

# A New Controller Structure for Robust Output Regulation

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**Abstract**—In this paper we employ a new controller structure in solving the robust output regulation problem for a linear distributed parameter system with finite or infinite-dimensional exosystems. In the case of an infinite-dimensional exosystem we also present additional conditions for achieving polynomial or logarithmic nonuniform decay rates for the closed-loop semigroup.

**Index Terms**—Output tracking, distributed parameter system, robustness.

## I. INTRODUCTION

The robust output regulation problem for an infinite-dimensional linear system

$$\dot{x}(t) = Ax(t) + Bu(t) + w(t), \quad x(0) = x_0 \in X \quad (1a)$$

$$y(t) = Cx(t) + Du(t) \quad (1b)$$

involves designing a controller in such a way that the output  $y(t)$  of the plant asymptotically tracks a given reference signal  $y_{ref}(t)$  despite the external disturbance signals  $w(t)$ . Robust output regulation of infinite-dimensional systems has been studied actively since the early 1980's, see [17], [16], [11], [13] and references therein.

In [11] the classical internal model principle of Francis and Wonham [5], and Davison [4] was extended for infinite-dimensional systems and for reference and disturbance signals generated by an infinite-dimensional exosystem. In its original form, the  $p$ -copy internal model principle is very much a finite-dimensional concept. Because of this, for infinite-dimensional systems the internal model has been redefined and has appeared in different forms in the literature [16], [7], [6]. In [11], three different definitions for an “internal model” were studied. Most notably, these included the *p-copy internal model principle*, which is very near the Jordan form version of the original definition by Francis and Wonham, and the so-called *G-conditions*. It was shown that under suitable conditions these concepts are equivalent, but they do possess differing properties. In particular, the *G-conditions* remain meaningful for plants whose input spaces are infinite-dimensional, whereas in such a situation the  $p$ -copy internal model becomes ambiguous.

In [6] the robust output regulation problem was solved with a controller that incorporates an internal model of the exosystem. The structure of the controller was chosen in such a way that the internal model could be inserted as a block in a triangular controller, and subsequently the remaining parameters of the controller were used in stabilizing the closed-loop system. The internal model in [6] was defined

using the *G-conditions*, and because of this, the form of the triangular structure was chosen in such a way that verifying the *G-conditions* for the controller became possible.

In this paper we solve the robust output regulation problem using an alternative controller structure. In particular, we choose a complementary triangular structure that is more natural for verifying that the controller incorporates the  $p$ -copy internal model of the exosystem.

Our alternative triangular structure also has two additional practical advantages. First of all, it allows controller design for plants that have more inputs than outputs. In addition, recently in [12] it was shown that if the class of admissible perturbations to the operators  $(A, B, C, D)$  of the plant (1) is restricted, the robust output regulation problem may be solvable with a controller that does not incorporate a full internal model. In order to design observer based controllers with reduced order internal models it becomes necessary to use the controller structure as defined in this paper [10]. For an example of application, see [10, Sec. VI].

The triangular controller structure used in this paper was first introduced in [10] where it was used in designing a controller with a reduced order internal model of a diagonal finite-dimensional exosystem. In this paper use it to solve the robust output regulation problem for two different types of full exosystems: A general finite-dimensional exosystem and a diagonal infinite-dimensional exosystem. For a finite-dimensional exosystem, the internal model based controller constructed in this paper achieves robust output regulation with exponential closed-loop stability. In the case of an infinite-dimensional exosystem, exponential stability is unachievable, and the closed-loop system is instead stabilized strongly. In the latter situation we show that under suitable additional assumptions it is possible to achieve nonuniform decay rates for the closed-loop semigroup. In particular, for systems on Hilbert spaces it is possible to achieve polynomial closed-loop stability.

If  $X$  and  $Y$  are Banach spaces and  $A : X \rightarrow Y$  is a linear operator, then  $\mathcal{D}(A)$  and  $\mathcal{N}(A)$  denote the domain and the null space of  $A$ , respectively. The space of bounded linear operators from  $X$  to  $Y$  is denoted by  $\mathcal{L}(X, Y)$ . If  $A : X \rightarrow X$ , then  $\sigma(A)$  and  $\rho(A) = \mathbb{C} \setminus \sigma(A)$  are its spectrum and its resolvent set, respectively. For  $\lambda \in \rho(A)$  the resolvent operator is given by  $R(\lambda, A) = (\lambda I - A)^{-1}$ .

## II. ASSUMPTIONS ON THE PLANT AND THE CONTROLLER

The operators of the plant (1) on a Banach space  $X$  are such that  $A : \mathcal{D}(A) \subset X \rightarrow X$  generates a strongly continuous semigroup  $T(t)$  on  $X$ . The input, output, and feedthrough operators are bounded in such a way that  $B \in$

$\mathcal{L}(U, X)$ ,  $C \in \mathcal{L}(X, Y)$ , and  $D \in \mathcal{L}(U, Y)$ , where the input space  $U$  is a Banach space and  $Y = \mathbb{C}^p$  is the output space. We assume that the pair  $(A, B)$  is exponentially stabilizable and the pair  $(C, A)$  is exponentially detectable. The transfer function of the plant is denoted by  $P(\lambda) = CR(\lambda, A)B + D$  for  $\lambda \in \rho(A)$ .

The reference and disturbance signals are generated by an exosystem of the form

$$\dot{v}(t) = Sv(t), \quad v(0) = v_0 \in W \quad (2a)$$

$$w(t) = Ev(t) \quad (2b)$$

$$y_{ref}(t) = -Fv(t) \quad (2c)$$

on a separable Hilbert space  $W$ , where  $S$  generates a strongly continuous group,  $E \in \mathcal{L}(W, X)$  and  $F \in \mathcal{L}(W, Y)$ . In this paper we consider two types of exosystems — one finite-dimensional and the other infinite-dimensional.

*Definition 1 (The Finite-Dimensional Exosystem):* For the finite-dimensional exosystem we have  $W = \mathbb{C}^r$  for some  $r \in \mathbb{N}$ ,  $S \in \mathcal{L}(W) = \mathbb{C}^{r \times r}$  is a matrix in its Jordan canonical form, and  $E \in \mathcal{L}(W, X)$  and  $F \in \mathcal{L}(W, Y) = \mathbb{C}^{p \times r}$ . The eigenvalues of  $\sigma(S) = \{i\omega_k\}_{k=1}^q \subset i\mathbb{R}$  are distinct.

*Definition 2 (The Infinite-Dimensional Exosystem):* For the infinite-dimensional diagonal exosystem we have that  $W = \ell^2(\mathbb{C})$  is a separable Hilbert space with the canonical orthonormal basis  $\{\phi_k\}_{k \in \mathbb{Z}}$ . The operator  $S$  is

$$Sv = \sum_{k \in \mathbb{Z}} i\omega_k \langle v, \phi_k \rangle \phi_k, \\ v \in \mathcal{D}(S) = \left\{ v \in W \mid \sum_{k \in \mathbb{Z}} |\omega_k|^2 |\langle v, \phi_k \rangle|^2 < \infty \right\},$$

where the eigenvalues  $\{i\omega_k\}_{k \in \mathbb{Z}} \subset i\mathbb{R}$  of  $S$  are distinct and have a uniform gap, i.e.,  $\inf_{k \neq l} |\omega_k - \omega_l| > 0$ . We also assume the sequence is ordered in such a way that  $\omega_k < \omega_l$  whenever  $k < l$ . The  $E$  and  $F$  are Hilbert-Schmidt operators, i.e.,  $(E\phi_k)_{k \in \mathbb{Z}} \in \ell^2(X)$  and  $(F\phi_k)_{k \in \mathbb{Z}} \in \ell^2(Y)$  ( $F$  automatically has this property due to the Riesz Representation Theorem).

For the finite-dimensional exosystem we denote by  $n_k \in \mathbb{N}$  the size of the Jordan block associated to  $i\omega_k \in \sigma(S)$ . For the infinite-dimensional exosystem we define  $n_k = 1$  for all  $k \in \mathbb{Z}$ . For both exosystems we assume that  $i\omega_k \in \rho(A)$  and that  $P(i\omega_k)$  is surjective for every  $k$ .

The regulation error is defined as  $e(t) = y(t) - y_{ref}(t)$ . We consider a dynamic error feedback controller

$$\dot{z}(t) = \mathcal{G}_1 z(t) + \mathcal{G}_2 e(t), \quad z(0) = z_0 \in Z \\ u(t) = Kz(t)$$

on a Banach space  $Z$ , where  $\mathcal{G}_1 : \mathcal{D}(\mathcal{G}_1) \subset Z \rightarrow Z$  generates a semigroup on  $Z$ ,  $\mathcal{G}_2 \in \mathcal{L}(Y, Z)$ , and  $K \in \mathcal{L}(Z, U)$ .

The plant and the controller can be written together as a closed-loop system on the product space  $X_e = X \times Z$  as

$$\dot{x}_e(t) = A_e x_e(t) + B_e v(t), \quad x_e(0) = x_{e0} = \begin{bmatrix} x_0 \\ z_0 \end{bmatrix} \\ e(t) = C_e x_e(t) + D_e v(t),$$

where  $C_e = [C \quad DK]$ ,  $D_e = -F$ ,

$$A_e = \begin{bmatrix} A & BK \\ \mathcal{G}_2 C & \mathcal{G}_1 + \mathcal{G}_2 DK \end{bmatrix} \quad \text{and} \quad B_e = \begin{bmatrix} E \\ -\mathcal{G}_2 F \end{bmatrix}.$$

Due to our assumptions, the operator  $A_e$  generates a strongly continuous semigroup  $T_e(t)$  on  $X_e$ .

### III. THE ROBUST OUTPUT REGULATION PROBLEM

Our main control problem is defined in the following.

*The Robust Output Regulation Problem:* Choose  $(\mathcal{G}_1, \mathcal{G}_2, K)$  in such a way that the following are satisfied:

1. The closed-loop system is exponentially/strongly stable, i.e., the semigroup  $T_e(t)$  generated by  $A_e$  is exponentially/strongly stable.
2. For all initial states  $v_0 \in W$  and  $x_{e0} \in X_e$  the regulation error goes to zero asymptotically, i.e.,  $\lim_{t \rightarrow \infty} e(t) = 0$ .
3. If the operators  $(A, B, C, D, E, F)$  are perturbed to  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{E}, \tilde{F})$  in such a way that  $\tilde{A}_e$  generates an exponentially/strongly stable semigroup,  $\sigma(\tilde{A}_e) \cap \sigma(S) = \emptyset$  and the Sylvester equation  $\Sigma S = \tilde{A}_e \Sigma + B_e$  has a solution, then  $\lim_{t \rightarrow \infty} e(t) = 0$  for all  $v_0 \in W$  and  $x_{e0} \in X_e$ .

The theory developed in [11] shows that the controllers solving the robust output regulation problem are characterized by the internal model principle in Theorem 4.

*Definition 3 (The p-Copy Internal Model):* A controller  $(\mathcal{G}_1, \mathcal{G}_2, K)$  is said to incorporate a  $p$ -copy internal model of the exosystem (2) if for all  $k$  we have

$$\dim \mathcal{N}(i\omega_k - \mathcal{G}_1) \geq \dim Y$$

and  $\mathcal{G}_1$  has at least  $\dim Y = p$  independent Jordan chains of length greater than or equal to  $n_k$  associated to the eigenvalue  $i\omega_k$ .

*Theorem 4:* Assume the controller  $(\mathcal{G}_1, \mathcal{G}_2, K)$  stabilizes the closed-loop system strongly in such a way that  $i\omega_k \in \rho(A_e)$  for all  $k$ , and the Sylvester equation  $\Sigma S = A_e \Sigma + B_e$  has a solution  $\Sigma \in \mathcal{L}(W, X_e)$ . Then the controller solves the robust output regulation problem if and only if it incorporates a  $p$ -copy internal model of the exosystem.

### IV. THE CONTROLLER WITH A P-COPY INTERNAL MODEL

In this section we define the controller structure used in solving the robust output regulation problem. As the state space of the controller we choose  $Z = Z_1 \times Z_2$ , where  $Z_1 = \mathbb{C}^{nz}$  and  $Z_2$  is a Banach space. The parameters  $(\mathcal{G}_1, \mathcal{G}_2, K)$  of the controller are of the form

$$\mathcal{G}_1 = \begin{bmatrix} G_1 & R_1 \\ 0 & R_2 \end{bmatrix}, \quad \mathcal{G}_2 = \begin{bmatrix} G_2 \\ R_3 \end{bmatrix}, \quad K = [K_1, K_2],$$

where  $G_1 \in \mathcal{L}(Z_1)$ ,  $R_1 \in \mathcal{L}(Z_2, Z_1)$ ,  $R_2 : \mathcal{D}(R_2) \subset Z_2 \rightarrow Z_2$  generates a semigroup on  $Z_2$ ,  $G_2 \in \mathcal{L}(Y, Z_1)$ ,  $R_3 \in \mathcal{L}(Y, Z_2)$ ,  $K_1 \in \mathcal{L}(Z_1, U)$ , and  $K_2 \in \mathcal{L}(Z_2, U)$ .

The operator  $G_1$  is called the *internal model* of the exosystem (2). We choose  $G_1$  to be a block diagonal operator

$$G_1 = \text{block diag} \underbrace{(S, S, \dots, S)}_{p \text{ times}} \quad (3)$$

with domain  $\mathcal{D}(G_1) = \mathcal{D}(S) \times \dots \times \mathcal{D}(S)$  on the space  $Z_1 = W^p$ . With this choice the controller incorporates a  $p$ -copy internal model, as is shown in the next theorem.

*Theorem 5:* The controller  $(\mathcal{G}_1, \mathcal{G}_2, K)$  incorporates a  $p$ -copy internal model of the exosystem.

*Proof:* For the finite-dimensional exosystem: Because the frequencies  $i\omega_k$  are distinct, we immediately see that for every  $k \in \{1, \dots, q\}$  we have  $\dim \mathcal{N}(i\omega_k I - G_1) = p$  and  $G_1$  has exactly  $p$  independent Jordan chains of length  $n_k$  associated to  $i\omega_k$ . The structure of  $\mathcal{G}_1$  further implies that the operator  $\mathcal{G}_1$  is such that  $\dim \mathcal{N}(i\omega_k I - \mathcal{G}_1) \geq p$  and  $\mathcal{G}_1$  has at least  $p$  independent Jordan chains of length  $\geq n_k$  associated to  $i\omega_k$ .

For the infinite-dimensional exosystem we similarly see that  $\dim \mathcal{N}(i\omega_k I - G_1) = p$ , and further  $\dim \mathcal{N}(i\omega_k I - \mathcal{G}_1) \geq p$ . ■

In the following theorem we choose the parameters of the controller  $(\mathcal{G}_1, \mathcal{G}_2, K)$  in such a way that the closed-loop system is stabilized in a suitable way. The type of stability available depends solely on the type of stability achievable for the semigroup generated by  $G_1 + G_2(CH_{e1} + DK_1)$ .

*Theorem 6:* Choose  $Z = Z_1 \times X$ ,  $K = [K_1, -K_2]$ ,

$$\mathcal{G}_1 = \begin{bmatrix} G_1 & G_2(C + DK_2) \\ 0 & A + BK_2 + L(C + DK_2) \end{bmatrix}, \quad \mathcal{G}_2 = \begin{bmatrix} G_2 \\ L \end{bmatrix},$$

where  $G_1$  is as in (3), and  $K_2 \in \mathcal{L}(X, U)$  and  $L_1 \in \mathcal{L}(Y, X)$  are chosen in such a way that  $A + BK_2$  and  $A + L_1 C$  generate exponentially stable semigroups.

For any  $K_1 \in \mathcal{L}(Z_1, U)$  the Sylvester equation

$$H_{e1}G_1 = (A + L_1C)H_{e1} + (B + L_1D)K_1. \quad (4)$$

has a unique solution  $H_{e1} \in \mathcal{L}(Z_1, X)$  satisfying  $H_{e1}(\mathcal{D}(G_1)) \subset \mathcal{D}(A)$ .

Assume then, that  $K_1 \in \mathcal{L}(Z_1, U)$  and  $G_2 \in \mathcal{L}(Y, Z_1)$  can be chosen in such a way that  $G_1 + G_2(CH_{e1} + DK_1)$  generates an exponentially/strongly stable semigroup. If we then choose  $L = L_1 + H_{e1}G_2$ , then the closed-loop system is exponentially/strongly stable. Moreover, there exist  $M \geq 1$  and  $\omega_0 > 0$  such that the resolvent operator of the closed-loop system satisfies

$$\|R(i\omega, A_e)\| \leq M \|R(i\omega, G_1 + G_2(C_1H_{e1} + DK_1))\| \quad (5)$$

for  $\omega \in \mathbb{R}$  with  $|\omega| \geq \omega_0$ .

*Proof:* The operator  $A + L_1C$  generates an exponentially stable semigroup and  $-G_1$  generates a semigroup with growth bound equal to zero. Now [15, Cor. 8] shows that for any  $K_1 \in \mathcal{L}(Z_1, U)$  the Sylvester equation (4) has a unique solution  $H_{e1} \in \mathcal{L}(Z_1, X)$  satisfying  $H_{e1}(\mathcal{D}(G_1)) \subset \mathcal{D}(A)$ .

If the controller  $(\mathcal{G}_1, \mathcal{G}_2, K)$  is chosen as suggested, then

$$A_e = \begin{bmatrix} A & BK_1 & -BK_2 \\ G_2C & G_1 + G_2DK_1 & G_2C \\ LC & LDK_1 & A + BK_2 + LC \end{bmatrix}.$$

If we choose a similarity transform  $Q_e \in \mathcal{L}(X \times Z_1 \times X)$  by

$$Q_e = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ I & 0 & I \end{bmatrix} \quad \text{and} \quad Q_e^{-1} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -I & 0 & I \end{bmatrix},$$

we can define  $\hat{A}_e = Q_e A_e Q_e^{-1}$  on  $X \times Z_1 \times X$ , and compute

$$\begin{aligned} \hat{A}_e &= Q_e \begin{bmatrix} A + BK_2 & BK_1 & -BK_2 \\ 0 & G_1 + G_2DK_1 & G_2C \\ -A - BK_2 & LDK_1 & A + BK_2 + LC \end{bmatrix} \\ &= \begin{bmatrix} A + BK_2 & BK_1 & -BK_2 \\ 0 & G_1 + G_2DK_1 & G_2C \\ 0 & (B + LD)K_1 & A + LC \end{bmatrix}. \end{aligned}$$

Denote

$$\hat{A}_{e1} = \begin{bmatrix} G_1 + G_2DK_1 & G_2C \\ (B + LD)K_1 & A + LC \end{bmatrix}.$$

Since  $A + BK_2$  is exponentially stable, the block triangular structure shows that  $\hat{A}_e$  (and hence also  $A_e$  by similarity) is exponentially/strongly stable if  $\hat{A}_{e1}$  is exponentially/strongly stable [6, Lem. 20]. Since  $L = L_1 + H_{e1}G_2$ , we have

$$\hat{A}_{e1} = \begin{bmatrix} G_1 & 0 \\ (B + L_1D)K_1 & A + L_1C \end{bmatrix} + \begin{bmatrix} G_2 \\ H_{e1}G_2 \end{bmatrix} [DK_1 \ C].$$

Define  $Q_{e1} = \begin{bmatrix} I & 0 \\ H_{e1} & -I \end{bmatrix} \in \mathcal{L}(Z_1 \times X)$  with  $Q_{e1}^{-1} = Q_{e1}$ . Since  $H_{e1}$  satisfies the equation (4), a direct computation yields

$$Q_{e1}^{-1} \begin{bmatrix} G_1 & 0 \\ (B + L_1D)K_1 & A + L_1C \end{bmatrix} Q_{e1} = \begin{bmatrix} G_1 & 0 \\ 0 & A + L_1C \end{bmatrix}.$$

Therefore, if we define  $A_{e1} = Q_{e1}^{-1} \hat{A}_{e1} Q_{e1}$ , then

$$\begin{aligned} A_{e1} &= \begin{bmatrix} G_1 & 0 \\ 0 & A + L_1C \end{bmatrix} + \begin{bmatrix} G_2 \\ 0 \end{bmatrix} [CH_{e1} + DK_1 \ -C] \\ &= \begin{bmatrix} G_1 + G_2(CH_{e1} + DK_1) & -G_2C \\ 0 & A + L_1C \end{bmatrix}. \end{aligned}$$

Since  $A + L_1C$  generates an exponentially stable semigroup and  $G_1 + G_2(CH_{e1} + DK_1)$  is exponentially/strongly stable, also the semigroup generated by  $A_{e1}$  is exponentially/strongly stable by [6, Lem. 20]. This finally concludes that the closed-loop system is exponentially/strongly stable.

Finally, the fact that there exist  $M \geq 1$  and  $\omega_0 > 0$  such that (5) holds for  $\omega \in \mathbb{R}$  with  $|\omega| \geq \omega_0$  follows from the triangular structures of operators  $A_{e1}$  and  $\hat{A}_e$  similarly as in the proof of [13, Lem. 19] ■

## V. STABILIZATION OF THE INTERNAL MODEL

Theorem 6 shows that in order to stabilize the closed-loop system, we must be able to choose the operators  $K_1 \in \mathcal{L}(Z_1, U)$  and  $G_2 \in \mathcal{L}(Y, Z_1)$  in such a way that the operator  $G_1 + G_2(CH_{e1} + DK_1)$  generates a suitably stable semigroup. These choices are the main topic of this section. We will complete the stabilization of the internal model separately for the two different types of exosystems.

Denote  $P_L(\lambda) = CR(\lambda, A + L_1C)(B + L_1D) + D$  for  $\lambda \in \mathbb{C}^+$ . This is the transfer function of the plant (1) after an output injection  $L_1y(t) = L_1(Cx(t) + Du(t))$ . Since  $\{i\omega_k\}_k \in \rho(A) \cap \rho(A + L_1C)$ , the surjectivity of  $P(i\omega_k)$  also implies that  $P_L(i\omega_k)$  is surjective for every  $k$ .

### A. Stabilization of the Finite-Dimensional Internal Model

Since we assumed  $S$  to be in its Jordan canonical form, we can denote the Euclidean basis vectors  $\{e_k\}_{k=1}^r$  of  $W = \mathbb{C}^r$  as

$$\{\phi_1^1, \dots, \phi_1^{n_1}, \phi_2^1, \dots, \phi_2^{n_2}, \dots, \phi_q^1, \dots, \phi_q^{n_q}\} = \{e_1, \dots, e_r\},$$

and then for every  $k \in \{1, \dots, q\}$  the sequence  $\{\phi_k^l\}_{l=1}^{n_k}$  is a Jordan chain of  $S$  associated to the eigenvalue  $i\omega_k$ . Moreover, define  $\varphi_{(k,l)}^j \in W^p$  by

$$\begin{aligned} \varphi_{(k,l)}^1 &= (\phi_k^l, 0, \dots, 0)^T, & \varphi_{(k,l)}^2 &= (0, \phi_k^l, 0, \dots, 0)^T, \dots, \\ \varphi_{(k,l)}^p &= (0, \dots, 0, \phi_k^l)^T. \end{aligned}$$

Then  $\{\varphi_{(k,l)}^j \mid k = 1, \dots, p, l = 1, \dots, n_k, j = 1, \dots, p\}$  is an orthonormal basis of  $Z_1 = \mathbb{C}^{pr}$ , and  $\mathcal{N}(i\omega_k - G_1) = \text{span}\{\varphi_{(k,1)}^j\}_{j=1}^p$ . Finally, for every  $k \in \{1, \dots, q\}$  the independent Jordan chains of  $G_1$  associated to  $i\omega_k$  are

$$\begin{aligned} &\{\varphi_{(k,1)}^1, \dots, \varphi_{(k,n_k)}^1\}, \quad \{\varphi_{(k,1)}^2, \dots, \varphi_{(k,n_k)}^2\}, \quad \dots, \\ &\{\varphi_{(k,1)}^p, \dots, \varphi_{(k,n_k)}^p\}. \end{aligned}$$

We choose the operator  $K_1 \in \mathcal{L}(Z_1, U)$  as

$$K_1 = \sum_{k=1}^q \sum_{j=1}^p \langle \cdot, \varphi_{(k,1)}^j \rangle P_L(i\omega_k)^\dagger e_j$$

where  $P_L(i\omega_k)^\dagger$  is the Moore–Penrose Pseudoinverse of  $P_L(i\omega_k)$ , and  $\{e_k\}_{k=1}^p \subset Y = \mathbb{C}^p$  are the Euclidean basis vectors of  $Y = \mathbb{C}^p$ .

We will now show that the pair  $(CH_{e1} + DK_1, G_1)$  is detectable. Since  $Z_1 = \mathbb{C}^{pr}$  and  $Y = \mathbb{C}^p$  are finite-dimensional and since  $\sigma(G_1) = \{i\omega_k\}_{k=1}^q$  and  $\mathcal{N}(i\omega_k - G_1) = \text{span}\{\varphi_{(k,1)}^j\}_{j=1}^p$  for every  $k \in \{1, \dots, q\}$ , the detectability of  $(CH_{e1} + DK_1, G_1)$  can be verified by showing that  $(CH_{e1} + DK_1)\varphi_{(k,1)}^j \neq 0$  for every  $k \in \{1, \dots, q\}$  and  $j \in \{1, \dots, p\}$  [8, Thm. 6.2-5]. To do this, we first need to compute  $H_{e1}\varphi_{(k,1)}^j$ . Applying both sides of the Sylvester equation (4) to  $\varphi_{(k,1)}^j$  (and using  $G_1\varphi_{(k,1)}^j = i\omega_k\varphi_{(k,1)}^j$ ) we get

$$\begin{aligned} (H_{e1}G_1 - (A + L_1C)H_{e1})\varphi_{(k,1)}^j &= (B + L_1D)K_1\varphi_{(k,1)}^j \\ \Rightarrow (i\omega_k I - A - L_1C)H_{e1}\varphi_{(k,1)}^j &= (B + L_1D)K_1\varphi_{(k,1)}^j \\ \Rightarrow H_{e1}\varphi_{(k,1)}^j &= R(i\omega_k, A + L_1C)(B + L_1D)K_1\varphi_{(k,1)}^j. \end{aligned}$$

A further computation shows

$$\begin{aligned} (CH_{e1} + DK_1)\varphi_{(k,1)}^j &= CR(i\omega_k, A + L_1C)(B + L_1D)K_1\varphi_{(k,1)}^j + DK_1\varphi_{(k,1)}^j \\ &= P_L(i\omega_k)K_1\varphi_{(k,1)}^j = P_L(i\omega_k)P_L(i\omega_k)^\dagger e_j = e_j \neq 0, \end{aligned}$$

since  $P_L(i\omega_k)$  is surjective and thus  $P_L(i\omega_k)^\dagger$  is a right inverse. This concludes that the pair  $(CH_{e1} + DK_1, G_1)$  is detectable, and therefore  $G_2$  can be chosen in such a way that  $G_1 + G_2(CH_{e1} + DK_1)$  generates an exponentially stable semigroup (i.e., the matrix is Hurwitz).

We have arrived that the following conclusion.

**Theorem 7:** For a finite-dimensional exosystem, the choices of the controller parameters in Theorem 6 together with the above internal model  $(G_1, G_2, K_1)$  solve the robust output regulation problem in such a way that the closed-loop system is exponentially stable.

*Proof:* Since  $S$  is a finite-dimensional operator with  $\sigma(S) \subset i\mathbb{R}$  and  $A_e$  generates an exponentially stable semigroup, the Sylvester equation  $\Sigma S = A_e \Sigma + B_e$  has a solution  $\Sigma \in \mathcal{L}(W, X_e)$  satisfying  $\mathcal{R}(\Sigma) \subset \mathcal{D}(A_e)$  [15]. The conclusion of the theorem now follows from Theorems 4 and 6.  $\blacksquare$

### B. Stabilization of the Infinite-Dimensional Internal Model

In the case of an infinite-dimensional exosystem we have  $Z_1 = W^p$ . For every  $k \in \mathbb{Z}$  we define  $\varphi_k^l \in W^p$  by

$$\begin{aligned} \varphi_k^1 &= (\phi_k, 0, \dots, 0)^T, & \varphi_k^2 &= (0, \phi_k, 0, \dots, 0)^T, \dots, \\ \varphi_k^p &= (0, \dots, 0, \phi_k)^T. \end{aligned}$$

Then  $\{\varphi_k^l \mid k \in \mathbb{Z}, l = 1, \dots, p\}$  is an orthonormal basis of  $Z_1$  and  $\mathcal{N}(i\omega_k - G_1) = \text{span}\{\varphi_k^l\}_{l=1}^p$ . Let  $(h_k)_{k \in \mathbb{Z}} \in \ell^2(\mathbb{C})$  be such that  $h_k \neq 0$  for all  $k \in \mathbb{Z}$ . The operator  $K_1$  is chosen to be

$$K_1 = \sum_{k \in \mathbb{Z}} \sum_{l=1}^p \langle \cdot, \varphi_k^l \rangle h_k \frac{P_L(i\omega_k)^\dagger e_l}{\|P_L(i\omega_k)^\dagger\|}, \quad (6)$$

where  $P_L(i\omega_k)^\dagger$  is the Moore–Penrose Pseudoinverse of  $P_L(i\omega_k)$ , and  $\{e_1, \dots, e_p\} \subset Y = \mathbb{C}^p$  are the Euclidean basis vectors of  $Y = \mathbb{C}^p$ . We clearly have  $K_1 \in \mathcal{L}(Z_1, U)$ .

As we saw in Theorem 6, the Sylvester equation (4) has a unique solution  $H_{e1} \in \mathcal{L}(Z_1, X)$  satisfying  $H_{e1}(\mathcal{D}(G_1)) \subset \mathcal{D}(A)$ . Applying both sides of (4) to  $\varphi_k^l$  for  $k \in \mathbb{Z}$  and  $l \in \{1, \dots, p\}$  yields

$$\begin{aligned} (H_{e1}G_1 - (A + L_1C)H_{e1})\varphi_k^l &= (B + L_1D)K_1\varphi_k^l \\ \Rightarrow (i\omega_k I - A - L_1C)H_{e1}\varphi_k^l &= (B + L_1D)K_1\varphi_k^l \\ \Rightarrow H_{e1}\varphi_k^l &= R(i\omega_k, A + L_1C)(B + L_1D)K_1\varphi_k^l. \end{aligned}$$

Since  $\{\varphi_k^l \mid k \in \mathbb{Z}, l = 1, \dots, p\}$  is a basis of  $Z_1$ , this concludes that  $H_{e1}$  is given by the formula

$$H_{e1} = \sum_{k \in \mathbb{Z}} \sum_{l=1}^p \langle \cdot, \varphi_k^l \rangle R(i\omega_k, A + L_1C)(B + L_1D)K_1\varphi_k^l.$$

We will now show that  $G_2$  can be chosen in such a way that the semigroup generated by  $G_1 + G_2(CH_{e1} + DK_1)$  is strongly stable. Using (6) and the formula for  $H_{e1}$  shows that

$$\begin{aligned} (CH_{e1} + DK_1)\varphi_k^l &= CR(i\omega_k, A + L_1C)(B + L_1D)K_1\varphi_k^l + DK_1\varphi_k^l \\ &= P_L(i\omega_k)K_1\varphi_k^l = h_k P_L(i\omega_k) \frac{P_L(i\omega_k)^\dagger e_l}{\|P_L(i\omega_k)^\dagger\|} \\ &= \frac{h_k}{\|P_L(i\omega_k)^\dagger\|} e_l. \end{aligned}$$

The sequence  $(\|P_L(i\omega_k)\|)_{k \in \mathbb{Z}}$  is uniformly bounded since  $A + L_1C$  is exponentially stable. Because

$$1 = \|I\| = \|P_L(i\omega_k)P_L(i\omega_k)^\dagger\| \leq \|P_L(i\omega_k)\| \|P_L(i\omega_k)^\dagger\| \\ \Leftrightarrow \frac{1}{\|P_L(i\omega_k)^\dagger\|} \leq \|P_L(i\omega_k)\|,$$

also the sequence  $(1/\|P_L(i\omega_k)^\dagger\|)_{k \in \mathbb{Z}}$  is uniformly bounded with respect to  $k \in \mathbb{Z}$ . Let  $z_1 \in Z_1$  be arbitrary. Since  $Z_1 = W^p$ ,  $z_1$  is of the form  $z_1 = (w_1, w_2, \dots, w_p)^T$ , where  $w_l \in W$  for every  $l \in \{1, \dots, p\}$ . Moreover, we have  $\langle z_1, \varphi_k^l \rangle = \langle w_l, \phi_k \rangle$  for every  $l \in \{1, \dots, p\}$ . If we denote  $c_1 = \sum_{k \in \mathbb{Z}} \frac{h_k}{\|P_L(i\omega_k)^\dagger\|} \phi_k \in W$ , then

$$(CH_{e1} + DK_1)z_1 = \sum_{k \in \mathbb{Z}} \sum_{l=1}^p \langle z_1, \varphi_k^l \rangle (CH_{e1} + DK_1) \varphi_k^l \\ = \sum_{l=1}^p \sum_{k \in \mathbb{Z}} \langle z_1, \varphi_k^l \rangle \frac{h_k}{\|P_L(i\omega_k)^\dagger\|} e_l \\ = \sum_{l=1}^p e_l \sum_{k \in \mathbb{Z}} \langle w_l, \phi_k \rangle \frac{h_k}{\|P_L(i\omega_k)^\dagger\|} \\ = \sum_{l=1}^p e_l \left\langle w_l, \sum_{k \in \mathbb{Z}} \frac{\bar{h}_k}{\|P_L(i\omega_k)^\dagger\|} \phi_k \right\rangle = \sum_{l=1}^p e_l \langle w_l, c_1 \rangle.$$

This concludes that the operator  $G_1 + G_2(CH_{e1} + DK_1)$  is of the form

$$G_1 + G_2(CH_{e1} + DK_1) \\ = \begin{bmatrix} S & & \\ & \ddots & \\ & & S \end{bmatrix} + \begin{bmatrix} G_2^{11} & \dots & G_2^{1p} \\ \vdots & \ddots & \vdots \\ G_2^{p1} & & G_2^{pp} \end{bmatrix} \begin{bmatrix} \langle \cdot, c_1 \rangle \\ \vdots \\ \langle \cdot, c_1 \rangle \end{bmatrix}$$

where  $G_2^{lj} \in \mathcal{L}(\mathbb{C}, W) = W$  for every  $l, j \in \{1, \dots, p\}$ . If we choose  $G_2^{lj} = 0$  whenever  $l \neq j$ , and  $G_2^{ll} = g_2 \in W$  for all  $l \in \{1, \dots, p\}$ , then the operator becomes diagonal, i.e.,

$$G_1 + G_2(CH_{e1} + DK_1) = \begin{bmatrix} S + g_2 \langle \cdot, c_1 \rangle & & \\ & \ddots & \\ & & S + g_2 \langle \cdot, c_1 \rangle \end{bmatrix}.$$

It is now clear that the properties of the semigroup generated by  $G_1 + G_2(CH_{e1} + DK_1)$  follow from those of the semigroup generated by the operator  $S + g_2 \langle \cdot, c_1 \rangle$ .

Since  $S$  has compact resolvent and generates a diagonal contraction semigroup, and since  $\langle c_1, \phi_k \rangle = \bar{h}_k / \|P(i\omega_k)^\dagger\| \neq 0$ , the internal model can be stabilized strongly with the choice  $g_2 = -c_1 \in W$  [2]. Theorems 9 and 10 show that under suitable assumptions on the growth of the norms  $\|P(i\omega_k)^\dagger\|$  we can achieve additional stability properties for the closed-loop system. The results use the following sufficient condition for the solvability of the Sylvester equation  $\Sigma S = A_e \Sigma + B_e$ .

**Lemma 8:** If the closed-loop system is strongly stable and  $(\|R(i\omega_k, A_e)\| \|B_e \phi_k\|)_{k \in \mathbb{Z}} \in \ell^2(\mathbb{C})$ , then  $\Sigma S = A_e \Sigma + B_e$  has a solution  $\Sigma \in \mathcal{L}(W, X_e)$  satisfying  $\Sigma(\mathcal{D}(S)) \subset \mathcal{D}(A_e)$ .

*Proof:* The results follows directly from [6, Lem. 6]. ■

**Theorem 9:** If  $\frac{\|P_L(i\omega_k)^\dagger\|}{|h_k|} = \mathcal{O}(|k|^\beta)$  for some  $\beta > 0$ , then for any  $\alpha > \beta + 1/2$  the operator  $G_2$  can be chosen in such a way that the closed-loop system is strongly stable,  $\sigma(A_e) \cap i\mathbb{R} = \emptyset$  and  $\|R(i\omega, A_e)\| = \mathcal{O}(|\omega|^\alpha)$ . The controller then solves the robust output regulation problem if  $B_e \in \mathcal{L}(W, X_e)$  satisfies  $(|\omega_k|^\alpha \|B_e \phi_k\|)_{k \in \mathbb{Z}} \in \ell^2(\mathbb{C})$ . Finally, if  $X$  is a Hilbert space, then the closed-loop is polynomially stable and there exists  $M_e \geq 1$  such that

$$\|T_e(t)x_e\| \leq \frac{M_e}{t^{1/\alpha}} \|A_e x_e\|, \quad t > 0 \quad (7)$$

for every  $x_e \in \mathcal{D}(A_e)$ .

*Proof:* From the structure of the operator  $G_1 + G_2(CH_{e1} + DK_1)$  it is clear that if  $S + g_2 \langle \cdot, c_1 \rangle$ , generates a strongly stable semigroup with the property  $\sigma(S + g_2 \langle \cdot, c_1 \rangle) \cap \sigma(S) = \emptyset$ , then the same is true for  $G_1 + G_2(CH_{e1} + DK_1)$ . Furthermore, since  $W$  is a Hilbert space and  $S^* = -S$ , this is equivalent to the operator  $-S + c_1 \langle \cdot, g_2 \rangle$  generating a strongly stable semigroup and  $\sigma(-S + c_1 \langle \cdot, g_2 \rangle) \cap \sigma(-S) = \emptyset$ . The operator  $-S$  is a diagonal operator with simple eigenvalues that have a uniform gap. Because of this, we can choose  $g_2 \in W$  using pole placement [18]. More precisely, we apply Theorem 15 in [14] (where a similar stabilization problem was considered for SISO systems). It is sufficient to replace  $|\langle g_2, \phi_k \rangle| \|P_K(i\omega_k)\|$  in [14, Thm. 17] by  $|h_k| / \|P_L(i\omega_k)^\dagger\|$ . After this modification the proof of [14, Thm. 15] gives a way of choosing  $g_2 \in W$  in such a way that  $\sigma(-S + c_1 \langle \cdot, g_2 \rangle) = \{\mu_k\}_{k \in \mathbb{Z}}$ , where

$$\mu_0 = -1 \quad \text{and} \quad \mu_k = -\frac{1}{|k|^\alpha} - i\omega_k.$$

The resulting operator  $-S + c_1 \langle \cdot, g_2 \rangle$  — and consequently also  $G_1 + G_2(CH_{e1} + DK_1)$  — is a Riesz-spectral operator with eigenvalues  $\{\mu_k\}_{k \in \mathbb{Z}}$  and all but a finite number of these eigenvalues are simple. Because of this, the semigroup generated by  $G_1 + G_2(CH_{e1} + DK_1)$  is strongly stable,  $\sigma(G_1 + G_2(CH_{e1} + DK_1)) \cap \sigma(S) = \emptyset$ . Moreover, a geometric argument can be used to show that for a large enough  $|\omega|$

$$\|R(i\omega, G_1 + G_2(CH_{e1} + DK_1))\| \\ \leq \frac{\tilde{M}_1}{\text{dist}(i\omega, \sigma(-S + c_1 \langle \cdot, g_2 \rangle))} = \frac{\tilde{M}_1}{\min_k |i\omega - \mu_k|} \\ \leq \tilde{M}_1 \tilde{M}_2 |\omega|^\alpha$$

Together with Theorem 6 this implies

$$\|R(i\omega, A_e)\| \leq M \|R(i\omega, G_1 + G_2(CH_{e1} + DK_1))\| \\ = \mathcal{O}(|\omega|^\alpha).$$

If  $X$  is a Hilbert space, we can also renorm  $X_e$  to be Hilbert, and we have from [3, Thm. 2.4] that there exists  $M_e \geq 1$  such that (7) holds.

The solvability of  $\Sigma S = A_e \Sigma + B_e$  follows from Lemma 8 and  $\|R(i\omega_k, A_e)\| = \mathcal{O}(|\omega_k|^\alpha)$ . By Theorems 4 and 6 the controller solves the robust output regulation problem. ■

**Theorem 10:** If  $\frac{\|P_L(i\omega_k)^\dagger\|}{|h_k|} = \mathcal{O}(e^{\beta|k|})$  for some  $\beta > 0$ , then for any  $\alpha > \beta$  the operator  $G_2$  can be chosen in

such a way that the closed-loop system is strongly stable,  $\sigma(A_e) \cap i\mathbb{R} = \emptyset$  and  $\|R(i\omega, A_e)\| = \mathcal{O}(e^{-\alpha|\omega|})$ . The controller then solves the robust output regulation problem if  $B_e \in \mathcal{L}(W, X_e)$  satisfies  $(e^{\alpha|\omega_k|} \|B_e \phi_k\|)_{k \in \mathbb{Z}} \in \ell^2(\mathbb{C})$ . Finally, the closed-loop is nonuniformly stable in such a way that there exist  $M_e \geq 1$  and  $t_0 > 1$  such that

$$\|T_e(t)x_e\| \leq \frac{M_e}{\ln t} \|A_e x_e\|, \quad t > t_0 \quad (8)$$

for every  $x_e \in \mathcal{D}(A_e)$ .

*Proof:* The stabilization can be completed using pole placement as in the proof of Theorem 9, but now the  $\{\mu_k\}_{k \in \mathbb{Z}}$  are chosen to be

$$\mu_0 = -1 \quad \text{and} \quad \mu_k = -e^{-\alpha|k|} - i\omega_k.$$

The possibility of choosing  $g_2$  in such a way that  $\sigma(-S + c_1(\cdot, g_2)) = \{\mu_k\}_{k \in \mathbb{Z}}$  and  $-S + c_1(\cdot, g_2)$  is a Riesz spectral operator with at most finite nonsimple eigenvalues now follows from verifying the required conditions in [18] (similarly as in, for example, [9, Sec. 3.3.3]). As in the proof of Theorem 9 we again have that  $G_1 + G_2(CH_{e1} + DK_1)$  is a Riesz spectral operator with  $\sigma(G_1 + G_2(CH_{e1} + DK_1)) = \{\mu_k\}_{k \in \mathbb{Z}}$  at most finite nonsimple eigenvalues. Theorem 6 and a geometric argument now show that for large  $|\omega|$  we have

$$\begin{aligned} \|R(i\omega, A_e)\| &\leq M \|R(i\omega, G_1 + G_2(CH_{e1} + DK_1))\| \\ &\leq M \frac{\tilde{M}_1}{\min_k |i\omega - \mu_k|} = \mathcal{O}(e^{-\alpha|\omega|}) \end{aligned}$$

Finally, we have from [1, Thm. 1.5 & Ex. 1.6] that there exist  $M_e \geq 1$  and  $t_0 > 1$  such that (8) holds.

The solvability of  $\Sigma S = A_e \Sigma + B_e$  follows from Lemma 8 and  $\|R(i\omega_k, A_e)\| = \mathcal{O}(e^{\alpha|\omega_k|})$ . By Theorems 4 and 6 the controller solves the robust output regulation problem. ■

## VI. EXAMPLE: CONTROL OF HARMONIC OSCILLATORS

In this example we control two undamped harmonic oscillators

$$\ddot{q}_1(t) + q_1(t) = F_1(t), \quad \ddot{q}_2(t) + 2q_2(t) = F_2(t).$$

The control inputs are the external forces  $F_1(t)$  and  $F_2(t)$ . The objective of the robust output regulation problem is to drive the positions of the oscillators to a constant state where  $q_1(t) - q_2(t) = 1$ . To achieve this, we can choose a measurement  $y(t) = q_1(t) - q_2(t)$  and track a constant reference signal  $y_{\text{ref}}(t) \equiv 1$  generated by a one-dimensional exosystem with  $S = 0 \in \mathbb{C}$ ,  $F = -1$ , and  $v_0 = 1$ . Figure 1 shows the behavior of  $q_1$  and  $q_2$  when the control law is chosen as in Sections IV and V. The matrices  $L_1$ ,  $K_2$ , and  $G_2$  are all chosen using pole placement so that the appropriate stabilized matrices have stability margins equal to 2. Even though it would have been possible to use only one control input to solve the control problem, the availability of two inputs should allow subsequent improvement of the performance of the control law.

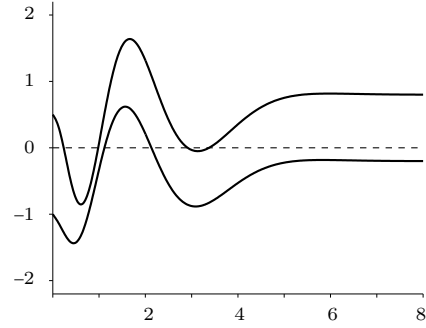


Fig. 1. Oscillator positions  $q_1(t)$  and  $q_2(t)$ .

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