On Decomposing of Infinite-Dimensional Sylvester Equations

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***Abstract*— In this paper we study certain infinite-dimensional Sylvester equations. The equations are closely related to robust output regulation of infinite-dimensional systems. If the signal generator is finite-dimensional or has discrete spectrum and a complete set of orthonormal eigenvectors, there are some known sufficient conditions for the decomposing of these Sylvester equations. In this paper we generalize these conditions to the case where the signal generator has discrete spectrum and a complete set of orthonormal generalized eigenvectors. We also study how these conditions are related to an infinite-dimensional version of the internal model of finite-dimensional control theory. We show that under certain assumptions on the spectra of the closed-loop system and the signal generator these conditions are equivalent to the concept of an internal model.**

I. Introduction

Recently there has been much work on infinite-dimensional robust regulation theory [1], [2]. In [3], [2] the finite-dimensional robust controller theory of Francis and Wonham represented by Huang [4] has been partly generalized to infinite-dimensional systems. The key idea is that the closed-loop state $x_e(t)$ approaches a dynamic steady state of the form $\Sigma v(t)$ as $t \rightarrow \infty$. Here Σ is the solution of the associated Sylvester equation and $v(t)$ is the state of the exosystem $\dot{v} = Sv$. The dynamic steady state operator Σ can be decomposed into two parts Π and Γ according to the decomposition of the extended state space to the state spaces of the system and the controller. The Sylvester equations can be decomposed accordingly into

$$\Pi S = A\Pi + BK\Gamma + E \tag{1a}$$
$$\Gamma S = \mathcal{G}_1\Gamma + \mathcal{G}_2(C\Pi + DK\Gamma + F) \tag{1b}$$

For stable closed-loop systems the regulation error $e(t)$ goes to zero as $t \rightarrow \infty$ if $C\Pi + DK\Gamma + F = 0$. To achieve this, the controller parameters $(\mathcal{G}_1, \mathcal{G}_2)$ are chosen such that the above equations decompose into

$$\Pi S = A\Pi + BK\Gamma + E \tag{2a}$$
$$\Gamma S = \mathcal{G}_1\Gamma \tag{2b}$$
$$0 = C\Pi + DK\Gamma + F \tag{2c}$$

This also leads naturally to robust regulation if we choose the controller parameters such that the equations (1) and (2) are equivalent for all suitable perturbations of operators A, B, C, D, E and F. The equation (2c) then implies that the regulation error goes to zero as $t \rightarrow \infty$ for all these perturbations and thus the regulation property is robust.

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The main purpose of this paper is to find necessary and sufficient conditions such that the equations (1) decompose into equations (2) for all operators (A, B, C, D, E, F). This problem is closely related to the Internal Model Principle, which states that any feedback controller which stabilizes the closed-loop system also solves the robust output regulation problem if and only if it contains a suitably reduplicated copy of the dynamics of the exosystem [5]. This is a well-known result for finite-dimensional systems and has also been studied in the case of distributed parameter systems with finite-dimensional exosystems [6] and infinite-dimensional exosystems with complete sets of eigenvectors [1].

It has already been shown in [3] that if the exosystem with system operator S is finite-dimensional, then the equations (1) and (2) are equivalent for all suitable operators if

$$\mathcal{R}(i\omega_k I - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) = \{0\}, \tag{3a}$$
$$\mathcal{N}(\mathcal{G}_2) = \{0\} \tag{3b}$$

and

$$\mathcal{N}(i\omega_k I - \mathcal{G}_1)^{d_k - 1} \subset \mathcal{R}(i\omega_k I - \mathcal{G}_1) \tag{3c}$$

for all $k \in \mathbb{Z}$, where $(i\omega_k)$ are the eigenvalues of the signal generator and d_k denotes the dimension of the largest Jordan block associated to the eigenvalue $i\omega_k$. In [2] this result has also been generalized to infinite-dimensional exosystems with complete sets of orthonormal eigenvectors. In this case the condition (3c) becomes redundant.

In [7] it has been shown that for S with discrete spectrum and a complete set of orthonormal eigenvectors the conditions (3) are both necessary and sufficient for the equations (1) and (2) to be equivalent. In this paper we extend these results for a more general exosystem capable of generating polynomially increasing infinite-dimensional signals. This type of signal generator is constructed by defining an operator with an infinite number of Jordan blocks.

We also compare the conditions (3) to the definition of internal model in finite-dimensional control theory. The classical definition uses the Jordan canonical forms of the system operators of the exosystem and the controller. Because of this, the definition cannot be used when these operators are infinite-dimensional. We generalize the definition for inifinite-dimensional systems and show that under certain assumptions the conditions (3) are equivalent to the definition of the internal model.

The equations (2) are often called the *regulator equations* related to the linear system consisting of operators (A, B, C, D, E, F) and the error feedback controller with operators $(\mathcal{G}_1, \mathcal{G}_2, K)$. Similar equations also exist for nonlinear systems [8]. Even though the Sylvester equations (1) and (2) are closely related to infinite-dimensional systems, we consider the equations for general operators without assumptions on the well-posedness of these systems.

In Section II we introduce the notation and state the basic assumptions on the considered operators. In Section III we show that conditions (3) are necessary and sufficient for the decomposing of the Sylvester equations. The main result of the section is Theorem 2. In Section IV we generalize the definition of the internal model for infinite-dimensional systems and compare it to the conditions (3). The main result in the section is Theorem 9 which states that under certain assumptions these conditions are equivalent to the definition of the internal model. Section V contains concluding remarks.

II. Notation and Definitions

If X and Y are Banach spaces and $A : X \to Y$ is a linear operator, we denote by $\mathcal{D}(A)$, $\mathcal{N}(A)$ and $\mathcal{R}(A)$ the domain, kernel and range of A, respectively. The space of bounded linear operators from X to Y is denoted by $\mathcal{L}(X,Y)$. If $A : X \to X$, then $\sigma(A)$, $\sigma_p(A)$ and $\rho(A)$ denote the spectrum, the point spectrum and the resolvent set of A, respectively. For $\lambda \in \rho(A)$ the resolvent operator is given by $R(\lambda, A) = (\lambda I - A)^{-1}$.

Let X, Y, U be Banach-spaces and let W be a Hilbert space. Let $\{i\omega_k\}_{k\in\mathbb{Z}} \in i\mathbb{R}$ be a set with no finite accumulation points such that the set $I_k = \{\, l \in \mathbb{Z} \mid \omega_l = \omega_k \,\}$ is finite for all $k \in \mathbb{Z}$.

Let $\{\, \phi_k^l \mid k \in \mathbb{Z}, l = 1, \ldots, n_k \,\} \subset W$ where $n_k < \infty$ for all $k \in \mathbb{Z}$ be an orthonormal basis of W, i.e.

$$W = \overline{\operatorname{span}}\{\phi_k^l\}_{kl}, \qquad \langle \phi_k^l, \phi_n^m \rangle = \delta_{kn}\delta_{lm}.$$

We assume there exists some $N_d \in \mathbb{N}$ such that for every $k \in \mathbb{Z}$ we have $n_k \le N_d$. For $k \in \mathbb{Z}$ define an operator $S_k \in \mathcal{L}(W)$ such that

$$S_k = i\omega_k \langle \cdot, \phi_k^1 \rangle \phi_k^1 + \sum_{l=2}^{n_k} \langle \cdot, \phi_k^l \rangle \left(i\omega_k \phi_k^l + \phi_k^{l-1} \right)$$

The operator S_k then satisfies $(i\omega_k I - S_k)\phi_k^1 = 0$ and $(S_k - i\omega_k I)\phi_k^l = \phi_k^{l-1}$ for all $l \in \{2, \ldots, n_k\}$ and thus corresponds to a Jordan block associated to the eigenvalue $i\omega_k$. For $k \in \mathbb{Z}$ define $d_k = \max\{\, n_l \mid l \in \mathbb{Z},\ \omega_l = \omega_k \,\}$. This corresponds to the dimension of the largest Jordan block associated to the eigenvalue $i\omega_k$.

Define operator $S : \mathcal{D}(S) \subset W \to W$ as

$$Sv = \sum_{k\in\mathbb{Z}} S_k v, \quad \mathcal{D}(S) = \{\, v \in W \mid \sum_{k\in\mathbb{Z}} \|S_k v\|^2 < \infty \,\}.$$

Also, for $k \in \mathbb{Z}$ denote by P_k the orthogonal projection $P_k = \sum_{l=1}^{n_k} \langle \cdot, \phi_k^l \rangle \phi_k^l$ onto the finite-dimensional subspace $W_k = \operatorname{span}\{\phi_k^l\}_{l=1}^{n_k}$.

Let (A, B, C, D, E, F) be a collection of operators such that $A : \mathcal{D}(A) \subset X \to X$, $B \in \mathcal{L}(U,X)$, $C \in \mathcal{L}(X,Y)$, $D \in \mathcal{L}(U,Y)$, $E \in \mathcal{L}(W,X)$ and $F \in \mathcal{L}(W,Y)$. These represent operators of a distributed parameter system

$$\begin{aligned} \dot{x} &= Ax + Bu + Ev, \qquad x(0) = x_0 \in X \\ e &= Cx + Du + Fv \end{aligned}$$

where e is the regulation error and v is the state of the exosystem with $\dot{v} = Sv$ and $v(0) = v_0 \in \mathcal{D}(S)$.

Let $(\mathcal{G}_1, \mathcal{G}_2, K)$ be a collection of operators such that $\mathcal{G}_1 : \mathcal{D}(\mathcal{G}_1) \subset Z \to Z$, $\mathcal{G}_2 \in \mathcal{L}(Y,Z)$ and $K \in \mathcal{L}(Z,U)$. They represent the operators of an error feedback controller

$$\begin{aligned} \dot{z} &= \mathcal{G}_1 z + \mathcal{G}_2 e, \qquad z(0) = z_0 \in Z \\ u &= Kz \end{aligned}$$

on the Banach space Z.

The closed-loop system with state $x_e(t) = (x(t), z(t))^T$ on the space $X \times Z$ is given by

$$\begin{aligned} \dot{x}_e &= A_e x_e + B_e v, \qquad x_e(0) = (x_0, z_0)^T \\ e &= C_e x_e + D_e v, \end{aligned}$$

where $C_e = \begin{bmatrix} C & DK \end{bmatrix}$, $D_e = F$,

$$A_e = \begin{bmatrix} A & BK \\ \mathcal{G}_2 C & \mathcal{G}_1 + \mathcal{G}_2 DK \end{bmatrix} \quad \text{and} \quad B_e = \begin{bmatrix} E \\ \mathcal{G}_2 F \end{bmatrix}.$$

For $\lambda \in \rho(A)$ the transfer function of the plant is defined as $P(\lambda) = CR(\lambda, A)B + D \in \mathcal{L}(U,Y)$.

Throughout this paper we denote by $\mathcal{O}$ a collection (A, B, C, D, E, F) of operators where $A : \mathcal{D}(A) \subset X \to X$, $B \in \mathcal{L}(U,X)$, $C \in \mathcal{L}(X,Y)$, $D \in \mathcal{L}(U,Y)$, $E \in \mathcal{L}(W,X)$ and $F \in \mathcal{L}(W,Y)$. It is worthwile to note that these operators are more general than the operators in a distributed parameter system, since we do not require that the operator A generates a C_0-semigroup on X or that the operator $\mathcal{G}_1$ generates a C_0-semigroup on Z.

III. Decomposing of the Sylvester Equations

In this section we study the infinite-dimensional Sylvester equations

$$\begin{aligned} \Pi S &= A\Pi + BK\Gamma + E && \text{(6a)} \\ \Gamma S &= \mathcal{G}_1 \Gamma + \mathcal{G}_2 (C\Pi + DK\Gamma + F) && \text{(6b)} \end{aligned}$$

and

$$\begin{aligned} \Pi S &= A\Pi + BK\Gamma + E && \text{(7a)} \\ \Gamma S &= \mathcal{G}_1 \Gamma && \text{(7b)} \\ 0 &= C\Pi + DK\Gamma + F, && \text{(7c)} \end{aligned}$$

for operator collections $\mathcal{O}$. The operator equations are considered on $\mathcal{D}(S)$ and the operators $\Pi \in \mathcal{L}(W,X)$ and $\Gamma \in \mathcal{L}(W,Z)$ are assumed to satisfy $\Pi(\mathcal{D}(S)) \subset \mathcal{D}(A)$ and $\Gamma(\mathcal{D}(S)) \subset \mathcal{D}(\mathcal{G}_1)$.

We are interested in finding necessary and sufficient conditions for the operators $\mathcal{G}_1$ and $\mathcal{G}_2$ such that the equations (6) and (7) are equivalent for all operator collections $\mathcal{O}$. By this equivalence we mean that if one of the equations (6) and (7) has a solution (Π, Γ) such that $\Pi \in \mathcal{L}(W,X)$, $\Gamma \in \mathcal{L}(W,Z)$, $\Pi(\mathcal{D}(S)) \subset \mathcal{D}(A)$ and $\Gamma(\mathcal{D}(S)) \subset \mathcal{D}(\mathcal{G}_1)$, then it is also a solution of the other equation.

To make this consideration meaningful, we present conditions under which the equations (6) have a solution. These assumptions are in no way minimal and they actually guarantee the existence of a unique bounded solution (Π, Γ). The proof of the lemma can be found in [9].

Lemma 1: If $\mathcal{O}$ is a collection of operators such that A_e generates a strongly stable C_0-semigroup, for all $k \in \mathbb{Z}$ and $l \in \{1, \dots, n_k\}$ we have $B_e\phi_k^l \in \mathcal{R}(i\omega_k I - A_e)^{n_k-l+1}$ and if

$$\sum_{k\in\mathbb{Z}} \sum_{l=1}^{n_k} \left(\sum_{j=1}^{l} \|R(i\omega_k, A_e)^{l+1-j} B_e \phi_k^j\| \right)^2 < \infty,$$

then there exist operators $\Pi \in \mathcal{L}(W, X)$ and $\Gamma \in \mathcal{L}(W, Z)$ with $\Pi(\mathcal{D}(S)) \subset \mathcal{D}(A)$ and $\Gamma(\mathcal{D}(S)) \subset \mathcal{D}(\mathcal{G}_1)$ such that the equations (6) are satisfied. ■

The following is the main result of this section.

Theorem 2: Assume $Z = \mathcal{R}(i\omega_k I - \mathcal{G}_1) + \mathcal{R}(\mathcal{G}_2)$ for all $k \in \mathbb{Z}$. The equations (6) and (7) are equivalent for all operator collections $\mathcal{O}$ if and only if

$$\mathcal{R}(i\omega_k I - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) = \{0\}, \quad \forall k \in \mathbb{Z} \tag{8a}$$

$$\mathcal{N}(\mathcal{G}_2) = \{0\} \tag{8b}$$

and

$$\mathcal{N}(i\omega_k I - \mathcal{G}_1)^{d_k-1} \subset \mathcal{R}(i\omega_k I - \mathcal{G}_1) \quad \forall k \in \mathbb{Z}. \tag{8c}$$

■

We will prove the theorem in parts. In Lemma 3 we present a simpler characterization for the equivalence of the Sylvester equations. This characterization is from Immonen [1]. The Lemmas 4, 5 and 6 prove that the decomposing of the Sylvester equations imply the conditions (8a), (8b) and (8c), respectively. Finally, Lemma 7 shows that the conditions (8) imply the decomposing of the Sylvester equations.

Lemma 3: The equations (6) and (7) are equivalent for all operator collections $\mathcal{O}$ if and only if

$$\forall \Lambda, \Delta: \qquad \Lambda S = \mathcal{G}_1\Lambda + \mathcal{G}_2\Delta \quad \Rightarrow \quad \Delta = 0, \tag{9}$$

where $\Lambda \in \mathcal{L}(W, Z)$ is such that $\Lambda(\mathcal{D}(S)) \subset \mathcal{D}(\mathcal{G}_1)$ and $\Delta \in \mathcal{L}(W, Y)$. ■

Proof: We first prove the necessity of condition (9). Assume the equations (6) and (7) are equivalent for all operator collections $\mathcal{O}$ and assume $\Lambda S = \mathcal{G}_1\Lambda + \mathcal{G}_2\Delta$ for $\Lambda \in \mathcal{L}(W, Z)$ and $\Delta \in \mathcal{L}(W, Y)$ with $\Lambda(\mathcal{D}(S)) \subset \mathcal{D}(\mathcal{G}_1)$. Let A, B, C and D be arbitrary operators and choose $\Gamma = \Lambda$, $\Pi = 0$, $E = -BK\Gamma$ and $F = \Delta - DK\Gamma$. We then have $\Pi(\mathcal{D}(S)) \subset \mathcal{D}(A)$, $\Gamma(\mathcal{D}(S)) \subset \mathcal{D}(\mathcal{G}_1)$ and the equations (6) are satisfied. Thus also the equations (7) are satisfied. The equation (7c) now implies

$$0 = C\Pi + DK\Gamma + F = DK\Gamma + \Delta - DK\Gamma = \Delta.$$

This concludes that (9) is satisfied.

It remains to prove the sufficiency of (9). Assume (9) is satisfied for all $\Lambda \in \mathcal{L}(W, Z)$ and $\Delta \in \mathcal{L}(W, Y)$ with $\Lambda(\mathcal{D}(S)) \subset \mathcal{D}(\mathcal{G}_1)$.

It is clear that if the equations (7) are satisfied for some operators A, B, C, D, E, F, Π and Γ, then also the equations (6) are satisfied for these operators.

Assume the equations (6) are satisfied for operators A, B, C, D, E, F, Π and Γ. If we choose $\Delta = C\Pi + DK\Gamma + F$ and $\Lambda = \Gamma$, then (9) implies $0 = \Delta = C\Pi + DK\Gamma + F$. Substituting this into equation (6b) we obtain $\Gamma S = \mathcal{G}_1\Gamma$. This concludes that the equations (7) are satisfied. ■

The following three lemmas prove the necessity of the conditions (8) for the equivalence of the Sylvester equations.

Lemma 4: If the equations (6) and (7) are equivalent for all operator collections $\mathcal{O}$, then $\mathcal{R}(i\omega_k I - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) = \{0\}$ for all $k \in \mathbb{Z}$. ■

Proof: Let $k \in \mathbb{Z}$ and let $w \in \mathcal{R}(i\omega_k I - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2)$. Then there exist $z \in \mathcal{D}(\mathcal{G}_1)$ and $y \in Y$ such that

$$w = (i\omega_k I - \mathcal{G}_1)z = \mathcal{G}_2 y.$$

Choose $\Lambda = \langle \cdot, \phi_k^{n_k} \rangle z$ and $\Delta = \langle \cdot, \phi_k^{n_k} \rangle y$. For $v \in \mathcal{D}(S)$

$$\begin{aligned}(\Lambda S - \mathcal{G}_1\Lambda)v &= \langle Sv, \phi_k^{n_k} \rangle z - \langle v, \phi_k^{n_k} \rangle \mathcal{G}_1 z \\ &= \langle v, \phi_k^{n_k} \rangle (i\omega_k I - \mathcal{G}_1)z = \langle v, \phi_k^{n_k} \rangle \mathcal{G}_2 y = \mathcal{G}_2\Delta v.\end{aligned}$$

Thus we have $\Lambda S = \mathcal{G}_1\Lambda + \mathcal{G}_2\Delta$ and our assumption together with Lemma 3 implies that $0 = \Delta\phi_k^{n_k} = \langle \phi_k^{n_k}, \phi_k^{n_k} \rangle y = y$. This further implies that $w = \mathcal{G}_2 y = 0$. ■

Lemma 5: If the equations (6) and (7) are equivalent for all operator collections $\mathcal{O}$, then $\mathcal{N}(\mathcal{G}_2) = \{0\}$. ■

Proof: Let $y \in \mathcal{N}(\mathcal{G}_2)$ and let $\phi \in \mathcal{D}(S)$ with $\|\phi\| = 1$. Choose $\Lambda = 0 \in \mathcal{L}(W, Z)$ and $\Delta = \langle \cdot, \phi \rangle y$. We then have $\mathcal{R}(\Lambda) = \{0\} \subset \mathcal{D}(\mathcal{G}_1)$, $\Lambda S v = 0$ and

$$\mathcal{G}_1\Lambda v + \mathcal{G}_2\Delta v = 0 + \langle v, \phi \rangle \mathcal{G}_2 y = 0$$

for any $v \in \mathcal{D}(S)$. Thus we have $\Lambda S = \mathcal{G}_1\Lambda + \mathcal{G}_2\Delta$ and our assumption together with Lemma 3 implies $\Delta = 0$. Now $0 = \Delta\phi = \langle \phi, \phi \rangle y = y$ and thus $\mathcal{N}(\mathcal{G}_2) = \{0\}$. ■

To prove the necessity of the condition (8c) we need the assumption that $Z = \mathcal{R}(i\omega_k I - \mathcal{G}_1) + \mathcal{R}(\mathcal{G}_2)$ for all $k \in \mathbb{Z}$.

Lemma 6: If $Z = \mathcal{R}(i\omega_k I - \mathcal{G}_1) + \mathcal{R}(\mathcal{G}_2)$ for all $k \in \mathbb{Z}$ and the equations (6) and (7) are equivalent for all operator collections $\mathcal{O}$, then $\mathcal{N}(i\omega_k I - \mathcal{G}_1)^{d_k-1} \subset \mathcal{R}(i\omega_k I - \mathcal{G}_1)$ for all $k \in \mathbb{Z}$. ■

Proof: Since $d_k = \max\{\, n_l \mid l \in \mathbb{Z},\ \omega_l = \omega_k \,\}$, it is sufficient to prove the condition (8c) with n_k in place of d_k. Let $k \in \mathbb{Z}$ and $z \in \mathcal{N}(i\omega_k I - \mathcal{G}_1)^{n_k-1}$. Since we have $Z = \mathcal{R}(i\omega_k I - \mathcal{G}_1) + \mathcal{R}(\mathcal{G}_2)$, there exist $z_1 \in \mathcal{D}(\mathcal{G}_1)$ and $y \in Y$ such that

$$z = (i\omega_k I - \mathcal{G}_1)z_1 + \mathcal{G}_2 y. \tag{10}$$

Choose $\Delta = (-1)^{n_k} \langle \cdot, \phi_k^{n_k} \rangle y$ and

$$\begin{aligned}\Lambda &= \left(\sum_{l=1}^{n_k-1} (-1)^{l-1} \langle \cdot, \phi_k^l \rangle (i\omega_k I - \mathcal{G}_1)^{n_k-1-l} z \right) \\ &\quad + (-1)^{n_k-1} \langle \cdot, \phi_k^{n_k} \rangle z_1.\end{aligned}$$

Since $z_1 \in \mathcal{D}(\mathcal{G}_1)$ and $z \in \mathcal{N}(i\omega_k I - \mathcal{G}_1)^{n_k-1}$, we have $\mathcal{R}(\Lambda) \subset \mathcal{D}(\mathcal{G}_1)$. Now for all $l \in \{2, \dots, n_k - 1\}$

$$\begin{aligned}&(\Lambda S - \mathcal{G}_1\Lambda)\phi_k^1 = (i\omega_k I - \mathcal{G}_1)\Lambda\phi_k^1 = (i\omega_k I - \mathcal{G}_1)^{n_k-1} z \\ &= 0 = \mathcal{G}_2\Delta\phi_k^1 \\ &(\Lambda S - \mathcal{G}_1\Lambda)\phi_k^l = (i\omega_k I - \mathcal{G}_1)\Lambda\phi_k^l + \Lambda\phi_k^{l-1} \\ &= (-1)^{l-1}(i\omega_k I - \mathcal{G}_1)(i\omega_k I - \mathcal{G}_1)^{n_k-1-l} z \\ &\quad + (-1)^{l-2}(i\omega_k I - \mathcal{G}_1)^{n_k-1-(l-1)} z = 0 = \mathcal{G}_2\Delta\phi_k^l\end{aligned}$$

and using (10)

$$\begin{aligned}(\Lambda S - \mathcal{G}_1\Lambda)\phi_k^{n_k} &= (i\omega_k I - \mathcal{G}_1)\Lambda\phi_k^{n_k} + \Lambda\phi_k^{n_k-1}\\ &= (-1)^{n_k-1}(i\omega_k I - \mathcal{G}_1)z_1 + (-1)^{n_k-2}z\\ &= (-1)^{n_k-1}\left((i\omega_k I - \mathcal{G}_1)z_1 - z\right) = (-1)^{n_k-1}(-\mathcal{G}_2 y)\\ &= \mathcal{G}_2\left((-1)^{n_k}\langle\phi_k^{n_k},\phi_k^{n_k}\rangle y\right) = \mathcal{G}_2\Delta\phi_k^{n_k}\end{aligned}$$

This concludes that $\Lambda S v = \mathcal{G}_1\Lambda v + \mathcal{G}_2\Delta v$ holds for all $v \in \operatorname{span}\{\phi_k^l\}_{l=1}^{n_k}$. Since clearly $\Lambda\phi_j^m = 0$ and $\Delta\phi_j^m = 0$ for all $j \neq k$ and $m \in \{1,\dots,n_j\}$, we have for all $v \in \mathcal{D}(S)$

$$\begin{aligned}\Lambda S v &= \Lambda P_k S v = \Lambda S P_k v = \mathcal{G}_1\Lambda P_k v + \mathcal{G}_2\Delta P_k v\\ &= \mathcal{G}_1\Lambda v + \mathcal{G}_2\Delta v\end{aligned}$$

and thus $\Lambda S = \mathcal{G}_1\Lambda + \mathcal{G}_2\Delta$ in $\mathcal{D}(S)$. Now Lemma 3 implies $0 = (-1)^{n_k-1}\Delta\phi_k^{n_k} = \|\phi_k^{n_k}\|^2 y = y$. Substituting this into (10) we get $z = (i\omega_k I - \mathcal{G}_1)z_1$ and thus $z \in \mathcal{R}(i\omega_k I - \mathcal{G}_1)$. ■

Finally, Lemma 7 proves the sufficiency of the conditions (8) for the equivalence of the Sylvester equations.

Lemma 7: If the conditions (8) are satisfied, then the equations (6) and (7) are equivalent for all operator collections $\mathcal{O}$. ■

Proof: By Lemma 3 it is sufficient to show that (9) holds. Let $\Lambda \in \mathcal{L}(W,Z)$ and $\Delta \in \mathcal{L}(W,Y)$ be such that $\Lambda(\mathcal{D}(S)) \subset \mathcal{D}(\mathcal{G}_1)$ and

$$\Lambda S = \mathcal{G}_1\Lambda + \mathcal{G}_2\Delta. \tag{11}$$

Let $k \in \mathbb{Z}$. Applying the both sides of (11) to ϕ_k^1 we obtain

$$(i\omega_k I - \mathcal{G}_1)\Lambda\phi_k^1 = \mathcal{G}_2\Delta\phi_k^1.$$

Now the conditions (8a) and (8b) imply that $\Delta\phi_k^1 = 0$ and $(i\omega_k I - \mathcal{G}_1)\Lambda\phi_k^1 = 0$ and the condition (8c) shows that

$$\Lambda\phi_k^1 \in \mathcal{N}(i\omega_k I - \mathcal{G}_1) \subset \mathcal{N}(i\omega_k I - \mathcal{G}_1)^{d_k-1} \subset \mathcal{R}(i\omega_k I - \mathcal{G}_1).$$

Applying the both sides of (11) to ϕ_k^2 we obtain

$$(i\omega_k I - \mathcal{G}_1)\Lambda\phi_k^2 + \Lambda\phi_k^1 = \mathcal{G}_2\Delta\phi_k^2.$$

Since $\Lambda\phi_k^1 \in \mathcal{R}(i\omega_k I - \mathcal{G}_1)$, the conditions (8a) and (8b) imply that

$$\Delta\phi_k^2 = 0, \qquad (i\omega_k I - \mathcal{G}_1)\Lambda\phi_k^2 + \Lambda\phi_k^1 = 0$$

Since $\Lambda\phi_k^1 \in \mathcal{D}(\mathcal{G}_1)$, we see that $\Lambda\phi_k^2 \in \mathcal{D}(i\omega_k I - \mathcal{G}_1)^2$. Applying $(i\omega_k I - \mathcal{G}_1)$ to the both sides of the latter equation and using $\Lambda\phi_k^1 \in \mathcal{N}(i\omega_k I - \mathcal{G}_1)$ we obtain $(i\omega_k I - \mathcal{G}_1)^2\Lambda\phi_k^2 = 0$ and the condition (8c) implies

$$\Lambda\phi_k^2 \in \mathcal{N}(i\omega_k I - \mathcal{G}_1)^2 \subset \mathcal{N}(i\omega_k I - \mathcal{G}_1)^{d_k-1} \subset \mathcal{R}(i\omega_k I - \mathcal{G}_1).$$

Continuing the same procedure we see that $\Delta\phi_k^l = 0$ and

$$\Lambda\phi_k^l \in \mathcal{N}(i\omega_k I - \mathcal{G}_1)^l \subset \mathcal{N}(i\omega_k I - \mathcal{G}_1)^{d_k-1} \subset \mathcal{R}(i\omega_k I - \mathcal{G}_1)$$

for all $l \in \{1,\dots,n_k-1\}$. Applying the both sides of (11) to $\phi_k^{n_k}$ we obtain

$$(i\omega_k I - \mathcal{G}_1)\Lambda\phi_k^{n_k} + \Lambda\phi_k^{n_k-1} = \mathcal{G}_2\Delta\phi_k^{n_k}$$

and the conditions (8a) and (8b) imply that $\Delta\phi_k^{n_k} = 0$. Since $k \in \mathbb{Z}$ was arbitrary, we have shown that $\Delta\phi_k^l = 0$ for all $k \in \mathbb{Z}$ and $l \in \{1,\dots,n_k\}$. Since $\{\phi_k^l\}$ is a basis of W this concludes that $\Delta = 0$. ■

IV. Connection to the Internal Model Principle

In this section we study the conditions

$$\mathcal{R}(i\omega_k I - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) = \{0\}, \quad \forall k \in \mathbb{Z} \tag{12a}$$

$$\mathcal{N}(\mathcal{G}_2) = \{0\} \tag{12b}$$

and

$$\mathcal{N}(i\omega_k I - \mathcal{G}_1)^{d_k-1} \subset \mathcal{R}(i\omega_k I - \mathcal{G}_1) \quad \forall k \in \mathbb{Z} \tag{12c}$$

in greater detail. In particular we want to compare them to the concept of internal model in finite-dimensional control theory. The classical definition states that if $\dim Y = p$, then a controller incorporates an internal model of the exosystem if the following is satisfied [5]: *If $s \in \sigma(S)$ is an eigenvalue of S such that $d(s)$ is the dimension of the largest Jordan block associated to s, then $s \in \sigma(\mathcal{G}_1)$ and $\mathcal{G}_1$ has at least p Jordan blocks of dimension $\geq d(s)$ associated to s.* Since the operators S_k can be seen as Jordan blocks of the operator S, this property can be expressed as

For all $k \in \mathbb{Z}$ we have $\dim\mathcal{N}(i\omega_k I - \mathcal{G}_1) \geq \dim Y$ and $\mathcal{G}_1$ has at least $\dim Y$ independent Jordan chains of length $\geq d_k$ associated to the eigenvalue $i\omega_k$. (13)

We can see that even though the original definition of the internal model is given using the Jordan normal form, condition (13) also makes sense if the the controller is infinite-dimensional. The main purpose of this section is to compare the conditions (12) to the property (13).

In this section we make the following standing assumption.

Assumption 8: $\sigma(S) \cap \sigma(A) = \varnothing$ for all considered operators A. ■

The following theorem is the main result of this section.

Theorem 9: Let $\sigma(S) \cap \sigma(A_e) = \varnothing$ and $\dim Y < \infty$. The conditions (12) are satisfied if and only if (13) holds. ■

We will prove this theorem by proving a series of lemmas. Since these are also useful results considered separately, we will prove them using weaker assumptions whenever possible. In particular it is interesting to see that the conditions (12) imply the property (13) even if the space Y is infinite-dimensional.

Lemma 11 proves that the conditions (12) imply that the condition (13) holds. Lemmas 13, 14 and 15 then prove that the condition (13) implies conditions (12a), (12b) and (12c), respectively.

We will start by proving the following useful lemma.

Lemma 10: If $\sigma(S) \cap \sigma_p(A_e) = \varnothing$, then the operator $(P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)}$ is injective for all $s \in \sigma(S)$. ■

Proof: Let $s \in \sigma(S)$ and let $z \in \mathcal{N}(sI - \mathcal{G}_1)$ be such that $P(s)Kz = 0$. Choose $x = R(s,A)BKz \in \mathcal{D}(A)$. Now

$$\begin{aligned}(sI - A_e)\begin{bmatrix}x\\ z\end{bmatrix} &= \begin{bmatrix}(sI-A)x - BKz\\ -\mathcal{G}_2 Cx + (sI - \mathcal{G}_1)z - \mathcal{G}_2 DKz\end{bmatrix}\\ &= \begin{bmatrix}BKz - BKz\\ -\mathcal{G}_2(CR(s,A)B + D)Kz + (sI - \mathcal{G}_1)z\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}\end{aligned}$$

Since $s \in \sigma(S)$, we know that $s \notin \sigma_p(A_e)$ and thus $sI - A_e$ is injective. This implies that $z = 0$. This concludes that the restriction of $P(s)K$ to $\mathcal{N}(sI - \mathcal{G}_1)$ is an injection. ∎

The following lemma states that if (12) are satisfied, then the spaces $\mathcal{N}(i\omega_k I - \mathcal{G}_1)$ are isomorphic to Y and $\mathcal{G}_1$ has $\dim Y$ independent Jordan chains of length $\geq d_k$ associated to the eigenvalue $i\omega_k$ for all $k \in \mathbb{Z}$. This concludes that under assumptions of Theorem 9 the conditions (12) imply that (13) holds, but the result is more general in the sense that it doesn't require Y to be finite-dimensional.

Lemma 11: If $\sigma(S) \cap \sigma(A_e) = \varnothing$ and the conditions (12) are satisfied, then for every $k \in \mathbb{Z}$ the operator $(P(i\omega_k)K)|_{\mathcal{N}(i\omega_k I - \mathcal{G}_1)}$ is an isomorphism between $\mathcal{N}(i\omega_k I - \mathcal{G}_1)$ and Y and $\mathcal{G}_1$ has $\dim Y$ independent Jordan chains of length $\geq d_k$ associated to the eigenvalue $i\omega_k$. ∎

Proof: Let $k \in \mathbb{Z}$ and denote $s = i\omega_k$. From Lemma 10 we see that $(P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)}$ is injective and thus it is sufficient to prove that it is also surjective.

Since $\sigma(S) \cap \sigma(A_e) = \varnothing$, we have $s \in \rho(A_e)$ and $sI - A_e$ is surjective. Thus for all $z \in Z$ there exist $x_1 \in \mathcal{D}(A)$ and $z_1 \in \mathcal{D}(\mathcal{G}_1)$ such that

$$\begin{aligned}\begin{bmatrix} 0 \\ z \end{bmatrix} &= (sI - A_e) \begin{bmatrix} x_1 \\ z_1 \end{bmatrix} \\ &= \begin{bmatrix} (sI - A)x_1 - BKz_1 \\ -\mathcal{G}_2 C x_1 + (sI - \mathcal{G}_1)z_1 - \mathcal{G}_2 DKz_1 \end{bmatrix}\end{aligned}$$

Since $\sigma(S) \cap \sigma(A) = \varnothing$, we have $s \in \rho(A)$ and we get from the first equation that $x_1 = R(s, A)BKz_1$. Thus

$$\begin{aligned} z &= -\mathcal{G}_2 C R(s, A) BKz_1 + (sI - \mathcal{G}_1)z_1 - \mathcal{G}_2 DKz_1 \\ &= (sI - \mathcal{G}_1)z_1 - \mathcal{G}_2 P(s)Kz_1. \end{aligned} \tag{14}$$

Let $y \in Y$. Then $z = -\mathcal{G}_2 y \in \mathcal{R}(\mathcal{G}_2) \subset Z$ and we can choose $z_1 \in \mathcal{D}(\mathcal{G}_1)$ such that (14) holds. Now

$$\begin{aligned} & -\mathcal{G}_2 y = (sI - \mathcal{G}_1)z_1 - \mathcal{G}_2 P(s)Kz_1 \\ \Leftrightarrow \quad & \underbrace{-\mathcal{G}_2 y + \mathcal{G}_2 P(s)Kz_1}_{\in \mathcal{R}(\mathcal{G}_2)} = \underbrace{(sI - \mathcal{G}_1)z_1}_{\in \mathcal{R}(sI - \mathcal{G}_1)} \\ \Leftrightarrow \quad & \begin{cases} \mathcal{G}_2 y = \mathcal{G}_2 P(s)Kz_1 \\ 0 = (sI - \mathcal{G}_1)z_1 \end{cases} \Leftrightarrow \quad \begin{cases} y = P(s)Kz_1 \\ 0 = (sI - \mathcal{G}_1)z_1 \end{cases} \end{aligned}$$

because $\mathcal{R}(sI - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) = \{0\}$ and $\mathcal{N}(\mathcal{G}_2) = \{0\}$. This means that for every $y \in Y$ there exists $z_1 \in \mathcal{N}(sI - \mathcal{G}_1)$ such that $y = P(s)Kz_1$ and thus $(P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)}$ is surjective.

This concludes that $\dim \mathcal{N}(sI - \mathcal{G}_1) = \dim Y$. Since Jordan chains related to linearly independent eigenvectors are independent, it remains to show that there exists a Jordan chain of length $\geq d_k$ related to every $\psi_1 \in \mathcal{N}(sI - \mathcal{G}_1)$.

Since $\mathcal{N}(sI - \mathcal{G}_1)^l \subset \mathcal{N}(sI - \mathcal{G}_1)^{l+1}$ for all $l \in \mathbb{N}$, the condition (12c) implies that we have

$$\mathcal{N}(sI - \mathcal{G}_1) \subset \cdots \subset \mathcal{N}(sI - \mathcal{G}_1)^{d_k - 1} \subset \mathcal{R}(sI - \mathcal{G}_1). \tag{15}$$

Choose $\psi_1 \in \mathcal{N}(sI - \mathcal{G}_1)$. Then (15) implies that there exists $\psi_2 \in \mathcal{D}(\mathcal{G}_1)$ such that $(\mathcal{G}_1 - sI)\psi_2 = \psi_1 \in \mathcal{D}(\mathcal{G}_1)$. This implies that $\psi_2 \in \mathcal{D}(\mathcal{G}_1^2) = \mathcal{D}(\mathcal{G}_1 - sI)^2$ and

$$(\mathcal{G}_1 - sI)^2 \psi_2 = (\mathcal{G}_1 - sI)\psi_1 = 0.$$

Thus we have $\psi_2 \in \mathcal{N}(sI - \mathcal{G}_1)^2$. Define $\{\psi_l\}_{l=3}^{d_k}$ recursively as follows:

Let $l \in \{3, \ldots, d_k\}$. Assume $\psi_{l-1} \in \mathcal{N}(sI - \mathcal{G}_1)^{l-1}$. We have from (15) that there exists $\psi_l \in \mathcal{D}(\mathcal{G}_1)$ such that

$$(\mathcal{G}_1 - sI)\psi_l = \psi_{l-1} \in \mathcal{N}(sI - \mathcal{G}_1)^{l-1} \subset \mathcal{D}(sI - \mathcal{G}_1)^{l-1}.$$

Thus we have $\psi_l \in \mathcal{D}(\mathcal{G}_1 - sI)^l$ and

$$(\mathcal{G}_1 - sI)^l \psi_l = (\mathcal{G}_1 - sI)^{l-1} \psi_{l-1} = 0.$$

This implies that $\psi_l \in \mathcal{N}(sI - \mathcal{G}_1)^l$.

The resulting set $\{\psi_l\}_{l=1}^{d_k}$ satisfies

$$(sI - \mathcal{G}_1)\psi_1 = 0, \quad (\mathcal{G}_1 - sI)\psi_l = \psi_{l-1}, \quad l \in \{2, \ldots, d_k\}.$$

and thus by possibly adding elements to this set we obtain a Jordan chain $\{\psi_l\}_{l=1}^{m}$ with length $m \geq d_k$. ∎

From the previous lemma we see that the conditions (12) imply that $\mathcal{G}_1$ has *exactly* $\dim Y$ independent Jordan chains. This follows from our assumption $\sigma(S) \cap \sigma_p(A_e) = \varnothing$ as is shown in the next lemma.

Lemma 12: If $\sigma(S) \cap \sigma_p(A_e) = \varnothing$, then for all $k \in \mathbb{Z}$ we have $\mathcal{N}(i\omega_k I - \mathcal{G}_1) \leq \dim Y$. ∎

Proof: Let $k \in \mathbb{Z}$ and denote $s = i\omega_k$. We have from Lemma 10 that $(P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)} \in \mathcal{L}(\mathcal{N}(sI - \mathcal{G}_1), Y)$ is injective. Using the Rank-Nullity Theorem [10, Thm 4.7.7] we can conclude that

$$\begin{aligned} \dim \mathcal{N}(sI - \mathcal{G}_1) &= \dim \mathcal{R}\left((P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)}\right) \\ &\quad + \dim \mathcal{N}\left((P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)}\right) \\ &= \dim \mathcal{R}\left((P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)}\right) \leq \dim Y. \end{aligned}$$

∎

We will now show that if $\dim Y < \infty$, then the property (13) also implies that the conditions (12) hold. For this it is sufficient to assume that $\sigma(S) \cap \sigma_p(A_e) = \varnothing$. This is satisfied, for example, whenever the operator A_e generates a strongly stable C_0-semigroup, because then $\sigma_p(A_e) \subset \mathbb{C}^-$ [11]. The proof is divided into the following three lemmas.

Lemma 13: If $\sigma(S) \cap \sigma_p(A_e) = \varnothing$, $\dim Y < \infty$ and (13) holds, then $\mathcal{R}(i\omega_k I - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) = \{0\}$ for all $k \in \mathbb{Z}$. ∎

Proof: Let $k \in \mathbb{Z}$ and denote $s = i\omega_k$. Lemma 12 shows that we must have $\dim \mathcal{N}(sI - \mathcal{G}_1) = \dim Y$. Let $v \in \mathcal{R}(sI - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2)$. Then there exist $y \in Y$ and $z \in \mathcal{D}(\mathcal{G}_1)$ such that $v = \mathcal{G}_2 y = (sI - \mathcal{G}_1)z$. We will first show that there exists $z_1 \in \mathcal{D}(\mathcal{G}_1)$ such that

$$v = \mathcal{G}_2 P(s)Kz_1 = (sI - \mathcal{G}_1)z_1.$$

We have from Lemma 10 that $(P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)}$ is injective and since $\dim \mathcal{N}(sI - \mathcal{G}_1) = \dim Y$, it is invertible. Because of this we can choose $z_0 \in \mathcal{N}(sI - \mathcal{G}_1)$ such that

$$P(s)Kz_0 = y - P(s)Kz \in Y \quad \Leftrightarrow \quad y = P(s)K(z + z_0).$$

We then have

$$\mathcal{G}_2 P(s)K(z + z_0) = \mathcal{G}_2 y = (sI - \mathcal{G}_1)z = (sI - \mathcal{G}_1)(z + z_0)$$

and thus we can choose $z_1 = z + z_0$.

Choose $x_1 = R(s,A)BKz_1 \in \mathcal{D}(A)$. As in the proof of Lemma 10, we see that

$$(sI - A_e)\begin{bmatrix} x_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ -\mathcal{G}_2 P(s) K z_1 + (sI - \mathcal{G}_1) z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since $sI - A_e$ is injective we have $z_1 = 0$, which implies $v = (sI - \mathcal{G}_1)z_1 = 0$. Thus $\mathcal{R}(sI - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) = \{0\}$. ■

Lemma 14: If $\sigma(S) \cap \sigma_p(A_e) = \varnothing$, $\dim Y < \infty$ and (13) holds, then $\mathcal{N}(\mathcal{G}_2) = \{0\}$. ■

Proof: Let $y \in \mathcal{N}(\mathcal{G}_2)$ and $k \in \mathbb{Z}$ and denote $s = i\omega_k$. Lemma 12 shows that $\dim \mathcal{N}(sI - \mathcal{G}_1) = \dim Y$. We have from Lemma 10 that $(P(s)K)|_{\mathcal{N}(sI - \mathcal{G}_1)}$ is injective and since $\dim \mathcal{N}(sI - \mathcal{G}_1) = \dim Y$, it is invertible. This implies that there exists $z_1 \in \mathcal{N}(sI - \mathcal{G}_1)$ such that $y = P(s)Kz_1$ and thus $\mathcal{G}_2 P(s) K z_1 = 0$. Choose $x_1 = R(s,A)BKz_1 \in \mathcal{D}(A)$. As in the proof of Lemma 10, we see that

$$(sI - A_e)\begin{bmatrix} x_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ -\mathcal{G}_2 P(s) K z_1 + (sI - \mathcal{G}_1) z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since $s \in \sigma(S)$ and $\sigma(S) \cap \sigma_p(A_e) = \varnothing$, we have that $sI - A_e$ is injective and thus $z_1 = 0$. This further implies that $y = P(s)Kz_1 = 0$ and thus $\mathcal{N}(\mathcal{G}_2) = \{0\}$. ■

Lemma 15: If $\sigma(S) \cap \sigma_p(A_e) = \varnothing$, $\dim Y < \infty$ and (13) holds, then $\mathcal{N}(i\omega_k I - \mathcal{G}_1)^{d_k - 1} \subset \mathcal{R}(i\omega_k I - \mathcal{G}_1)$ for all $k \in \mathbb{Z}$. ■

Proof: Let $k \in \mathbb{Z}$ and denote $s = i\omega_k$ and $N = \dim Y$. Lemma 12 shows that we must have $\dim \mathcal{N}(sI - \mathcal{G}_1) = N$.

By our assumption $\mathcal{G}_1$ has N independent Jordan chains $\{\psi_n^l\}_{l=1}^{m_n}$ with $m_n \geq d_k$ associated to s. Because by the definition of the Jordan chain we have $\psi_n^k \in \mathcal{R}(sI - \mathcal{G}_1)$ for all $n \in \{1, \ldots, N\}$ and $k \in \{1, \ldots, d_k - 1\}$, to prove the lemma it is sufficient to show that

$$\mathcal{N}(sI - \mathcal{G}_1)^m \subset \operatorname{span}\left\{ \psi_n^l \;\middle|\; n \leq N,\ l \leq m \right\} \tag{16}$$

for $m \in \{1, \ldots, d_k - 1\}$. We will do this using induction. Since the set $\{\psi_n^1\}_{n=1}^N$ is linearly independent and $\psi_n^1 \in \mathcal{N}(sI - \mathcal{G}_1)$ for all $n \in \{1, \ldots, N\}$, we have

$$\mathcal{N}(sI - \mathcal{G}_1) = \operatorname{span}\{\psi_n^1\}_{n=1}^N \tag{17}$$

and thus (16) holds for $m = 1$.

Assume (16) holds for $m = j \in \{1, \ldots, d_k - 2\}$. Let $z \in \mathcal{N}(sI - \mathcal{G}_1)^{j+1}$. Then $z \in \mathcal{D}(\mathcal{G}_1)$ and

$$(sI - \mathcal{G}_1)z \in \mathcal{N}(sI - \mathcal{G}_1)^j.$$

Since we assumed (16) holds for $m = j$, there exist constants $\left\{ \alpha_n^l \;\middle|\; n = 1, \ldots, N,\ l = 1, \ldots, j \right\}$ such that

$$(sI - \mathcal{G}_1)z = \sum_{n=1}^N \sum_{l=1}^j \alpha_n^l \psi_n^l = \sum_{n=1}^N \sum_{l=1}^j \alpha_n^l (\mathcal{G}_1 - sI)\psi_n^{l+1}$$

where the second equality follows from the fact that $\{\psi_n^l\}_l$ are Jordan chains of $\mathcal{G}_1$ associated to s. This implies

$$(sI - \mathcal{G}_1)(z + \sum_{n=1}^N \sum_{l=1}^j \alpha_n^l \psi_n^{l+1}) = 0.$$

We now have from (17) that there exist constants $\{\alpha_n^0\}_{n=1}^N$ such that

$$z + \sum_{n=1}^N \sum_{l=1}^j \alpha_n^l \psi_n^{l+1} = \sum_{n=1}^N \alpha_n^0 \psi_n^1$$

and thus $z \in \operatorname{span}\left\{ \psi_n^l \;\middle|\; n \leq N,\ l \leq j+1 \right\}$. This implies that (16) holds for $m = j+1$. This completes the proof. ■

V. Conclusions

Necessary and sufficient conditions for decomposing of certain infinite-dimensional Sylvester equations have been presented. This property is closely related to robust output regulation of infinite-dimensional systems. The definition of the internal model of finite-dimensional control theory was generalized for infinite-dimensional systems. Using this generalization it was shown that under certain assumptions the conditions for the decomposing of the Sylvester equations are equivalent to the definition of the internal model.

In this paper we have only considered bounded solutions of the Sylvester equations. A more general theory of robust regulation may require the dynamic steady state operator Σ to be unbounded. Because of this, the results presented in this paper should be extended to allow unbounded operators Π and Γ.

It was also assumed that the exosystem has pure point spectrum. One area of further research is considering more general classes of exosystems.

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