinite-dimensional Banach spaces. Such classes of systems are required in the study of models described by linear partial differential equations, as well as systems with delays. However, to the authors’ knowledge, the main results of this paper are new also for finite-dimensional systems. In such a case the parameters  $A$ ,  $B$ ,  $C$ , and  $D$  of the plant (1) are matrices of appropriate sizes and many of the technical assumptions made in Section II become redundant.

We illustrate the theoretic results with two examples. In the first example we consider a system of two identical and independent shock absorber models. We first design a one-dimensional error feedback controller for output tracking of constant reference signals. We can then use our results to examine the robustness properties of the control law. In particular we will see that the perturbations in the damping coefficients of the shock absorbers do not affect the transfer function  $P(\lambda)$  of the system at  $\lambda = 0$ , and therefore the controller is robust with respect to small changes in these parameters. As an example of control of an infinite-dimensional system we consider tracking of constant reference signals for a one-dimensional heat equation. We use our theoretic results to derive conditions for the perturbations to preserve the output tracking property.

In this paper we assume that the matrix  $S$  in the exosystem (2) is diagonalizable. The case of a non-diagonalizable  $S$  in the signal generator is studied in [8].

## II. MATHEMATICAL PRELIMINARIES

In this section we introduce the notation used in the paper and state the basic assumptions on the system, the exosystem and the controller. In the case of a finite-dimensional plant we have  $X = \mathbb{C}^n$  and the operators in the system, controller and the closed-loop system are matrices of appropriate sizes.

If  $X$  and  $Y$  are Banach spaces and  $A : X \rightarrow Y$  is a linear operator, we denote by  $\mathcal{D}(A)$ ,  $\mathcal{N}(A)$  and  $\mathcal{R}(A)$  the domain, kernel and range of  $A$ , respectively. The space of bounded linear operators from  $X$  to  $Y$  is denoted by  $\mathcal{L}(X, Y)$ . If  $A : X \rightarrow X$ , then  $\sigma(A)$ ,  $\sigma_p(A)$  and  $\rho(A)$  denote the spectrum, the point spectrum and the resolvent set of  $A$ , respectively. For  $\lambda \in \rho(A)$  the resolvent operator is given by  $R(\lambda, A) = (\lambda - A)^{-1}$ . The inner product on a Hilbert space is denoted by  $\langle \cdot, \cdot \rangle$ .

We consider plants of the form (1), where  $x(t) \in X$  is the state of the system,  $u(t) \in U = \mathbb{C}^m$  the input and  $y(t) \in Y = \mathbb{C}^p$  the output. The dimensions of the input space and the output space satisfy  $p \leq m$ . We assume that  $A$  generates a strongly continuous semigroup on  $X$  and that the rest of the operators are bounded in such a way that  $B \in \mathcal{L}(U, X)$ ,  $C \in \mathcal{L}(X, Y)$  and  $D \in \mathcal{L}(U, Y)$ . For  $\lambda \in \rho(A)$  the transfer function of the plant is given by  $P(\lambda) = CR(\lambda, A)B + D \in \mathcal{L}(U, Y)$ .

The reference signals  $y_{ref}(t)$  are generated by finite-dimensional exosystem

$$\dot{v}(t) = Sv(t), \quad v(0) = v_0 \in W \quad (3a)$$

$$y_{ref}(t) = -Fv(t). \quad (3b)$$

on  $W = \mathbb{C}^q$ . In this paper we assume that  $S$  is a diagonal matrix  $S = \text{diag}(i\omega_1, \dots, i\omega_q)$  and  $F \in \mathcal{L}(W, Y) = \mathbb{C}^{p \times q}$ . We denote by  $(\phi_1, \dots, \phi_q)$  the Euclidean basis vectors, which are also the eigenvectors of  $S$ .

The plant can be written in a standard form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \in X \quad (4a)$$

$$e(t) = Cx(t) + Du(t) + Fv(t) \quad (4b)$$

where  $e(t) = y(t) - y_{ref}(t) \in Y$  is the regulation error and  $v(t) \in W$  is the state of the exosystem (3). We assume that  $\sigma(A) \cap \sigma(S) = \emptyset$  and that  $P(i\omega_k)$  is surjective for all  $k \in \{1, \dots, q\}$ .

We consider a dynamic error feedback controller

$$\dot{z}(t) = \mathcal{G}_1 z(t) + \mathcal{G}_2 e(t), \quad z(0) = z_0 \in Z$$

$$u(t) = Kz(t)$$

on a Banach-space  $Z$ . Here  $z(t) \in Z$  is the state of the controller,  $\mathcal{G}_1 : \mathcal{D}(\mathcal{G}_1) \subset Z \rightarrow Z$  generates a strongly continuous semigroup on  $Z$ ,  $\mathcal{G}_2 \in \mathcal{L}(Y, Z)$  and  $K \in \mathcal{L}(Z, U)$ . The closed-loop system consisting of the plant and the controller on  $X_e = X \times Z$  with state  $x_e(t) = (x(t), z(t))^T$  is given by

$$\dot{x}_e(t) = A_e x_e(t) + B_e v(t), \quad x_e(0) = x_{e0} = (x_0, z_0)^T$$

$$e(t) = C_e x_e(t) + D_e v(t),$$

where  $C_e = (C \ D K)$ ,  $D_e = F$ ,

$$A_e = \begin{pmatrix} A & BK \\ \mathcal{G}_2 C & \mathcal{G}_1 + \mathcal{G}_2 DK \end{pmatrix} \quad \text{and} \quad B_e = \begin{pmatrix} 0 \\ \mathcal{G}_2 F \end{pmatrix}.$$

The operator  $A_e : \mathcal{D}(A) \times \mathcal{D}(\mathcal{G}_1) \subset X_e \rightarrow X_e$  generates a strongly continuous semigroup  $T_{A_e}(t)$  on  $X_e$ .

### A. The Classes of Perturbations to the Plant

In this paper we consider a situation where parameters of the plant (1) are perturbed in such a way that the operators  $A$ ,  $B$ ,  $C$ , and  $D$  are changed into  $\tilde{A} : \mathcal{D}(\tilde{A}) \subset X \rightarrow X$ ,  $\tilde{B} \in \mathcal{L}(U, X)$ ,  $\tilde{C} \in \mathcal{L}(X, Y)$ , and  $\tilde{D} \in \mathcal{L}(U, Y)$ , respectively. For  $\lambda \in \rho(\tilde{A})$  the transfer function of the perturbed plant is denoted  $\tilde{P}(\lambda) = \tilde{C}R(\lambda, \tilde{A})\tilde{B} + \tilde{D}$ . We likewise denote the operators of the closed-loop system consisting of the perturbed plant and the controller by  $\tilde{C}_e = \begin{pmatrix} \tilde{C} & \tilde{D}K \end{pmatrix}$ , and

$$\tilde{A}_e = \begin{pmatrix} \tilde{A} & \tilde{B}K \\ \mathcal{G}_2 \tilde{C} & \mathcal{G}_1 + \mathcal{G}_2 \tilde{D}K \end{pmatrix}, \quad \mathcal{D}(\tilde{A}_e) = \mathcal{D}(\tilde{A}) \times \mathcal{D}(\mathcal{G}_1).$$

The perturbations to the operators  $A$ ,  $B$ ,  $C$ , and  $D$  do not affect the operators  $B_e$  and  $D_e$  of the closed-loop system. Whenever  $\tilde{A}$  generates a semigroup on  $X$ , the semigroup generated by  $\tilde{A}_e$  is denoted by  $\tilde{T}_e(t)$ .

The perturbations  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$  in the class  $\mathcal{O}$  of perturbations are assumed to satisfy the following conditions:

- The perturbed system operator  $\tilde{A}$  generates a strongly continuous semigroup on  $X$  and satisfies  $i\omega_k \in \rho(\tilde{A})$  for all  $k \in \{1, \dots, q\}$
- The perturbed closed-loop system is exponentially stable, and  $\tilde{P}(i\omega_k)$  is surjective for all  $k \in \{1, \dots, q\}$ .

If the unperturbed closed-loop system is exponentially stable, then the above conditions are satisfied, for example, for any bounded perturbations of small enough norms.

## III. THE ROBUST OUTPUT REGULATION PROBLEM

The robust output regulation problem is formulated as follows.

*The Robust Output Regulation Problem:* Choose the parameters  $(\mathcal{G}_1, \mathcal{G}_2, K)$  in such a way that the following are satisfied:

- The closed-loop system operator  $A_e$  generates an exponentially stable semigroup on  $X_e$ .
- For all initial states  $v_0 \in W$  and  $x_{e0} \in X_e$  the regulation error goes to zero asymptotically, i.e.,  $\lim_{t \rightarrow \infty} e(t) = 0$ .
- If the operators  $(A, B, C, D)$  are perturbed to  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) \in \mathcal{O}$ , then  $\lim_{t \rightarrow \infty} e(t) = 0$  for all initial states  $v_0 \in W$  and  $x_{e0} \in X_e$ .

Parts 1 and 2 of the problem, i.e. output tracking without robustness, are called the *output regulation problem*. The following terminology is used in the rest of the paper.

*Definition 1:* If the controller  $(\mathcal{G}_1, \mathcal{G}_2, K)$  satisfies parts 1 and 2 of the robust output regulation problem, it is said to *solve the output regulation problem*. If the controller solves the robust output regulation problem (with respect to

given perturbations), it is called *robust* (with respect to given perturbations).

The robustness of a controller with respect to given perturbations can be characterized using the solvability of the Sylvester type regulator equations [5], [4].

*Theorem 2:* A controller solving the output regulation problem is robust with respect to given perturbations  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) \in \mathcal{O}$  if and only if the perturbed regulator equations

$$\Sigma S = \tilde{A}_e \Sigma + B_e \quad (6a)$$

$$0 = \tilde{C}_e \Sigma + D_e \quad (6b)$$

have a solution  $\Sigma \in \mathcal{L}(W, X_e)$  satisfying  $\mathcal{R}(\Sigma) \subset \mathcal{D}(\tilde{A}_e)$ .

*Proof:* This is a direct consequence of Theorem 3.1 in [7]. ■

#### IV. TESTING ROBUSTNESS WITH RESPECT TO GIVEN PERTURBATIONS

In this section we introduce a new characterization for the robustness of a control law. The conditions can also be used as a method for testing a robustness of a control law with respect to given perturbations. In particular, the result shows that the perturbations in the operators  $(A, B, C, D)$  only affect the regulation property of the controller through the change of the transfer function of the plant at the frequencies of the exosystem.

*Theorem 3:* A controller  $(\mathcal{G}_1, \mathcal{G}_2, K)$  solving the output regulation problem is robust with respect to given perturbations  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) \in \mathcal{O}$  if and only if the equations

$$\tilde{P}(i\omega_k)Kz^k = -F\phi_k \quad (7a)$$

$$(i\omega_k - \mathcal{G}_1)z^k = 0 \quad (7b)$$

have a solution  $z^k \in \mathcal{D}(\mathcal{G}_1)$  for all  $k \in \{1, \dots, q\}$ . Moreover, the solution of (7) is unique.

The proof of the theorem is based on the following properties of the regulator equations.

*Lemma 4:* Let  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) \in \mathcal{O}$ . An operator  $\Sigma = (\Pi, \Gamma)^T \in \mathcal{L}(W, X_e)$  with  $\mathcal{R}(\Sigma) \subset \mathcal{D}(\tilde{A}_e) = \mathcal{D}(\tilde{A}) \times \mathcal{D}(\mathcal{G}_1)$  satisfies the Sylvester equation  $\Sigma S = \tilde{A}_e \Sigma + B_e$  if and only if

$$(i\omega_k - \mathcal{G}_1)\Gamma\phi_k = \mathcal{G}_2 \left( \tilde{P}(i\omega_k)K\Gamma\phi_k + F\phi_k \right) \quad (8a)$$

$$\Pi\phi_k = R(i\omega_k, \tilde{A})\tilde{B}K\Gamma\phi_k \quad (8b)$$

for all  $k \in \{1, \dots, q\}$ . In this case we have

$$\tilde{C}_e \Sigma \phi_k + D_e \phi_k = \tilde{P}(i\omega_k)K\Gamma\phi_k + F\phi_k \quad (8c)$$

for all  $k \in \{1, \dots, q\}$ .

*Proof:* The conclusion of the lemma can be verified by applying both sides of the regulator equations (6) to the eigenvectors  $\phi_k$  of  $S$ . See [8] for a detailed proof. ■

*Proof of Theorem 3:* Let  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) \in \mathcal{O}$ .

We begin by showing that robustness of a controller with respect to the given perturbations implies that the equations (7) have solutions for all  $k \in \{1, \dots, q\}$ . The perturbed

closed-loop system is stable, and thus the robustness of the controller together with Theorem 2 implies that the perturbed regulator equations (6) have a solution  $\Sigma = (\Pi, \Gamma) \in \mathcal{L}(W, X_e)$  such that  $\mathcal{R}(\Pi) \subset \mathcal{D}(\tilde{A})$  and  $\mathcal{R}(\Gamma) \subset \mathcal{D}(\mathcal{G}_1)$ . Let  $k \in \{1, \dots, q\}$ . We have from (8a) and (8c) in Lemma 4 that the perturbed regulator equations (6) imply

$$\begin{aligned} (i\omega_k - \mathcal{G}_1)\Gamma\phi_k &= \mathcal{G}_2 \left( \tilde{P}(i\omega_k)K\Gamma\phi_k + F\phi_k \right) \\ 0 &= \tilde{P}(i\omega_k)K\Gamma\phi_k + F\phi_k. \end{aligned}$$

If we choose  $z^k = \Gamma\phi_k \in \mathcal{D}(\mathcal{G}_1)^{n_k}$ , then (7a) follows immediately from the second equation. Furthermore, substituting the second equation into the right-hand side of the first shows that  $(i\omega_k - \mathcal{G}_1)z^k = 0$ , and thus  $z^k$  is the solution of the equations (7). Since  $k \in \{1, \dots, q\}$  was arbitrary, the first part of the proof is concluded.

We now assume that for all  $k \in \{1, \dots, q\}$  the equations (7) have solutions  $z^k \in \mathcal{D}(\mathcal{G}_1)$ . We define operators  $\Pi \in \mathcal{L}(W, X)$ ,  $\Gamma \in \mathcal{L}(W, Z)$ , and  $\Sigma \in \mathcal{L}(W, X_e)$  by

$$\Gamma = \sum_{k=1}^q \langle \cdot, \phi_k \rangle z^k, \quad \Pi = \sum_{k=1}^q \langle \cdot, \phi_k \rangle R(i\omega_k, \tilde{A})\tilde{B}Kz^k, \quad (9)$$

and  $\Sigma = (\Pi, \Gamma)$ . We will show that  $\Sigma$  is a solution of the perturbed regulator equations (6). First of all, we have  $\mathcal{R}(\Sigma) \subset \mathcal{D}(\tilde{A}) \times \mathcal{D}(\mathcal{G}_1) = \mathcal{D}(\tilde{A}_e)$ . Let  $k \in \{1, \dots, q\}$ . We have  $\Gamma\phi_k = z^k$ , which together with the definition of  $\Pi$  implies that (8b) is satisfied. Furthermore, since  $\Gamma\phi_k = z^k$ , we have from (7) that

$$\begin{aligned} (i\omega_k - \mathcal{G}_1)\Gamma\phi_k &= (i\omega_k - \mathcal{G}_1)z^k \\ &= 0 = \mathcal{G}_2 \left( \tilde{P}(i\omega_k)Kz^k + F\phi_k \right) \\ &= \mathcal{G}_2 \left( \tilde{P}(i\omega_k)K\Gamma\phi_k + F\phi_k \right), \end{aligned}$$

which is precisely (8a). Lemma 4 now implies that  $\Sigma$  is a solution of the Sylvester equation  $\Sigma S = \tilde{A}_e \Sigma + B_e$ . Furthermore, Lemma 4 and equation (7a) imply that we have

$$\begin{aligned} \tilde{C}_e \Sigma \phi_k + D_e \phi_k &= \tilde{P}(i\omega_k)K\Gamma\phi_k + F\phi_k \\ &= \tilde{P}(i\omega_k)Kz^k + F\phi_k = 0. \end{aligned}$$

Therefore  $\Sigma$  is a solution of the perturbed regulator equations, and thus Theorem 2 concludes that the controller is robust with respect to the given perturbations.

It remains to show that the equations (7) may have at most one solution. We first note that since the perturbed closed-loop system is exponentially stable and since  $S$  is a finite-dimensional operator with  $\sigma(S) \subset i\mathbb{R}$ , the solution of the Sylvester equation (6a) is unique [9].

Assume that for some  $k_0 \in \{1, \dots, q\}$  the equations (7) have solutions  $z^{k_0}, \tilde{z}^{k_0} \in \mathcal{D}(\mathcal{G}_1)$ . Let  $z^k$  be solutions of (7) for  $k \neq k_0$ . As in the second part of this proof, we can define  $\Sigma = (\Pi, \Gamma)^T$  and  $\tilde{\Sigma} = (\tilde{\Pi}, \tilde{\Gamma})^T$  with the formulas in (9) using the sets  $(z^1, \dots, z^{k_0}, \dots, z^q)$  and  $(z^1, \dots, \tilde{z}^{k_0}, \dots, z^q)$  of elements, respectively. As above, we have from Lemma 4 that  $\Sigma$  and  $\tilde{\Sigma}$  are solutions of the Sylvester equation (6a). Since the solution of this equation is unique, we must have

$\Sigma = \tilde{\Sigma}$ . Due to the definitions of  $\Gamma$  and  $\tilde{\Gamma}$  this is only possible if  $z^{k_0} = \tilde{z}^{k_0}$ . ■

Theorem 3 also enables us to present a new proof for the result that a stabilizing controller that incorporates a p-copy internal model of the exosystem solves the robust output regulation problem. For distributed parameter systems this result was proved in [7] by showing the equivalence of three different definitions for an internal model, which finally concluded that the p-copy internal model was necessary and sufficient for robustness. Theorem 3 can be used to give a more direct proof for the sufficiency of the p-copy internal model for robustness.

For a diagonal exosystem, the p-copy internal model in a controller  $(\mathcal{G}_1, \mathcal{G}_1, K)$  can be defined as follows.

*Definition 5:* A controller is said to *incorporate a p-copy internal model of the exosystem* if

$$\dim \mathcal{N}(i\omega_k - \mathcal{G}_1) \geq p \quad \forall k \in \{1, \dots, q\}. \quad (10)$$

*Theorem 6:* If the controller solves the output regulation problem and incorporates a p-copy internal model of the exosystem, then it solves the robust output regulation problem.

*Proof:* Assume the controller  $(\mathcal{G}_1, \mathcal{G}_1, K)$  solves the output regulation problem and that (10) is satisfied. Let  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$  be such that the perturbed closed-loop system is exponentially stable.

Since  $\sigma_p(\tilde{A}_e) \cap \sigma(S) = \emptyset$ , have as in [7, Lem. 6.3] that the operator  $(\tilde{P}(i\omega_k)K)|_{\mathcal{N}(i\omega_k - \mathcal{G}_1)}$  is injective for every  $k \in \{1, \dots, q\}$ . Furthermore, the condition (10) together with the Rank-Nullity Theorem [10, Thm. 4.7.7] imply that  $(\tilde{P}(i\omega_k)K)|_{\mathcal{N}(i\omega_k - \mathcal{G}_1)}$  is also surjective for every  $k \in \{1, \dots, q\}$ . In particular, the surjectivity of this operator means that for every  $k$  we can find  $z^k \in \mathcal{N}(i\omega_k - \mathcal{G}_1)$  such that (7a) is satisfied. This concludes that for all  $k \in \{1, \dots, q\}$  the equations (7) are satisfied, and thus by Theorem 3 the controller is robust with respect to perturbations  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ . Since the perturbations were arbitrary, this concludes the proof. ■

## V. SHOCK ABSORBER MODEL

In this section we consider control of a system consisting of two independent shock absorbers. We begin by building a one-dimensional feedback controller to achieve tracking of constant reference signals. We will then study the robustness properties of the control law using Theorem 3.

The behavior of an individual shock absorber is described by the equations

$$\ddot{q}(t) + r\dot{q}(t) + q(t) = F(t).$$

where  $r > 0$  is the damping coefficient. If we control the external force  $F(t)$  and observe the position  $q(t)$ , the standard form for a single system becomes

$$\begin{aligned} \dot{x}_k(t) &= \begin{pmatrix} 0 & 1 \\ -1 & -r \end{pmatrix} x_k(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_k(t), \quad x_k(0) = \begin{pmatrix} q(0) \\ \dot{q}(0) \end{pmatrix} \\ y_k(t) &= (1 \ 0) x_k(t). \end{aligned}$$

We consider a situation where the nominal plant has a damping coefficient  $r = 1$ , and later in the example consider the effects of uncertainty in this parameter. In the nominal situation we therefore have 2 identical and independent systems, and the system equations become

$$\begin{aligned} \dot{x}(t) &= \text{diag}(A_0, A_0) x(t) + \text{diag}(B_0, B_0) u(t) \\ y(t) &= \text{diag}(C_0, C_0) x(t), \end{aligned}$$

where we have denoted  $x(t) = (x_1^1(t), x_1^2(t), x_2^1(t), x_2^2(t))^T$ ,  $u(t) = (u_1(t), u_2(t))^T$ ,  $y(t) = (y_1(t), y_2(t))^T$ , and

$$A_0 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C_0 = (1 \ 0).$$

The plant is exponentially stable, and its transfer function is given by

$$\begin{aligned} P(\lambda) &= CR(\lambda, A)B = \text{diag}(C_0R(\lambda, A_0)B_0) \\ &= \text{diag}\left(\frac{1}{\lambda^2 + \lambda + 1}\right) = \frac{1}{\lambda^2 + \lambda + 1}I \end{aligned}$$

for all  $\lambda \notin \sigma_p(A_0)$ .

We consider output tracking of constant signals. As an exosystem we use

$$\begin{aligned} \dot{v}(t) &= 0, \quad v(0) = v_0 \\ y_{ref}(t) &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} v(t) = \mathbf{1}v_0. \end{aligned}$$

We now have  $i\omega_0 = 0$ , and  $\phi = 1 \in \mathcal{N}(i\omega_0 - \mathcal{G}) = \mathbb{C}$ . Furthermore, with this choice we have  $F = -\mathbf{1}$  in the standard form, and  $P(0) = I$  is invertible.

As the controller we consider

$$\begin{aligned} \dot{z}(t) &= 0 \cdot z(t) + (1 \ 0) e(t) \\ u(t) &= Kz(t) = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} z(t), \end{aligned}$$

where we choose the elements of  $K$  in such a way that the controller solves the output regulation problem for the nominal plant. This can be done by ensuring that the closed-loop system is stable and the regulator equations have a solution. In particular, since we have  $\mathcal{G}_1 = 0$ , Lemma 4 implies that the regulator equations are satisfied if

$$0 = C_e \Sigma + D_e = P(0)K\Gamma + F$$

is satisfied for some  $\Gamma \in \mathcal{N}(0 - \mathcal{G}_1) = \mathbb{C}$ .

Since the closed-loop system is finite-dimensional, its stability can be determined from the eigenvalues of  $A_e$ . Since the plant is stable, the inverse of  $\lambda - A_e$  can be computed using the inverse of  $\lambda - A$  and that of its Schur complement  $S_A(\lambda) = \lambda - \mathcal{G}_1 - \mathcal{G}_2P(\lambda)K$ . Therefore, the closed-loop system is exponentially stable if we choose  $K$  in such a way that  $\lambda - \mathcal{G}_1 - \mathcal{G}_2P(\lambda)K$  has no roots in  $\mathbb{C}^+$ . A direct computation yields

$$\begin{aligned} S_A(\lambda) &= \lambda - \mathcal{G}_1 - \mathcal{G}_2P(\lambda)K = \lambda - \frac{k_1}{\lambda^2 + \lambda + 1} \\ &= \frac{\lambda^3 + \lambda^2 + \lambda - k_1}{\lambda^2 + \lambda + 1} \end{aligned}$$

and if we choose  $k_1 = -1/2$ , then  $S_A(\lambda) \neq 0$  for all  $\lambda \in \overline{\mathbb{C}^+}$ , and the closed-loop system is exponentially stable. The regulator equations have a solution if the regulation constraint is satisfied, i.e. if there exists  $\Gamma \in \mathbb{C}$  such that

$$\begin{aligned} P(0)K\Gamma + F = 0 &\Leftrightarrow \begin{pmatrix} -1/2 \\ k_2 \end{pmatrix} \Gamma + (-\mathbf{1}) = 0 \\ \Leftrightarrow \begin{pmatrix} -1/2 \\ k_2 \end{pmatrix} \Gamma = \mathbf{1}. \end{aligned}$$

This implies we can choose  $\Gamma = -2$  and  $k_2 = -1/2$ , and then the regulator equations are satisfied. This concludes that with the choice  $K = -1/2 \cdot \mathbf{1}$  the controller solves the output regulation problem.

#### A. Robustness Properties of the Control Law

Since  $\dim Y = 2 > 1 = \dim \mathcal{N}(0 - \mathcal{G}_1)$ , our controller is not guaranteed to be robust with respect to arbitrary perturbations in the parameters of the plant. Theorem 3 states that the system is robust with respect to any perturbations for which the closed-loop system stability is preserved and for which the equations

$$\begin{aligned} \tilde{P}(0)Kz &= -F \\ (0 - \mathcal{G}_1)z &= 0 \end{aligned}$$

have a solution  $z \in \mathbb{C}$ . Since  $0 - \mathcal{G}_1 = 0$ , the second equation is satisfied for all  $z \in \mathbb{C}$ .

If we first consider a situation where the damping coefficients  $r$  of the different subsystems are perturbed independently of each others, the perturbed system operator of the plant is given by

$$\tilde{A}_0^j = \begin{pmatrix} 0 & 1 \\ -1 & -r_j \end{pmatrix}, \quad A = \text{diag}(\tilde{A}_0^1, \tilde{A}_0^2)$$

and the perturbed transfer function becomes

$$\begin{aligned} \tilde{P}(\lambda) &= CR(\lambda, \tilde{A})B \\ &= \text{diag}(C_0R(\lambda, \tilde{A}_0^1)B_0, C_0R(\lambda, \tilde{A}_0^2)B_0) \\ &= \text{diag}\left(\frac{1}{\lambda^2 + r_1\lambda + 1}, \frac{1}{\lambda^2 + r_2\lambda + 1}\right) \end{aligned}$$

for all  $\lambda \notin \sigma_p(\tilde{A})$ . Since changing the values of the damping coefficients can be written as an additive perturbation to  $A$ , we know that for small enough changes (i.e.,  $r_j \approx 1$ ) the closed-loop system remains exponentially stable and  $0 \in \rho(\tilde{A})$ . However, for any such perturbed values  $r_j$  the perturbed transfer function at the frequency  $\lambda = 0$  is given by

$$\begin{aligned} \tilde{P}(0) &= \text{diag}\left(\frac{1}{0^2 + r_1 \cdot 0 + 1}, \frac{1}{0^2 + r_2 \cdot 0 + 1}\right) \\ &= I = P(0). \end{aligned}$$

This means that the uncertainties damping coefficients do not affect the transfer function of the plant at  $\lambda = 0$ . In particular this concludes that the 1-dimensional controller is robust with respect to any changes in the values  $r_j$  for which the closed-loop system is exponentially stable and  $0 \in \rho(\tilde{A})$ .

On the other hand, if we consider a more general case where the two subsystems are perturbed independently, then the perturbed transfer function can be written in the form

$$\begin{aligned} \tilde{P}(\lambda) &= \text{diag}(\tilde{P}_1(\lambda), \tilde{P}_2(\lambda)) \\ &= P(\lambda) + \text{diag}(\delta_1(\lambda), \delta_2(\lambda)), \end{aligned}$$

where  $\delta_j(\lambda)$  are functions that are analytic in  $\overline{\mathbb{C}^+}$ . Now, writing  $\alpha = 1/z \in \mathbb{C}$  the first equation in (7) becomes

$$\begin{aligned} \tilde{P}(0)K &= -\alpha F \\ \Leftrightarrow \text{diag}(\delta_1(0), \delta_2(0))K &= -\alpha F - K \\ \Leftrightarrow -1/2 \cdot \begin{pmatrix} \delta_1(0) \\ \delta_2(0) \end{pmatrix} &= (\alpha + 1/2) \cdot \mathbf{1} \\ \Leftrightarrow \delta_j(0) &= -1 - 2\alpha, \quad j = 1, 2. \end{aligned}$$

This shows that the control law is robust with respect to precisely those perturbations that affect the transfer functions  $P_j(\lambda)$  of the subsystems at  $\lambda = 0$  in the same way.

## VI. CONTROLLED HEAT EQUATION

In this section we consider controlling the heat distribution of a metal bar. We assume the heating can be controlled on intervals  $[1/4, 1/2]$  and  $[3/4, 1]$ , and observed on the intervals  $[0, 1/2]$  and  $[1/2, 3/4]$ , as illustrated in Figure 1.

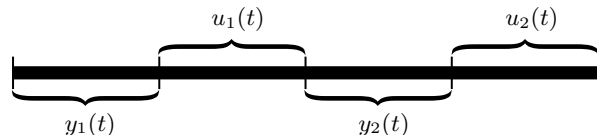


Fig. 1. Heating of a metal bar.

We assume that there is no heat flow through the ends of the bar. The parameters of the plant can be chosen as  $X = L^2(0, 1)$ ,  $U = Y = \mathbb{C}^2$ , and

$$\begin{aligned} Ax(z) &= x''(z) - x(z), \\ \mathcal{D}(A) &= \{x \in X \mid x, x' \text{ abs. cont.}, x'' \in X, \\ &\quad x'(0) = x'(1) = 0\}, \\ Bu &= 4(\chi_{[1/4, 1/2]}(z), \chi_{[3/4, 1]}(z)) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \\ Cx &= 4 \begin{pmatrix} \int_0^1 \chi_{[0, 1/4]}(z)x(z)dz \\ \int_0^1 \chi_{[3/4, 1]}(z)x(z)dz \end{pmatrix}, \end{aligned}$$

where  $\chi_{[a,b]}(z)$  is the characteristic function on an interval  $[a, b] \subset [0, 1]$ . The operator  $A$  generates an exponentially stable semigroup on  $X$ . The explicit expression of the transfer function  $P(\lambda)$  of the plant can be found in [11, Ex. 3.13]. In particular, we have that  $P(0) \in \mathbb{C}^{2 \times 2}$  is invertible, and

$$P(0) \approx \begin{pmatrix} 0.2590 & 4.7008 \\ 0.2453 & 0.2590 \end{pmatrix}.$$

We consider output tracking of constant reference signals. To this end, we choose the exosystem on  $W = \mathbb{C}$  to be

$$\begin{aligned} \dot{v}(t) &= 0, \quad v(0) = v_0 \in W \\ y_{ref}(t) &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} v(t). \end{aligned}$$

We then have  $S = 0 \in \mathbb{C}$  and  $F = (-1, -2)^T$ . As the controller, we choose a one-dimensional dynamic error feedback controller

$$\begin{aligned} \dot{z}(t) &= \varepsilon(1, 0) e(t), & z(0) &= z_0 \in Z \\ u(t) &= -P(0)^{-1} F z(t). \end{aligned}$$

on  $Z = \mathbb{C}$ . With this controller, we have  $\mathcal{G}_1 = 0 \in \mathbb{C}$ ,  $\mathcal{G}_2 = \varepsilon(1, 0)$ , and  $K = -P(0)^{-1} F \approx (8.4167, -0.2510)^T$  with  $\varepsilon > 0$ . For  $z = 1 \in \mathcal{N}(0 - \mathcal{G}_1) = \mathbb{C}$  we have  $P(0)Kz = P(0)(-P(0)^{-1} F) \cdot 1 = -F$ , and similarly as in Theorem 3 we can conclude that the controller solves the output regulation problem provided that the closed-loop system is exponentially stable. Perturbation techniques such as the ones in [12, App. B] can be used to show that if we choose  $\varepsilon > 0$  small enough, then the controller stabilizes the closed-loop system exponentially.

We can now investigate the robustness of the control law. Since  $\dim Y = 2 > 1 = \dim \mathcal{N}(0 - \mathcal{G}_1)$ , the controller is not guaranteed to be robust with respect to all perturbations in the parameters of the plant. The effects of the perturbations to the transfer function of the plant can be written as  $\tilde{P}(\lambda) = P(\lambda) + \Delta(\lambda)$  where  $\Delta(\lambda) = \begin{pmatrix} \delta_{11}(\lambda) & \delta_{12}(\lambda) \\ \delta_{21}(\lambda) & \delta_{22}(\lambda) \end{pmatrix}$ . If we write  $\alpha = 1/z \in \mathbb{C}$  for  $z \in \mathcal{N}(0 - \mathcal{G}_1) = \mathbb{C}$ , then the solvability of the equations in Theorem 3 are equivalent to

$$\begin{aligned} \tilde{P}(0)Kz &= -F \\ \Leftrightarrow (P(0) + \Delta(0))(-P(0)^{-1}F) &= -\alpha F \\ \Leftrightarrow \Delta(0)P(0)^{-1}F &= (\alpha - 1)F \end{aligned}$$

for some  $\alpha \in \mathbb{C}$ . This concludes that the controller is robust with respect to any small enough perturbations to the parameters of the plant if and only if the perturbation  $\Delta(0) = \tilde{P}(0) - P(0)$  is such that the application of  $\Delta(0)P(0)^{-1}$  does not change the direction of  $F$ , i.e.,  $\Delta(0)P(0)^{-1}F \subset \text{span } F$ . This condition, in turn, can be written as a set of conditions on the relationships between the components  $\delta_{ij}(0)$  of the perturbing function  $\Delta(\cdot)$  evaluated at  $\lambda = 0$ . In particular, the one-dimensional controller is robust with respect to any perturbations in the parameters of the plant that do not affect the behaviour of the transfer function at  $\lambda = 0$ .

## VII. CONCLUSIONS

In this paper we have considered the robust output regulation problem for infinite-dimensional linear systems and finite-dimensional diagonal signal generators. We have in particular concentrated in studying robustness properties of controllers that do not incorporate full p-copy internal models. We have shown that the perturbations in the parameters of the system affect the tracking of reference signals only through the change of the transfer function of the plant at the frequencies  $i\omega_k$  of the exosystem. Therefore, a feedback controller is in particular robust with respect to any perturbations that do not affect these values of the transfer function.

Future research topics include extending the results for nondiagonal exosystems and considering rejection of disturbance signals to the state of the plant.

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